ON PRINCIPAL IDEALS AND BANDS IN l-ALGEBRA

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Let A be a uniformly complete l-algebra with unit element e. Some results are obtained about algebra structure of principal ideal and principal band.

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1. INTRODUCTION

An ordered vector spaces E is called a Riesz space (or a vector lattice) if $\sup\{x, y\} = x \vee y$ (or $\inf\{x, y\} = x \wedge y$) exits in E, for all $x, y \in E$. Sets of the form $[x, y] = \{z \in E : x \leq z \leq y\}$ are called order intervals or simply intervals. The subset M of E is said to be order bounded if M is included in some order interval. A linear map T, between E and L, is said to be order bounded whenever T maps order bounded sets into order bounded sets. Order bounded linear maps between E and L will be denoted by $L_b(E, L)$. We denote $L_b(E)$ the order bounded operators from E into itself. A mapping $\pi \in L_b(E)$ is called an orthomorphism if $x \perp y$ (i.e., $|x| \wedge |y| = 0$) implies $\pi x \perp y$. An order bounded operator T on an Archimedean Riesz space E is orthomorphism if and only if $T(B) \subseteq B$ holds for each band B of E. The set of orthomorphisms of E will be denoted by $Orth(E)$. The principal order ideal generated by the identity operator I in $Orth(E)$ is called the ideal center of E and is denoted by $Z(E)$ (i.e., $Z(E) = \{\pi \in Orth(E) : |\pi| \leq \lambda I$, for some $\lambda \in \mathbb{R}_+\}$). If E is an Dedekind complete Riesz space then $Z(E)$ is the ideal generated by I and $Orth(E)$ is the band generated by I in $L_b(E)$. Morever, if E is a Banach lattice then $Z(E) = Orth(E)$.

Let A be a Riesz algebra (lattice ordered algebra), i.e., A is a Riesz space which is simultaneously an associative algebra with the additional property that $a, b \in A^+$ implies that $a.b \in A^+$. An f-algebra A is a Riesz algebra which satisfies the extra requirement that $a \perp b$ implies that $ac \perp b = ca \perp b$, for all $c \in A^+$. Every Archimedean f-algebra is commutative. $Orth(E)$ and $Z(E)$ are f-algebras under pointwise order and composition of operators.

In [4], Huijsmans has given some properties for principal ideal and band generated by unit in a uniformly complete l-algebra A with unit element $e > 0$, which are similar properties to $Z(E)$ and $Orth(E)$. Alpay and Uyar have studied algebra structure of principal ideals and bands generated by order bounded operators in $L_b(E)$ [3]. In this paper, we will obtain some results about algebra structure of principal ideals and bands generated by an element in a uniformly complete l-algebra A with unit element $e > 0$.

The Riesz spaces in this paper are assumed to be Archimedean. For all undefined terminology concerning vector lattice and l-algebra we shall adhere to the definitions in [1], [5], [6] and [8].

2. MAIN RESULTS

Let A be a l-algebra with unit element $e > 0$. We shall now investigate whether I_a (principal ideal generated by a), and B_a (principal band generated by a) are subalgebras of A.

Theorem 2.1 ([4], Theorem 1). *Let A be a uniformly complete l-algebra with unit element $e > 0$. Then, the following holds:*

(1) *The principal band B_e generated by e in A is a projection band, i.e., $A = B_e \oplus B_e^d$.*

(2) *For any $a \in B_e$, the left multiplication π_a^l by a and right multiplication π_a^r by a are orthomorphisms.*

Theorem 2.2 ([4], Theorem 2). *Let A be a Banach l-algebra with unit element $e > 0$. Then, the principal ideal I_e generated by e satisfies $I_e = B_e$, and so, $A = I_e \oplus I_e^d$.*

We now give a simple necessary and sufficient condition for I_a to be subalgebra.

Lemma 2.3. *Let A be an l-algebra and $a \in A$. Then, I_a is an subalgebra if and only if $|a|^2 \leq \lambda |a|$, for some $\lambda \in \mathbb{R}$.*

Corollary 2.4. *Let A be a uniformly complete l-algebra with unit element $e > 0$. Then, the following holds:*

(1) *I_e and B_e are Archimedean f-algebras with unit element e;*

(2) *$I_e = Z(I_e)$ and $B_e = Orth(B_e)$ (lattice and algebra isomorphic).*

Proof. (1) I_e is a subalgebra from Lemma 2.3. As $\pi_a^l \in Orth(A)$, $\pi_a^r \in Orth(A)$ and B_e is a band, B_e is an f-algebra.

(2) It follows from Theorem 141.1 in [8]. □

Proposition 2.5. *Let A be a uniformly complete l-algebra with unit element $e > 0$, and a^{-1} exists and is positive for $a \in A^+$. Then, the principal* *band B_a generated by a is a projection band, i.e., $A = B_a \oplus B_a^d$. If A is a Banach l-algebra then, $I_a = B_a$ is satisfied, and so, $A = I_a \oplus I_a^d$.*

Proof. In view of Theorem 24.7 in [5], it is sufficient to show that $\sup_n(b\wedge na)$ exists in B_a for each $b \in A^+$. Since B_e is a projection band, $\sup_n(a^{-1}b \wedge ne) = v$ exist in B_e, for each $b \in A^+$. The mapping π_a^l is evidently a positive isomorphism. Furthermore, the invers mapping $(\pi_a^l)^{-1} = \pi_{a^{-1}}^l$ is also positive. Hence, π_a^l is an order continous lattice isomorphism. We have $\sup_n(b \wedge na) = av$ and $av \in B_a$ since $\pi_v^r \in Orth(A)$. Let A be a Banach lattice and $0 \leq b \in B_a$. We have to prove that $b \in I_a$. By Theorem 3.4 in [1], $\sup_n(b \wedge na) = b$. Therefore, $\sup_n(a^{-1}b\wedge ne) = a^{-1}b$. This implies $a^{-1}b \in I_e$ as B_e is a projection band and $I_e = B_e$. We find $b \in I_a$ as desired. □

Although a^{-1} exists and is positive for $0 \leq a \in A$, I_a (or B_a) may not be subalgebra.

Example 2.6. Let A denote the set of all 2×2 real matrix. It is well known that A is Dedekind complete (hence, uniformly complete) l-algebra with matrix operations and the order "$[a_{i,j}] \leq [b_{i,j}] \Leftrightarrow a_{i,j} \leq b_{i,j}$, for all i and j".

Let $0 \leq a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then, a^{-1} exists in A and is positive but a^2 is not an element of I_a (or B_a). Hence, I_a (or B_a) is not a subalgebra.

Proposition 2.7. *Let A be a Dedekind complete l-algebra with unit element $e > 0$ and $a \in A^+$. Then, the following are equivalent:*
(i) *$a \in B_e$ and I_a is a subalgebra;*
(ii) *$a \in I_e$.*

Proof. (i) $\Rightarrow$ (ii) Let $a \in B_e$ and I_a be a subalgebra. As I_a be a subalgebra $0 \leq a^2 \leq \lambda a$, for some $\lambda \in \mathbb{R}^+$. By the Theorem 8.15 in [1] and Corollary 2.4 (2) there exists $b \in B_e$ such that $0 \leq b \leq e$ and $\lambda ab = a^2$. We have $a(a-\lambda b) = 0$ and consequently $a\wedge|a-\lambda b| = 0$ because of B_e is an f-algebra. But $a-\lambda b \leq a$ implies that $a - \lambda b \leq a \wedge |a - \lambda b| = 0$. Hence, $a \leq \lambda b \leq \lambda e$, so $a \in I_e$.
(ii) $\Rightarrow$ (i) It is evident. □

Proposition 2.8. *Let A be an l-algebra with unit element $e > 0$, and a^{-1} exists and is positive for $a \in A^+$. Then, B_a is a subalgebra if and only if $a \in B_e$.*

Proof. Suppose B_a is a subalgebra. As $a^2 \in B_a$, we have $\sup_n(a^2 \wedge na) = a^2$. Therefore, $\sup_n(a \wedge ne) = a$ and this yields $a \in B_e$. Let now $a \in B_e$. Since $\pi_a^l \in Orth(A)$, we get $\pi_a^l(B_a) \subseteq B_a$. Hence, B_a is a subalgebra.
The hypothesis that a^{-1} be positive is indispensable. □

Example 2.9. Let A be the l-algebra in Example 2.6 and $0 \leq a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then, a^{-1} exists in A and B_a is a subalgebra. But, a is not element of B_e.

PROPOSITION 2.10. *Let A be an l-algebra with unit element $e > 0$, and a^{-1} exists and is positive for $a \in A^+$. Then,*

$$B_a = \{b \in A : b = ac, \text{ for some } c \in B_e\}.$$

Proof. Since π_a^l is a lattice homomorphism, the set $\{b \in A : b = ac$, for some $c \in B_e\}$ is Riesz subspace of A. Furthermore, it is easy to see that

$$b \in B_a \Leftrightarrow \sup_n (b \wedge na) = b \Leftrightarrow \sup_n (a^{-1}b \wedge ne) = a^{-1}b \Leftrightarrow a^{-1}b \in B_e,$$

and $b = a(a^{-1}b)$, for each $0 \leq b$. Thus, the equality follows.

Let A be an algebra and M be a non-empty subset of A. The commutant of M is denoted by $M_c = \{a \in A : am = ma$, for all $m \in M\}$. Suppose E is a Riesz space with separating order dual. Let $x \in E$ be arbitrary and $0 \leq y \leq x$. E is said to have topologically full centre (or topologically full) if there exists a net $0 \leq \pi_\alpha \leq I$ in $Z(E)$ with $\pi_\alpha x \to y$ in $\sigma(E, E^\sim)$. The class of Riesz spaces that have topologically full centre are quite large. Each σ-Dedekind complete Riesz spaces and each unital f-algebras with separating order dual are topologically full [7]. In [2] we have shown that if E is a topologically full Riesz space, then the commutant $Z(E)$ in $L_b(E)$ is $Orth(E)$, i.e., $Z(E)_c = Orth(E)$. Now, we will give similar properties for the commutant of I_a in a l-algebra A.

After then, $A^\sim$ (or $E^\sim$) will be assumed to separate the points of A (or E). This assumption implies that A (or E) is Archimedean.

Let E, F and G be Riesz space and $\mu : E \times F \to G$ be a bilinear map. If for each $x \in E^+$ and $y \in F^+$, the maps $\mu_x : F \to G : z \to \mu_x(z) = \mu(x, z)$ and $\mu_y : E \to G : u \to \mu_y(u) = \mu(u, y)$ are positive (lattice homomorphism), then μ is called bipositive (bilattice homomorphism). Let E be a Riesz space. We can define a bilinear map

$$\mu : E \times E^\sim \to Z(E)^\sim : (x, f) \to \mu_{x,f}, \mu_{x,f}(\pi) = f(\pi x), \quad \text{for } \pi \in Z(E).$$

Some properties of this map is studied in [7]. We will study similar a bilenear map.

If A is an l-algebra and $a \in A^+$ then, the following bilenear map is considered

$$\mu : A \times A^\sim \to I_a^\sim : (x, f) \to \mu_{x,f}, \mu_{x,f}(b) = f(bx), \quad \text{for } b \in I_a.$$

For each $x \in A^+$, $0 \leq f \in A^\sim$, the map $\mu_x : A^\sim \to I_a^\sim$ and the map $\mu_f : A \to I_a^\sim$ are positive and we have $|\mu_{x,f}| \leq \mu_{|x|,|f|}$, for each each $(x, f) \in A \times A^\sim$. If

A is a topologically full algebra we can say more about the positivity of the map μ.

THEOREM 2.11. *If A is a topologically full algebra and $a \in A^+$ then, $\mu_f : A \to I_a^\sim$ is a lattice homomorphism, for each $0 \le f \in A^\sim$.*

Proof. Let $0 \le f \in A^\sim$. It is enough to show that $\mu_f(x) \wedge \mu_f(y) = 0$ for each x, y in A satisfying $x \wedge y = 0$. It is well known that

$$\begin{aligned}[\mu_f(x) \wedge \mu_f(y)](b) &= [\mu_{x,f} \wedge \mu_{y,f}](b) \\ &= \inf\{\mu_{x,f}(c) + \mu_{y,f}(d) : 0 \le c, d \in I_a,\ c + d = b\} \\ &= \inf\{f(cx) + f(dy) : 0 \le c, d \in I_a,\ c + d = b\}.\end{aligned}$$

Let $z = x + y$ and I_x, I_y and I_z be respectively, the order ideals generated by x, y and z. Then, I_z is actually the order direct sum of I_x and I_y by Theorem 1.7.6 in [5]. We denote p the order projection of I_z onto I_x. Let J be the restriction to I_z of order bounded functionals on A. Then, J is an order ideal in $I_z^\sim$ because if $f \in I_z^\sim$ satisfies $0 \le f \le g/_{I_z}$, for some $g \in A^\sim$ then f has extension to a positive functional on A by Theorem 2.3 in [1]. The adjoint $p^\sim : I_z^\sim \to I_z^\sim$ of p satisfies $0 \le p^\sim \le I$ and as a consequence we obtain $p^\sim(J) \subseteq J$. As a result of these simple observations, the pair (I_z, J) constitutes a Riesz pair and $p : (I_z, \sigma(I_z, J)) \to (I_z, \sigma(I_z, J))$ is continuous. Since $0 \le p(z) \le z$ there exists (π_α) in $Z(A)$ such that $0 \le \pi_\alpha \le I$ and $\pi_\alpha(z) \to p(z) = x$, and also the continuity of p $p(\pi_\alpha(z)) = \pi_\alpha(p(z)) \to p(x)$ now yields $\pi_\alpha(x) \to x$ in $\sigma(I_z, J)$. Since $\pi_\alpha(z) = \pi_\alpha(x) + \pi_\alpha(y)$ for each α, we have $\pi_\alpha(y) \to 0$ in $\sigma(I_z, J)$. As

$$[\mu_f(x) \wedge \mu_f(y)](b) \le f((b - \pi_\alpha(b))x + \pi_\alpha(b)y)$$

for each α, we obtain

$$[\mu_f(x) \wedge \mu_f(y)](b) \le \lim_\alpha f((b - \pi_\alpha(b))x + \pi_\alpha(b)y) = 0,$$

which completes the proof. □

THEOREM 2.12. *Let A be a topologically full l-algebra with unit element $e > 0$, and a^{-1} exists for $a \in A^+$. Then, $(I_a)_c \subseteq B_e$.*

Proof. Let $b \in (I_a)_c$ be arbitrary. We show that $\pi_b^l \in Orth(A)$. If $x \perp y$ in A, then the positivity of the map μ implies

$$|\mu_{x,f}| \le \mu_{|x|,|f|} \le \mu_{|x|,|f| \vee |g|} \quad \text{and} \quad |\mu_{y,g}| \le \mu_{|y|,|g|} \le \mu_{|y|,|f| \vee |g|}$$

for each $f, g \in A^\sim$. Hence,

$$0 \le |\mu_{x,f}| \wedge |\mu_{y,g}| \le \mu_{|x|,|f| \vee |g|} \wedge \mu_{|y|,|f| \vee |g|} = \mu_{|x| \wedge |y|,|f| \vee |g|} = 0$$

by Theorem 2.11. Thus, if $x \perp y$ in A, then $\mu_{x,f} \perp \mu_{y,g}$ in $I_a^{\sim}$, for each $f, g \in A^{\sim}$. On the other hand, as $b \in (I_a)_c$ we have

$$\mu_{\pi_b^l(x),f}(z) = \mu_{bx,f}(z) = f(z(bx)) = f(b(zx)) = (\pi_b^l)^{\sim}(f)(zx) = \mu_{x,(\pi_b^l)^{\sim}(f)}(z),$$

for each $z \in I_a$. Hence, $\mu_{\pi_b^l(x),f} \perp \mu_{y,f}$, for each $f \in A^{\sim}$. As μ_f is a lattice homomorphism for each $0 \leq f \in A^{\sim}$, $\mu_{|\pi_b^l(x)| \wedge |y|,f} = 0$. Thus, $\mu_{|\pi_b^l(x)| \wedge |y|,f}(a) = 0$, for each $f \in A^{\sim}$. Since $A^{\sim}$ separetes the points of A, we have $a(|\pi_b^l(x)| \wedge |y|) = 0$ which yields $|\pi_b^l(x)| \wedge |y| = 0$ as a^{-1} exists. So, $\pi_b^l \in Orth(A)$. It follows that $\pi_b^l(e) = b \in B_e$ as $\pi_b^l(B_e) \subseteq B_e$.

Corollary 2.4 suggests that if A is a uniformly complete l-algebra with unit element $e > 0$ then, B_e is an Archimedean f-algebra. Hence, B_e is a commutative algebra. By this observation and Theorem 2.12 we can give a corollary.

Corollary 2.13. *If A is a topologically full uniformly complete l-algebra with unit element $e > 0$, then $(I_e)_c = (B_e)_c = B_e$.*

Corollary 2.14. *If A is a topologically full uniformly complete l-algebra with unit element $e > 0$, and a^{-1} exists for $a \in A^+$. Then $(I_a)_c = B_e$ if and only if $a \in B_e$.*

Proof. If $a \in B_e$, then it is clear that $(I_a)_c = B_e$. Let $(I_a)_c = B_e$. This implies that $ba = ab$, for each $b \in B_e$. Hence, $a \in (B_e)_c = B_e$. □

Corollary 2.15. *If A is a topologically full uniformly complete l-algebra with unit element $e > 0$, then B_e is a full subalgebra (i.e., if $c \in B_e$ and c^{-1} exists in A, then $c^{-1} \in B_e$).*

Proof. Since $(B_e)_c = B_e$ and $(B_e)_c$ is a full subalgebra, then B_e is a full subalgebra. □

Corollary 2.16. *Let A be a topologically full uniformly complete l-algebra with unit element $e > 0$, and a^{-1} exists and is positive for $a \in A^+$. If B_a is an algebra, then it is full subalgebra.*

Proof. Let B_a be an algebra, $b \in B_a$ and b^{-1} exists in A. As B_e is full algebra, we have

$$\begin{aligned} b \in B_a &\Rightarrow b \wedge na \uparrow b \Rightarrow ba^{-1} \wedge ne \uparrow ba^{-1} \Rightarrow ba^{-1} \in B_e \Rightarrow ab^{-1} \in B_e \\ &\Rightarrow ab^{-1} \wedge ne \uparrow ab^{-1} \Rightarrow a^2b^{-1} \wedge na \uparrow a^2b^{-1} \Rightarrow a^2b^{-1} \in B_a. \end{aligned}$$

Since B_a is an algebra, $a \in B_e$ by Proposition 2.8. Hence, $a^{-1} \in B_e$. This implies $\pi_{a^{-1}}^l \in Orth(A)$. It follows that $b^{-1} \in B_a$ as $\pi_{a^{-1}}^l(B_a) \subseteq B_a$. □

Corollary 2.17. *Let A be a topologically full uniformly complete l-algebra with unit element $e > 0$, and a^{-1} exists and is positive for $a \in A^+$. Then, B_a is an algebra if and only if $B_a = B_e$.*

Proof. By Proposition 2.8 and Corollary 2.16, the proof is clear. □

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