HILBERT PRO-C^*-BIMODULES AND APPLICATIONS

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In this paper, we introduce the notion of a Hilbert pro-C^*-bimodule over a pro-C^*-algebra and study its structure. Examples and constructions of this kind of bimodules are also given. As applications, we present a result of automatic continuity for derivations in Hilbert pro-C^*-bimodules and a realization of the so-called "compact" operators of a Hilbert pro-C^*-bimodule over a pro-C^*-algebra.

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1. INTRODUCTION

The concept of a Hilbert C^*-module was first introduced by I. Kaplansky in 1953, while developing the theory of commutative unital algebras. This concept generalizes the notion of a Hilbert space, which in its turn constitutes a generalization of a Euclidean space. Since 1953, a continuous development of the theory of Hilbert C^*-modules has started, which increased in the last forty years, having offered a very rich literature and useful tools in various important fields of Mathematics. In the 1970's, the theory was extended independently by W.L. Paschke and M.A. Rieffel to non commutative C^*-algebras and the latter author used it to construct the theory of "induced representations of C^*-algebras". Moreover, Hilbert C^*-modules gave the right context for the extension of the notion of Morita equivalence to C^*-algebras and have played a crucial role in Kasparov's KK-theory. Finally, they may be considered as a generalization of vector bundles to non-commutative C^*-algebras, therefore they play a significant role in non-commutative geometry and, in particular, in C^*-algebraic quantum group theory and groupoid C^*-algebras. The extension of such a richness in results concept, to the case of pro-C^*-algebras (:inverse limits of C^*-algebras) could not be disregarded. It was A. Mallios who first considered in 1985 (see [15]) (finitely generated) modules, over a topological $*$-algebra A, endowed with an A-valued inner product and used the standard Hilbert module H_A over a pro-C^*-algebra A. In 1988, N.C. Phillips considers Hilbert modules over pro-C^*-algebras, in [18]; in this regard, also see [24] and

[13]. But the main body of the work on Hilbert modules over pro-C^*-algebras is due to M. Joiţa; all her work on the subject can be found in her book, under the homonym title "Hilbert modules over locally C^*-algebras" (see [9]). Our reference for Hilbert C^*-modules will be [14]. A detailed list of references on the theory of Hilbert C^*-modules is exhibited in "Hilbert C^*-modules Homepage".

Brown, Mingo and Shen introduced the notion of a Hilbert C^*-bimodule E over a C^*-algebra A and proved that the two topologies inherited on E by its left and right A-module structure coincide [4, Corollary 1.11]. As far as we know the notion of a Hilbert C^*-bimodule over a pro-C^*-algebra has not yet been studied. In this paper, we first consider the question whether an analogous result to that of Brown, Mingo and Shen, mentioned before, holds true in case A is a pro-C^*-algebra. Towards an affirmative answer to this question we give the definition of a Hilbert pro-C^*-bimodule over a pro-C^*-algebra, and study some aspects of its structure (Section 3). Furthermore, we give some examples (Section 4) and construct Hilbert pro-C^*-bimodules as inverse limits of Hilbert C^*-bimodules (Section 5). Finally, we give two applications in Section 6. The first one concerns the continuity of a derivation from a pro-C^*-algebra $A[\tau_\Gamma]$ in a Hilbert bimodule $E[\tau]$ over A. Moreover, we remark that any such derivation $\delta : A[\tau_\Gamma] \to E[\tau]$ whose the factorization $\delta_\lambda : A_\lambda \to E^A_\lambda$ is inner for every $\lambda \in \Lambda$, is approximately inner, where $\Gamma = (p_\lambda)_{\lambda\in\Lambda}$ is the directed family of C^*-seminorms defining the topology τ_Γ and $(p^A_\lambda)_{\lambda\in\Lambda}$, the family of seminorms defining the locally convex topology τ of E, through its inner product(s). These results extend previous results due to J.R. Ringrose in [20] and R. Becker in [1]. The second application concerns a realization of "compact" operators on a Hilbert pro-C^*-bimodule E over a σ-C^*-algebra $A[\tau_\Gamma]$, by the closed two-sided $*$-ideal of $A[\tau_\Gamma]$ generated by the set $\{{}_A\langle \xi, \eta\rangle : \xi, \eta \in E\}$. The latter extends a relevant result of Brown, Mingo and Shen [4, Proposition 1.10] for Hilbert C^*-bimodules.

2. PRELIMINARIES AND DEFINITIONS

Throughout this paper all algebras are considered over the field $\mathbb{C}$ of complexes and all topological spaces are assumed to be Hausdorff.

A *pro-C^*-algebra* $A[\tau_\Gamma]$ is a complete topological $*$-algebra for which there exists an upward directed family Γ of C^*-seminorms $(p_\lambda)_{\lambda\in\Lambda}$ defining the topology τ_Γ [7, Definition 7.5]. In [7], pro-C^*-algebras are called locally C^*-algebras (A. Inoue), whereas in [1], they are named LMC^*-algebras. A pro-C^*-algebra, whose topology is defined by an upward directed countable family of C*-seminorms, will be called a σ-C^*-algebra as, for example in [19]. Every pro-C^*-algebra has jointly continuous multiplication (Sebestyén, see [7, Theorem 7.2]).

For a pro-C^*-algebra $A[\tau_\Gamma]$, and every $\lambda \in \Lambda$, the quotient normed $*$-algebra $A_\lambda := A/N_\lambda$, where $N_\lambda := \{a \in A : p_\lambda(a) = 0\}$, is already complete, hence a C^*-algebra in the norm $\dot{p}_\lambda(a + N_\lambda) \equiv \|a + N_\lambda\| := p_\lambda(a)$, $a \in A$ (Apostol, see [7, Theorem 10.24]). The Arens-Michael decomposition gives us the representation of $A[\tau_\Gamma]$ as an inverse limit of C^*-algebras; namely $A[\tau_\Gamma] = \varprojlim A/N_\lambda$, up to a topological $*$-isomorphism [7, pp. 15–16]. We refer the reader to [7] for further information about pro-C^*-algebras.

For clarity's sake, we first recall the definition of a right (left) Hilbert C^*-module from [14]. Let A be a C^*-algebra and E a complex vector space and a right A-module equipped with an A-valued inner product, that is a sesqui-linear map $\langle\, , \rangle_A : E \times E \to A$, which is conjugate linear in the first variable and linear in the second variable, such that, for all x, y in E and $a \in A$ the following properties hold:

$$\begin{aligned} &\text{(i)} && \langle x, x\rangle_A \geq 0, \\ &\text{(ii)} && \langle x, x\rangle_A = 0 \Rightarrow x = 0, \\ &\text{(iii)} && \langle x, y\rangle_A^* = \langle y, x\rangle_A, \\ &\text{(iv)} && \langle x, ya\rangle_A = \langle x, y\rangle_A a. \end{aligned}$$

If E is complete with respect to the norm $\|x\|_A := \|\langle x, x\rangle_A\|^{\frac{1}{2}}$, $x \in E$, then E is called a *right Hilbert A-module* or a *right Hilbert C^*-module over the C^*-algebra A* [14]. The notion of a *left Hilbert C^*-module E over a C^*-algebra B* is defined in an analogous way. That is, E is a left B-module, equipped with a B-valued inner product, which is a sesqui-linear map ${}_B\langle\, , \rangle : E \times E \to B$, linear in the first variable and conjugate linear in the second variable, satisfying analogous properties to (i)–(iv) above, where for instance (iv) becomes now

$${}_B\langle bx, y\rangle = b({}_B\langle x, y\rangle), \quad \forall\, x, y \in E,\, b \in B,$$

and E is complete with respect to the norm ${}_B\|x\| = \|{}_B\langle x, x\rangle\|^{\frac{1}{2}}$. In case E is both a left Hilbert B-module and a right Hilbert A-module, such that the following relation is satisfied

$${}_B\langle x, y\rangle z = x\langle y, z\rangle_A, \quad \forall\, x, y, z \in E,$$

then E is called a *Hilbert B-A-bimodule.* In [4, Corollary 1.11] it is proved that

$$\|x\|_A = {}_B\|x\|, \quad \forall\, x \in E. \tag{2.1}$$

Let now $A[\tau_\Gamma]$ be a pro-C^*-algebra and E a complex vector space and a right A-module equipped with an A-valued sesqui-linear map, conjugate linear in the first and linear in the second variable, satisfying the conditions (i)–(iv) above. Then, for every $p_\lambda \in \Gamma$, a seminorm p_λ^A is defined on E, as follows [9, Corollary 1.2.3]

$$p_\lambda^A(x) := p_\lambda(\langle x,x\rangle_A)^{\frac{1}{2}}, \quad \forall\, x \in E.$$

If E is complete with respect to the locally convex topology defined by the family of seminorms $\{p_\lambda^A\}_{\lambda\in\Lambda}$, then E is called a *right Hilbert A-module* [9, Definition 1.2.5], but we shall call it a *right Hilbert pro-C^*-module over A*. Similarly, the notion of a left Hilbert pro-C^*-module over a pro-C^*-algebra B is defined. In order to speak of a Hilbert B-A-bimodule E, we will see in the next section that we have to impose an extra "natural" condition concerning the continuity of the module actions (see (T)), so as to be able to prove the coincidence of the two respective topologies defined on E, as in (2.1).

3. STRUCTURE OF HILBERT PRO-C^*-BIMODULES

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra. Let E be both a left and right Hilbert pro-C^*-module over A, where ${}_A\langle\,,\rangle$ and $\langle\,,\rangle_A$ denote the respective left and right A-valued inner products on E. Then, there are two locally convex topologies defined on E. One, denoted by τ^A, induced by the seminorms $\{\,p_\lambda^A\}_{\lambda\in\Lambda}$ corresponding to its structure as a right Hilbert A-module and the other, denoted by ${}^A\tau$, induced by the seminorms $\{{}^Ap_\lambda\}_{\lambda\in\Lambda}$, corresponding to its structure as a left Hilbert A-module. Namely,

$$p_\lambda^A(x) := p_\lambda(\langle x,x\rangle_A)^{\frac{1}{2}}, \quad \forall\, x \in E,\ \lambda\in\Lambda, \tag{3.1}$$

$${}^Ap_\lambda(x) := p_\lambda({}_A\langle x,x\rangle)^{\frac{1}{2}}, \quad \forall\, x \in E,\ \lambda\in\Lambda. \tag{3.2}$$

We assume continuity of the left (resp. right) module action, in the sense that

$$(T) \quad p_\lambda^A(ax)\le p_\lambda(a)\, p_\lambda^A(x),\ {}^Ap_\lambda(xa) \le {}^Ap_\lambda(x)\, p_\lambda(a), \quad \forall\, x\in E,\ a\in A,\ \lambda\in\Lambda.$$

That is, the left action is (smoothly) continuous with respect to the topology τ^A defined on E by its right module structure and vice-versa. It is easily seen that the above inequalities always hold in the normed case (see e.g., [5, p. 239]). Also (T) is always true when $p_\lambda^A = {}^Ap_\lambda$, for every $\lambda\in\Lambda$; relevant examples can be seen in Section 4. A kind of converse to this situation is provided by Corollary 3.2, below. It is clear from (3.1), (3.2) that whenever $A[\tau_\Gamma]$ is Hausdorff, both $E[\tau^A]$ and $E[{}^A\tau]$ are Hausdorff as locally convex spaces.

The following result is of independent interest.

THEOREM 3.1. *Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and E a left and right Hilbert pro-C^*-module over A, such that ${}_A\langle x,y\rangle z = x\langle y,z\rangle_A$, for all $x,y,z\in E$. If the condition (T) is also satisfied, then each C^*-seminorm p_λ of A, is realized on the inner product elements $\langle x,x\rangle_A$, ${}_A\langle x,x\rangle$, $x\in E$, of A, by* *the norm of the "adjointable" operators induced by the left, respectively right, action of A on the elements of E* (see (3.5), (3.6) below).

Proof. For every $\lambda \in \Lambda$, let

$$N_\lambda^A := \{x \in E : p_\lambda^A(x) = 0\}, \quad E_\lambda^A := E/N_\lambda^A.$$

It is known that E_λ^A is a right Hilbert A_λ-module [9, Theorem 1.3.9], with module action and inner product well defined by

$$(x + N_\lambda^A)\,(a + N_\lambda) := xa + N_\lambda^A, \quad \forall\, x \in E,\, a \in A,$$
$$\langle x + N_\lambda^A,\, y + N_\lambda^A\rangle_{A_\lambda} := \langle x, y\rangle_A + N_\lambda, \quad \forall\, x, y \in E,$$

and norm by

$$\|x + N_\lambda^A\|_{A_\lambda}^2 := \|\langle x + N_\lambda^A, x + N_\lambda^A\rangle_{A_\lambda}\| = p_\lambda(\langle x, x\rangle_A). \tag{3.3}$$

For all $\lambda \in \Lambda$, consider the correspondence

$$\kappa_\lambda :\, A_\lambda \to L_{A_\lambda}(E_\lambda^A) \text{ with } \kappa_\lambda(a + N_\lambda)(x + N_\lambda^A) = ax + N_\lambda^A, \quad \forall\, x \in A,$$

where $L_{A_\lambda}(E_\lambda^A)$ is the set of all maps $S : E_\lambda^A \to E_\lambda^A$ which have an adjoint, with respect to the inner product $\langle\,,\rangle_{A_\lambda}$. It is a C^*-algebra, under the norm $\|S\| = \sup\{\|S(x + N_\lambda^A)\|_{A_\lambda} : \|x + N_\lambda^A\|_{A_\lambda} \le 1;\, x \in E\}$, $S \in L_{A_\lambda}(E_\lambda^A)$ (see [14, p. 8]). The map κ_λ is well defined due to the first inequality in (T). We show that $\kappa_\lambda(a + N_\lambda) \in L_{A_\lambda}(E_\lambda^A)$.

For this, let $I := \operatorname{span}\{\,\langle x, y\rangle_A :\, x, y \in E\}$. Then I is a two-sided ideal of A. Its closure I_A is a $*$-ideal [7, Theorem 11.7], therefore it is a pro-C^*-algebra and thus contains an approximate identity (u_α) [7, Theorem 11.5]. If $b \in I_A$ such that for all $z \in E$, $zb = 0$, we get $\langle E, E\rangle_A\, b = \langle E, Eb\rangle_A = 0$, so $u_\alpha b = 0$ for all α and thus $b = 0$. Therefore, since for all $x, y, z \in E$, $a \in A$

$$x\,\langle ay, z\rangle_A = {}_A\langle x, ay\rangle z = {}_A\langle x, y\rangle a^* z = x\,\langle y, a^* z\rangle_A,$$

we take

$$\begin{aligned} &\text{(i)} \quad \langle ax, y\rangle_A = \langle x, a^* y\rangle_A \text{ and similarly} \\ &\text{(ii)} \quad {}_A\langle xa, y\rangle \;=\; {}_A\langle x, ya^*\rangle. \end{aligned} \tag{3.4}$$

Then, for all $x, y \in E$, $a \in A$ we have that

$$\langle \kappa_\lambda(a + N_\lambda)(x + N_\lambda^A),\, y + N_\lambda^A\rangle_{A_\lambda} = \;\langle x + N_\lambda^A,\, a^* y + N_\lambda^A\rangle_{A_\lambda}.$$

Therefore, $\kappa_\lambda(a + N_\lambda)^* \;=\; \kappa_\lambda(a^* + N_\lambda)$. Since κ_λ is a $*$-morphism between C^*-algebras, then $\|\kappa_\lambda(a + N_\lambda)\| \le \|a + N_\lambda\|$, for every $a \in A$. In addition, if

$a \equiv_A \langle x, x\rangle$, $x \in E$, we have the following calculation

$$\begin{aligned}
&\|\kappa_\lambda({}_A\langle x,x\rangle + N_\lambda)\|^2\\
&\qquad = \sup\{\|\kappa_\lambda({}_A\langle x,x\rangle + N_\lambda)(\omega + N_\lambda^A)\|^2_{A_\lambda} : \omega \in E;\ \|\omega + N_\lambda^A\|_{A_\lambda} \le 1\}\\
&\qquad = \sup\{\|x\langle x,\omega\rangle_A + N_\lambda^A\|^2_{A_\lambda} : \omega \in E;\ p_\lambda^A(\omega) \le 1\}\\
&\qquad = \sup\{p_\lambda(\langle x\langle x,x\rangle_A{}^{\frac{1}{2}}, \omega\rangle_A)^2 : \omega \in E;\ p_\lambda^A(\omega) \le 1\}\\
&\qquad = p_\lambda^A(x\langle x,x\rangle_A^{\frac{1}{2}})^2 = p_\lambda(\langle x,x\rangle_A^{\frac{1}{2}}\langle x,x\rangle_A\langle x,x\rangle_A^{\frac{1}{2}})\\
&\qquad = p_\lambda(\langle x,x\rangle_A^2) = p_\lambda(\langle x,x\rangle_A)^2.
\end{aligned}$$

The first equality in the last but one line is a consequence of the Cauchy-Schwarz inequality, as this is applied for Hilbert pro-C^*-modules (see [9, Proposition 1.2.2]). Hence

$$p_\lambda(\langle x,x\rangle_A) = \|\kappa_\lambda({}_A\langle x,x\rangle + N_\lambda)\|, \quad \forall\, x \in E \text{ and } \lambda \in \Lambda. \tag{3.5}$$

On the other hand, if we define

$${}^AN_\lambda := \{x \in E : {}^Ap_\lambda(x) = 0\}, \quad {}^AE_\lambda := E/\,{}^AN_\lambda,$$

then by [9, Theorem 1.3.9], ${}^AE_\lambda$ is a left Hilbert A_λ-module with module action, inner product and norm defined in a similar way as in the case of the right module action.

Now, for every $\lambda \in \Lambda$ we consider the assignment

$$\rho_\lambda : A_\lambda \to {}_{A_\lambda}L({}^AE_\lambda) : \rho_\lambda(a + N_\lambda)(x + {}^AN_\lambda) := xa + {}^AN_\lambda,$$

where ${}_{A_\lambda}L({}^AE_\lambda)$ is the C^*-algebra of all maps $\phi : {}^AE_\lambda \to {}^AE_\lambda$, for which there is a map $\phi^* : {}^AE_\lambda \to {}^AE_\lambda$, such that

$${}_{A_\lambda}\langle \phi(x + {}^AN_\lambda), y + {}^AN_\lambda\rangle = {}_{A_\lambda}\langle x + {}^AN_\lambda, \phi^*(y + {}^AN_\lambda)\rangle, \quad \forall\, x, y \in E.$$

For $\phi \in {}_{A_\lambda}L({}^AE_\lambda)$ the norm is given by

$$\|\phi\| = \sup\{{}_{A_\lambda}\|\phi(x + {}^AN_\lambda)\| : {}_{A_\lambda}\|x + {}^AN_\lambda\| \le 1;\ x \in E\}.$$

The map ρ_λ is well defined due to the second inequality in (T) and (3.4)(ii). Moreover, it is a $*$-morphism between C^*-algebras, when we consider the opposite multiplication in ${}_{A_\lambda}L({}^AE_\lambda)$. Therefore, we get that

$$\|\rho_\lambda(a + N_\lambda)\| \le \|a + N_\lambda\|, \quad \forall\, a \in A.$$

Furthermore, by considering the element $\langle x,x\rangle_A$, $x \in E$, in A, by similar calculations as above we conclude that

$$\begin{aligned}
\|\rho_\lambda(\langle x,x\rangle_A + N_\lambda)\|^2 &= \sup\{{}_{A_\lambda}\|\omega\langle x,x\rangle_A + {}^AN_\lambda\|^2 : \omega \in E;\ {}^Ap_\lambda(\omega) \le 1\}\\
&= \sup\{\, p_\lambda({}_A\langle {}_A\langle x,x\rangle^{\frac{1}{2}}x, \omega\rangle)^2 : \omega \in E;\ {}^Ap_\lambda(\omega) \le 1\}\\
&= {}^Ap_\lambda({}_A\langle x,x\rangle^{\frac{1}{2}}x)^2 = p_\lambda({}_A\langle x,x\rangle)^2.
\end{aligned}$$

Consequently,

(3.6) $$p_\lambda({}_A\langle x,x\rangle) = \|\rho_\lambda(\langle x,x\rangle_A + N_\lambda)\|, \quad \forall\, x \in E \text{ and } \lambda \in \Lambda.$$

The proof of theorem is complete. □

Corollary 3.2. *Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and E a left and right Hilbert pro-C^*-module over A, such that ${}_A\langle x,y\rangle z = x\langle y,z\rangle_A$, for all $x,y,z \in E$. If the condition (T) is also satisfied, then $p^A_\lambda(x) = {}^Ap_\lambda(x)$, for all $\lambda \in \Lambda$, $x \in E$.*

Proof. By (3.5), (3.6) and (3.1), (3.2), we conclude that

$$p_\lambda(\langle x,x\rangle_A) = \|\kappa_\lambda({}_A\langle x,x\rangle + N_\lambda)\| \le \|{}_A\langle x,x\rangle + N_\lambda\| = p_\lambda({}_A\langle x,x\rangle),$$
$$p_\lambda({}_A\langle x,x\rangle) = \|\rho_\lambda(\langle x,x\rangle_A + N_\lambda)\| \le \|\langle x,x\rangle_A + N_\lambda\| = p_\lambda(\langle x,x\rangle_A),$$

for all $\lambda \in \Lambda$ and $x \in E$. Thus, $p^A_\lambda(x) \le {}^Ap_\lambda(x) \le p^A_\lambda(x)$, for all $x \in E$ and $\lambda \in \Lambda$, which completes the proof. □

Remarks 3.3. (1) In an electronic correspondence with Professor Maria Joiţa, she suggested me a direct proof of Corollary 3.2, independent of Theorem 3.1. We present it here:

Use the C^*-property for p_λ's, the properties of the A-inner product(s) from Section 2, the assumptions of Corollary 3.2 and the Cauchy-Schwarz inequality [9, Proposition 1.2.2]. Let $\lambda \in \Lambda$ and $x \in E$ with ${}^Ap_\lambda(x) \neq 0$. Then, since ${}_A\langle x,x\rangle^* = {}_A \langle x,x\rangle$, we get

$$\begin{aligned}
{}^Ap_\lambda(x)^4 &= p_\lambda({}_A\langle x,x\rangle)^2 = p_\lambda({}_A\langle x,x\rangle{}_A\langle x,x\rangle)\\
&= p_\lambda({}_A\langle {}_A\langle x,x\rangle x,x\rangle) = p_\lambda({}_A\langle x\langle x,x\rangle_A,x\rangle)\\
&\le \text{(Cauchy-Schwarz) } {}^Ap_\lambda(x\langle x,x\rangle_A)\, {}^Ap_\lambda(x)\\
&\le \text{(from } (T)) \ {}^Ap_\lambda(x)p_\lambda(\langle x,x\rangle_A)\, {}^Ap_\lambda(x) = {}^Ap_\lambda(x)^2 p^A_\lambda(x)^2,
\end{aligned}$$

therefore ${}^Ap_\lambda(x) \le p^A_\lambda(x)$. If $x \in E$ with ${}^Ap_\lambda(x) = 0$, then clearly the inequality is true. Hence, ${}^Ap_\lambda(x) \le p^A_\lambda(x)$, for all $x \in E$.

Using the same arguments, one also gets $p^A_\lambda(x) \le {}^Ap_\lambda(x)$, for all $x \in E$, so that $p^A_\lambda(x) = {}^Ap_\lambda(x)$, for all $x \in E$.

(2) Let $B[\tau_{\Gamma'}]$ and $A[\tau_\Gamma]$ be two pro-C^*-algebras, where $\Gamma' = \{q_\lambda\}_{\lambda\in\Lambda}$ and $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ are the families of C^*-seminorms defining the topology of B and A respectively, indexed by the same index set Λ. If E is a left Hilbert pro-C^*-module over B and a right Hilbert pro-C^*-module over A, such that

$$_B\langle x,y\rangle z = x\langle y,z\rangle_A, \quad \forall\, x,y,z \in E$$

and relation (T) is respectively given by

(T') $$p^A_\lambda(bx) \le q_\lambda(b)\, p^A_\lambda(x), \quad {}^Bq_\lambda(xa) \le {}^Bq_\lambda(x)\, p_\lambda(a),$$

for all $x \in E$, $a \in A$, $b \in B$, $\lambda \in \Lambda$. Then by the above proof, with obvious modifications, we get the equality of the seminorms induced on E by A and B, i.e.

$$^{B}q_\lambda(x) = p_\lambda^A(x), \quad \forall\, x \in E,\, \lambda \in \Lambda.$$

(3) Here are some examples of different pro-C^*-algebras $B[\tau_{\Gamma'}]$, $A[\tau_\Gamma]$ where, Γ, Γ' are indexed by the same index set; $A[\tau_\Gamma]$ and its unitization $A_1[\tau_1]$ (for $A_1[\tau_1]$ see [7, Theorem 8.3]), $A[\tau_\Gamma]$, $L_A(E)$ and $A[\tau_\Gamma]$, $K_A(E)$. For the last two pairs, see subsection 6.(2).

(4) Note that the equality

$$_{A}\langle x, y\rangle z = x\langle y, z\rangle_A, \quad \forall\, x, y, z \in E$$

in Theorem 3.1, *clearly makes the two A-inner products on E, compatible.*

On the other hand, condition (T) in the same theorem *provides a relation of compatibility of the right module action on E with the topology on E induced by the left Hilbert pro-C^*-module structure of E and vice-versa.*

According to Corollary 3.2, the two locally convex topologies $^{A}\tau$, τ^A defined on the Hilbert pro-C^*-bimodule E over A coincide. In what follows we shall use the notation τ for the topology $^{A}\tau = \tau^A$ on E. In this way, we can work with one topology on E compatible with both the left and the right Hilbert structure of our module. Based on the preceding we can now give the following

Definition 3.4. Let $B[\tau_{\Gamma'}]$ and $A[\tau_\Gamma]$ be two pro-C^*-algebras, where Γ, Γ' have the same index set, say Λ. Let E be a left Hilbert pro-C^*-module over B and a right Hilbert pro-C^*-module over A. Suppose that condition (T') is satisfied and that

$$_{B}\langle x, y\rangle z = x\langle y, z\rangle_A, \quad \forall\, x, y, z \in E.$$

Then E will be called a *Hilbert B-A-bimodule*, or a *Hilbert pro-C^*-bimodule over B, A.*

By Corollary 3.2, we have that $N_\lambda^A = {}^{A}N_\lambda$, so the quotient space $E_\lambda^A = {}^{A}E_\lambda$ becomes a Hilbert A_λ-bimodule, under the actions and A_λ-inner products defined at the beginning of the proof of Theorem 3.1 (see also [4, Definition 1.8]). The equality condition between the A_λ-valued inner products is immediate by the very definitions.

4. SOME EXAMPLES

In this section we present some examples of Hilbert bimodules over a pro-C^*-algebra.

Example 4.1. Every pro-C^*-algebra A is a Hilbert pro-C^*-bimodule over itself. Indeed if ${}_A\langle\, ,\rangle$ and $\langle\, ,\rangle_A$ are defined by

$${}_A\langle a, b\rangle = ab^*, \quad \langle a, b\rangle_A = a^*b, \quad \forall\, a, b \in A,$$

then it is straightforward to check that $a\,\langle b, c\rangle_A = {}_A\langle a, b\rangle\, c$, for all $a, b, c \in A$ and that

$${}^A p_\lambda(a)^2 := p_\lambda({}_A\langle a, a\rangle) = p_\lambda(a^*)^2 = p_\lambda(a)^2 = p_\lambda^A(a)^2, \quad \forall\, a \in A,\, \lambda \in \Lambda.$$

Example 4.2. Let X be a locally compact Hausdorff space and A a C^*-algebra. Let E be a Hilbert A-bimodule with ${}_A\langle\, ,\rangle$ and $\langle\, ,\rangle_A$ A-valued inner products defined on E as in Section 2. Let $C(X, A)$ denote the algebra of all A-valued continuous functions from X and $\mathcal{K}$ the family of all compact subsets of X. As usual $C(X, A)$ is endowed with the compact open topology c defined by the C^*-seminorms

$$p_K(f) := \sup\{\|f(t)\|_A : t \in K\}, \quad f \in C(X, A),\, K \in \mathcal{K}$$

(see e.g., [16, p. 387, (1.1)]). Then $C_c(X, A) = \varprojlim_{K\in\mathcal{K}} C(K, A)$ and $C_c(X, A)$ is a pro-C^*-algebra. Completeness stems from the fact that X as a locally compact space is a k-space (see e.g., [7, p. 35]). Consider the set $C(X, E)$ of all E-valued continuous functions on X equipped with the following actions:

$$C(X, A) \times C(X, E) \to C(X, E) : (f, \phi) \mapsto f\phi \text{ with } (f\phi)(t) := f(t)\phi(t),\, t \in X,$$

$$C(X, E) \times C(X, A) \to C(X, E) : (\phi, f) \mapsto \phi f \text{ with } (\phi\, f)(t) = \phi(t)f(t),\, t \in X.$$

Denote by ${}_{C(X,A)}\langle\, ,\rangle$ and $\langle\, ,\rangle_{C(X,A)}$ the $C(X, A)$-valued well-defined inner products of $C(X, E)$, given by

$$\langle\phi, \psi\rangle_{C(X,A)}(t) := \langle\phi(t), \psi(t)\rangle_A, \quad t \in X$$

and

$${}_{C(X,A)}\langle\phi, \psi\rangle(t) := {}_A\langle\phi(t), \psi(t)\rangle, \quad t \in X.$$

These inner products turn $C(X, E)$ into a right and left $C(X, A)$-module, such that

$$\begin{aligned}(\phi\langle\psi, \omega\rangle_{C(X,A)})(t) &= \phi(t)\langle\psi(t), \omega(t)\rangle_A = {}_A\langle\phi(t), \psi(t)\rangle\omega(t)\\ &= {}_{C(X,A)}\langle\phi, \psi\rangle(t)\omega(t) = ({}_{C(X,A)}\langle\phi, \psi\rangle\omega)(t),\end{aligned}$$

for all $\phi, \psi, \omega \in C(X, E)$, $t \in X$. Thus

$$\phi\langle\psi, \omega\rangle_{C(X,A)} = {}_{C(X,A)}\langle\phi, \psi\rangle\omega, \quad \forall\, \phi, \psi, \omega \in C(X, E).$$

Moreover, considering the families of seminorms $\{p_K^{C(X,A)}\}$, $\{{}^{C(X,A)}p_K\}$ we get easily that

$${}^{C(X,A)}p_K(\phi)^2 : = p_K({}_{C(X,A)}\langle\phi, \phi\rangle) = p_K^{C(X,A)}(\phi)^2, \quad \forall\, \phi \in C(X, E).$$

Now, along the lines of the proof of [16, p. 390, (1.12), and the discussion after it], we have that $C(X,E) = \varprojlim C(K,E), K \in \mathcal{K}$, within an isomorphism of locally convex spaces, where each $C(K,E)$ is a Hilbert C^*-bimodule over $C(K,A)$. This yields completeness for $C(X,E)$, hence the conditions of the Definition 3.4 are met, and so $C(X,E)$ becomes a Hilbert pro-C^*-bimodule over the pro-C^*-algebra $C(X,A)$.

Example 4.3. Let $A[\tau_\Gamma]$ be a commutative pro-C^*-algebra and $M(A)$ its multiplier algebra, which is also a pro-C^*-algebra [18, Theorem 3.14]. For the definition of $M(A)$, in the case where A is an arbitrary pro-C^*-algebra see [18, Definition 3.13]. We consider $M(A)^{op}$, that is $M(A)$ with the opposite multiplication. In $M(A)^{op}$ the following module actions are well defined

$$M(A)^{op} \times A \to A : (l,r)a =: r(a) \quad \text{and} \quad A \times M(A)^{op} \to A : a(l,r) := l(a).$$

The opposite multiplication in $M(A)$ is considered so as to ensure that $a\big((l_1,r_1) \circ(l_2,r_2)\big) = \big(a(l_1,r_1)\big)(l_2,r_2)$, for all $a \in A$, $(l_i,r_i) \in M(A)$, $i = 1,2$. Also, the following $M(A)^{op}$-valued maps are defined

$$_{M(A)^{op}}\langle\, ,\rangle : A \times A \to M(A)^{op} : {}_{M(A)^{op}}\langle a,b\rangle := (l_{ab^*}, r_{ab^*})$$

and

$$\langle\, ,\rangle_{M(A)^{op}} : A \times A \to M(A)^{op} : \langle a,b\rangle_{M(A)^{op}} := (l_{a^*b}, r_{a^*b}).$$

It can be checked, using the commutativity of A, that the above maps are $M(A)^{op}$-valued inner products on A, under which A is a left and right Hilbert pro-C^*-module over $M(A)^{op}$, such that

$$_{M(A)^{op}}\langle a,b\rangle c = a\langle b,c\rangle_{M(A)^{op}}, \quad \forall\, a,b,c \in A.$$

Considering on A the respective families of seminorms $\{q_\lambda^{M(A)^{op}}\}$, $\{^{M(A)^{op}}q_\lambda\}$, $\lambda \in \Lambda$, where $q_\lambda(l,r) := \sup\{p_\lambda(l(a)) : p_\lambda(a) \le 1\}$, $\lambda \in \Lambda$, $(l,r) \in M(A)$, is the family of seminorms, making $M(A)$ a pro-C^*-algebra [18, Definition 3.13], we have the following

$$\begin{aligned}{}^{M(A)^{op}}q_\lambda(a)^2 &= q_\lambda({}_{M(A)^{op}}\langle a,a\rangle) = q_\lambda\big((l_{aa^*}, r_{aa^*})\big) = p_\lambda(aa^*) = p_\lambda(a)^2 = p_\lambda(a^*a)\\ &= q_\lambda\big((l_{a^*a}, r_{a^*a})\big) = q_\lambda(\langle a,a\rangle_{M(A)^{op}}) = q_\lambda^{M(A)^{op}}(a)^2, \quad \forall\, a \in A.\end{aligned}$$

Therefore, the topology inherited on A from its $M(A)^{op}$-bimodule structure coincides with its topology τ_Γ as a pro-C^*-algebra and A is a Hilbert pro-C^*-bimodule over $M(A)^{op}$. Note that, in general, $A \subset M(A)$ and $A = M(A)$ in case A is unital. In this sense, it can be said that the present example generalizes Example 4.1, in case A is commutative and unital.

5. HILBERT PRO-C^*-BIMODULES VIA INVERSE LIMITS OF HILBERT C^*-BIMODULES

In this section we investigate how a Hilbert pro-C^*-bimodule can be realized via an inverse limit of Hilbert C^*-bimodules.

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and E a Hilbert pro-C^*-bimodule over A. Then $p_\lambda^A(x) = {}^A p_\lambda(x)$, for all $x \in E$ (see Corollary 3.2) and $E_\lambda^A = {}^A E_\lambda$ is a Hilbert A_λ-bimodule (see comments after Definition 3.4). For all $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$ consider the well-defined, surjective A_λ-A_μ-bimodule morphisms (i.e., (5.1), (5.2) below hold true)

$$\sigma_{\lambda\mu} : E_\lambda^A \to E_\mu^A : x + N_\lambda^A \mapsto x + N_\mu^A, \quad \forall\, x \in E,$$

which are continuous as it follows from (3.3). If $\{A_\lambda,\, \pi_{\lambda\mu}\}_{\mu\leq\lambda}$ is the inverse system of the C^*-algebras corresponding to $A[\tau_\Gamma]$, where $\pi_{\lambda\mu}(a+N_\lambda) := a+N_\mu$, $a \in A$, $\mu \leq \lambda$ in Λ. Then for any $x, y \in E$, $a \in A$, $\mu \leq \lambda$ in Λ the following conditions are satisfied

$$\sigma_{\lambda\mu}\big(\,(x + N_\lambda^A)\,(a + N_\lambda)\,\big) = \sigma_{\lambda\mu}(x + N_\lambda^A)\,\pi_{\lambda\mu}(a + N_\lambda), \tag{5.1}$$

$$\sigma_{\lambda\mu}\big((a + N_\lambda)(x + N_\lambda^A)\big) = \pi_{\lambda\mu}(a + N_\lambda)\sigma_{\lambda\mu}(x + N_\lambda^A), \tag{5.2}$$

$$\begin{aligned} \langle\sigma_{\lambda\mu}(x + N_\lambda^A), \sigma_{\lambda\mu}(y + N_\lambda^A)\rangle_{A_\mu} &= \langle x + N_\mu^A, y + N_\mu^A\rangle_{A_\mu} = \langle x, y\rangle_A + N_\mu \\ &= \pi_{\lambda\mu}(\langle x + N_\lambda^A, y + N_\lambda^A\rangle_{A_\lambda}), \end{aligned} \tag{5.3}$$

$${}_{A_\mu}\langle\sigma_{\lambda\mu}(x + N_\lambda^A), \sigma_{\lambda\mu}(y + N_\lambda^A)\rangle = \pi_{\lambda\mu}({}_{A_\lambda}\langle x + N_\lambda^A, y + N_\lambda^A\rangle). \tag{5.4}$$

From all the above, it is clear that the family $\{E_\lambda^A\}_{\lambda\in\Lambda}$ constitutes an inverse system of A_λ-bimodules with connecting maps $\sigma_{\lambda\mu}$, $\mu \leq \lambda \in \Lambda$ [9, Definition 1.2.20]. Thus we can form the inverse limit $\varprojlim E_\lambda^A$, which is non empty according to [2, p. 198, Proposition 5], and consider it as a left and right Hilbert pro-C^*-module over A, with module actions and A-valued inner products defined in the obvious manner, i.e.,

$$\begin{gathered} a(\sigma_\lambda(x))_\lambda := (\pi_\lambda(a)\sigma_\lambda(x))_\lambda,\ (\sigma_\lambda(x))_\lambda\, a := (\sigma_\lambda(x)\pi_\lambda(a))_\lambda \\ {}_A\langle(\sigma_\lambda(x))_\lambda, (\sigma_\lambda(y))_\lambda\rangle := \big(\,{}_{A_\lambda}\langle\, \sigma_\lambda(x), \sigma_\lambda(y)\rangle\big)_\lambda \\ \langle(\sigma_\lambda(x))_\lambda, (\sigma_\lambda(y))_\lambda\rangle_A := \big(\langle\sigma_\lambda(x), \sigma_\lambda(y)\rangle_{A_\lambda}\big)_\lambda, \end{gathered}$$

for all $x, y \in \varprojlim E_\lambda^A$, $a \in A$, where $\sigma_\lambda : \varprojlim E_\lambda^A \to E_\lambda^A$ and $\pi_\lambda : A \to A_\lambda$, $\lambda \in \Lambda$, are the projection maps of the respective inverse limits. The above actions and inner products are well-defined, due to the relations (5.1)–(5.4). Moreover, for any $x, y, z \in \varprojlim E_\lambda^A$ we get the following equality

$${}_A\langle(\sigma_\lambda(x))_\lambda,\ (\sigma_\lambda(y))_\lambda\rangle\,(\sigma_\lambda(z))_\lambda = (\sigma_\lambda(x))_\lambda\langle(\sigma_\lambda(y))_\lambda, (\sigma_\lambda(z))_\lambda\rangle_A.$$

The inverse limit $\varprojlim E_\lambda^A$ inherits two locally convex topologies, determined by the families of seminorms $\{p_\lambda^A\}$, $\{{}^A p_\lambda\}$, with respect to which $\varprojlim E_\lambda^A$ is a right and a left Hilbert pro-C^*-module over A respectively [9, Proposition 1.2.21]. Also, for any $\lambda \in \Lambda$, $x \in \varprojlim E_\lambda^A$, we get the following

$$\begin{aligned} p_\lambda^A\big((\sigma_\lambda(x))_\lambda\big)^2 &:= p_\lambda(\langle(\sigma_\lambda(x))_\lambda, (\sigma_\lambda(x))_\lambda\rangle_A) = p_\lambda\big((\langle\sigma_\lambda(x), \sigma_\lambda(x)\rangle_{A_\lambda})_\lambda\big) \\ &= \|\sigma_\lambda(x)\|_{A_\lambda}^2 = {}_{A_\lambda}\|\sigma_\lambda(x)\|^2 = {}^A p_\lambda\big((\sigma_\lambda(x))_\lambda\big)^2. \end{aligned}$$

Therefore, according to Definition 3.4, $\varprojlim E_\lambda^A$ *is a Hilbert pro-C^*-bimodule over the pro-C^*-algebra* $A[\tau_\Gamma]$.

Consider now the map

$$\Phi : E \to \varprojlim E_\lambda^A : \, x \mapsto (x + N_\lambda^A)_\lambda,$$

which is well-defined by the definition of the connecting maps $\sigma_{\lambda\mu}$, $\lambda \geq \mu$. The map Φ is an A-bimodule morphism and for any x, y in E we have that

$$\begin{aligned} \langle\Phi(x), \Phi(y)\rangle_A &= \langle(x + N_\lambda^A)_\lambda, (y + N_\lambda^A)_\lambda\rangle_A \\ &= \big(\langle x + N_\lambda^A, y + N_\lambda^A\rangle_{A_\lambda}\big)_\lambda = \langle x, y\rangle_A. \end{aligned}$$

Similarly, we get that ${}_A\langle\Phi(x), \Phi(y)\rangle = {}_A\langle x, y\rangle$. Thus, $\Phi(E)$ *is a closed A-sub-bimodule* of $\varprojlim E_\lambda^A$, *therefore* $E = \varprojlim E_\lambda^A$ (see proof of [9, Proposition 1.3.10]), *up to a topological isomorphism of A-bimodules.*

6. APPLICATIONS

In this section we give two applications. The first one deals with the continuity of a derivation from a pro-C^*-algebra $A[\tau_\Gamma]$ into a Hilbert pro-C^*-bimodule E over A (see 6.(1)), while the second one gives a realization of "compact" operators on Hilbert pro-C^*-bimodules by means of a specific closed $*$-ideal of the pro-C^*-algebra involved (see 6.(2)).

6.(1) Continuity of derivations in Hilbert pro-C^*-bimodules.

If A is an algebra and E an A-bimodule, a linear map $\delta : A \to E$ is called *a derivation* if it satisfies the Leibniz rule, i.e.,

$$\delta(ab) = \delta(a)b + a\delta(b), \quad \forall\, a, b \in A.$$

The derivation δ is said to be *inner*, if

$$\exists\, x \in E \text{ such that } \delta(a) = ax - xa, \ \forall\, a \in A.$$

A lot of automatic continuity results for derivations of Banach, C^*-algebras and von Neumann algebras are given in [5]. In the case of a C^*-algebra A, every derivation $\delta : A \to A$ is continuous [21, Theorem 2.3.1]. J.R. Ringrose extended this result in 1972, showing that every derivation from a C^*-algebra

A into a Banach bimodule X over A is continuous [20]. In the case of pro-C^*-algebras, Becker proved in 1992, that if A is a pro-C^*-algebra, every derivation $\delta : A \to A$ is continuous [1]. In 1995, N.C. Phillips proved that every derivation $\delta : A[\tau_\Gamma] \to A[\tau_\Gamma]$ of a pro-C^*-algebra is approximately inner [19]. In this subsection we prove (Theorem 6.1) that every derivation of a pro-C^*-algebra in a Hilbert pro-C^*-bimodule is continuous, generalizing thus Becker's result, and giving at the same time a version of Ringrose theorem in our setting. In a paper, joint with M. Weigt, we present various generalizations of Ringrose's result, using a complete locally convex bimodule, over a pro-C^*-algebra (see [23]).

THEOREM 6.1. *Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and $E[\tau]$ (see Remarks 3.3(4)) a Hilbert pro-C^*-bimodule over $A[\tau_\Gamma]$. Then every derivation $\delta : A[\tau_\Gamma] \to E[\tau]$ is continuous.*

Proof. For all $\lambda \in \Lambda$, consider the correspondence

$$\delta_\lambda : A_\lambda \to E_\lambda^A : \delta_\lambda(\pi_\lambda(a)) := \sigma_\lambda(\delta(a)), \quad \forall\, a \in A,$$

where $\pi_\lambda : A \to A_\lambda$, $\sigma_\lambda : E \to E_\lambda^A$ are the natural quotient maps. Then for every λ, the map δ_λ is well defined, since if $\pi_\lambda(\alpha) = 0$, then

$$\begin{aligned}\|\sigma_\lambda(\delta(a))\|^2_{A_\lambda} &= \|\langle \sigma_\lambda(\delta(a)), \sigma_\lambda(\delta(a))\rangle_{A_\lambda}\| = \|\langle \delta(a), \delta(a)\rangle_A + N_\lambda\| \\ &= p_\lambda(\langle \delta(a), \delta(a)\rangle).\end{aligned}$$

Now, since $a \in N_\lambda$, from [1, Lemma 1], there are y_1, y_2, y_3, $y_4 \in N_\lambda$, such that $a = \sum\limits_{k=1}^{4} i^k y_k^2$, where i is the imaginary unit. Therefore, for each $p_\lambda \in \Gamma$ we have

$$p_\lambda(\langle \delta(a), \delta(a)\rangle_A) = p_\lambda\left(\left\langle \delta\Big(\sum_{k=1}^{4} i^k y_k^2\Big), \delta\Big(\sum_{m=1}^{4} i^m y_m^2\Big)\right\rangle_A\right) \leq$$

$$\leq \sum_{k,m=1}^{4} p_\lambda(\langle \delta(y_k^2), \delta(y_m^2)\rangle_A) \leq$$

$$\leq \sum_{k,m=1}^{4} \{p_\lambda(y_k^*) p_\lambda(\langle \delta(y_k), \delta(y_m)\rangle_A) p_\lambda(y_m) + p_\lambda(\langle y_k \delta(y_k), \delta(y_m)\rangle_A) p_\lambda(y_m) +$$

$$+ p_\lambda(y_k^*) p_\lambda(\langle \delta(y_k), y_m \delta(y_m)\rangle_A) + p_\lambda(\langle \delta(y_k), y_k^* y_m \delta(y_m)\rangle_A)\} \leq$$

$$\leq \sum_{k,m=1}^{4} p_\lambda(\langle \delta(y_k), \delta(y_k)\rangle_A)^{\frac{1}{2}} p_\lambda(\langle y_k^* y_m \delta(y_m), y_k^* y_m \delta(y_m)\rangle_A)^{\frac{1}{2}} =$$

$$= \sum_{k,m=1}^{4} p_\lambda^A(\delta(y_k)) p_\lambda^A(y_k^* y_m \delta(y_m)) \leq$$

$$\leq \sum_{k,m=1}^{4} p_\lambda^A(\delta(y_k))p_\lambda(y_k)p_\lambda(y_m)p_\lambda^A(\delta(y_m)) = 0.$$

The last but one inequality follows from Cauchy-Schwarz inequality and the last inequality follows from the first inequality in (T). Thus, we get that $\|\sigma_\lambda(\delta(a))\|_{A_\lambda} = 0$, therefore δ_λ is well defined. Moreover, it is easily checked that δ_λ is a derivation from A_λ in E_λ^A, for every $\lambda \in \Lambda$. Since every E_λ^A is a Hilbert A_λ-bimodule, it follows from Ringrose's result [20, Theorem 2] that every δ_λ is continuous. That is, for each $p_\lambda \in \Gamma$, there is $C_{p_\lambda} > 0$ such that: $\|\delta_\lambda(\pi_\lambda(a))\| \leq C_{p_\lambda}\|\pi_\lambda(a)\|$, for all $a \in A$, or equivalently

$$p_\lambda(\langle\delta(a), \delta(a)\rangle_A)^{\frac{1}{2}} = p_\lambda^A(\delta(a)) \leq C_{p_\lambda} p_\lambda(a), \quad \forall\, a \in A,$$

therefore, δ is continuous. $\square$

Proposition 6.3 below is a restatement of a result of Becker in the case of Hilbert bimodules. For this, we modify the definition of approximate innerness [1, Definition 11] in the following way.

Definition 6.2. A derivation $\delta : A \to E$ from a pro-C^*-algebra $A[\tau_\Gamma]$ into a Hilbert pro-C^*-bimodule E over $A[\tau_\Gamma]$ is called *approximately inner*, if there exists a net $(x_j)_{j\in J}$ in E, such that $\delta(a) = \lim\limits_j (x_j a - a x_j)$, for all $a \in A$.

If $E[\tau]$ is a Hilbert A-bimodule over a pro-C^*-algebra $A[\tau_\Gamma]$, then $A[\tau_\Gamma] = \varprojlim_\lambda A_\lambda$ and $E[\tau] = \varprojlim_\lambda E_\lambda^A$ (see Section 5). The inverse limit $E[\tau]$ fulfils the relations (5.1)–(5.4) in Section 5. Based on these properties the following proposition is easily proved (see [1, Proposition 12]).

Proposition 6.3. *Let $\delta : A[\tau_\Gamma] \to E[\tau]$ be a derivation, such that every induced derivation $\delta_\lambda : A_\lambda \to E_\lambda^A$, $\lambda \in \Lambda$, is inner. Then the derivation δ is approximately inner.*

We remark that if we take $A[\tau]$ to be the pro-C^*-algebra $\varprojlim_n M_n(\mathbb{C}) = \prod_n M_n(\mathbb{C})$, $M_n(\mathbb{C})$ are all $n \times n$ matrices with entries from $\mathbb{C}$, $E[\tau]$ a Hilbert A-bimodule and δ a derivation of $A[\tau_\Gamma]$ in $E[\tau]$, then every $\delta_n : M_n(\mathbb{C}) \to E_n^A$, $n \in \mathbb{N}$, is inner, since $M_n(\mathbb{C})$ is semisimple and finite-dimensional (see [5, Theorem 1.9.21]). For a survey account on derivations of locally convex $(*$-)algebras, see [8].

6.(2) A realization of "compact" operators on Hilbert pro-C^*-bimodules.

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra and E a Hilbert pro-C^*-bimodule over A. $L_A(E)$ stands for the set of all maps $T : E \to E$, for which there is a map $T^* : E \to E$, such that

$$\langle T(x), y\rangle_A = \langle x, T^*y\rangle_A, \quad \forall x, y \in E.$$

It is a pro-C^*-algebra, where the C^*-seminorms $\{\tilde{p}_\lambda\}_{\lambda\in\Lambda}$ determining its topology are given by

$$\tilde{p}_\lambda(T) := \sup\{p_\lambda^A(Tx) : x \in E,\, p_\lambda^A(x) \le 1\}, \quad \lambda \in \Lambda,\, T \in L_A(E);$$

moreover, $L_A(E) = \varprojlim L_{A_\lambda}(E_\lambda^A)$, as pro-$C^*$-algebras [18, Theorem 4.2, Proposition 4.7]. For $x, y \in E$, $\theta_{x,y}$ is defined to be the element of $L_A(E)$, with

$$\theta_{x,y}(z) = x\langle y, z\rangle_A, \quad z \in E.$$

Then, denote by $K_A(E)$ the closed linear span of $\{\theta_{x,y} : x,\, y \in E\}$. $K_A(E)$ is a closed two-sided $*$-ideal of $L_A(E)$, whose elements are usually called *"compact" operators*. Note that if E, F are Hilbert pro C^*-bimodules over $A[\tau_\Gamma]$, the "compact" operators $K_A(E, F)$ are defined in a similar way (ibid.). If E, F are Hilbert C^*-modules over a C^*-algebra A, the elements of $K_A(E, F)$ considered as operators between the Banach spaces E, F need not be compact [14, p. 10]. For this reason, some authors do not use the preceding terminology. Coming back to $K_A(E)$ as before, note that this is a pro-C^*-algebra, topologically $*$-isomorphic to the pro-C^*-algebra $\varprojlim K_{A_\lambda}(E_\lambda)$ [18, Theorem 4.2, Proposition 4.7]. Let now ${}_AI$ be the closure of the two-sided ideal $span\{\,{}_A\langle x, y\rangle : x, y \in E\}$; the notation is analogous to that of I_A in the proof of Theorem 3.1. By [7, Theorem 11.7] ${}_AI$ is a $*$-ideal. Theorem 6.5, below, gives a realization of the "compact" operators $K_A(E)$, through the closed two-sided $*$-ideal ${}_AI$ of A, in case A is a σ-C^*-algebra. For this, we will need a particular form for the elements of the approximate identity that ${}_AI$ gets as a pro-C^*-algebra, which for the C^*-case stems from [6, Proposition 1.7.2] and [3, Theorem 2.1], as indicated in [4, Remark 1.9]. In the following lemma, for clarity's sake, we give a detailed proof of the same particular result in our setting.

Lemma 6.4. *Let $A[\tau_\Gamma]$ be a σ-C^*-algebra, E a Hilbert A-bimodule and ${}_AI = \overline{span}\{{}_A\langle \xi, \eta\rangle : \xi,\, \eta \in E\}$. Then ${}_AI$ has an approximate identity $\{u_\alpha\}$ with $u_\alpha = \sum_{i=1}^{n} {}_A\langle \eta_i^\alpha, \eta_i^\alpha\rangle$, where $\alpha = \{\xi_1, \dots, \xi_n\} \subset E$ ranges over finite subsets of E and $\eta_i^\alpha = \Big(\sum_{j=1}^{n} {}_A\langle \xi_j, \xi_j\rangle + \frac{1}{n} 1\Big)^{-\frac{1}{2}} \xi_i$, $i = 1, \dots, n$, where 1 is the identity of the unitization A_1 of A* [7, 8.3 Theorem].

Proof. ${}_AI$ as a closed ideal of $A[\tau_\Gamma]$ is a $*$-ideal [7, Theorem 11.7] and thus as a closed $*$-subalgebra of $A[\tau_\Gamma]$ is a σ-C^*-algebra. Let $\Gamma = \{p_n\}_{n\in\mathbb{N}}$ be the family of C^*-seminorms defining τ_Γ. We consider the right ideal R of ${}_AI$ generated by the set $\{{}_A\langle \xi, \xi\rangle^{\frac{1}{2}} : \xi \in E\}$. We note that from the functional calculus in pro-C^*-algebras [7, Theorem 10.2 and Proposition 10.13] ${}_A\langle \xi, \xi\rangle^{\frac{1}{2}} \in {}_AI$, $\xi \in E$. Then, ${}_A\langle \xi, \xi\rangle \in R^*R$, where R^*R is a two-sided ideal in ${}_AI$ and $R^* = \{r^* : r \in R\}$. From the Polarization identity we then get that ${}_A\langle \xi, \eta\rangle \in$

R^*R and thus $span\{\,_A\langle \xi, \eta\rangle : \xi, \eta \in E\} \subseteq R^*R$. Therefore, R^*R is dense in $_AI$. Then from [7, Theorem 11.5] we know that $_AI$ has an approximate identity e_λ of the form $e_\lambda = \sum_{i=1}^{n} x_i^* x_i \Big(\frac{1}{n}1 + \sum_{i=1}^{n} x_i^* x_i\Big)^{-1}$ where $\lambda = \{x_1, \ldots, x_n\}$ ranges over all finite subsets of R^*R. Now, we note that for any self-adjoint element c of R^*R there are finite many elements $c_j \in R$ such that $c \leq \sum c_j^* c_j$. This is due to the inequality

$$2Re(a^*b) \leq a^*a + b^*b, \quad \forall\, a, b \in R, \tag{6.1}$$

which is considered in the proof of [3, Theorem 2.1]. Therefore, if $c = \sum_{k=1}^{p} a_k^* b_k$, a_k, b_k in R, is a self-adjoint element of R^*R, we have that

$$c = Re(c) = Re\Big(\sum_{k=1}^{p} a_k^* b_k\Big) \leq \sum_{k=1}^{2p} c_k^* c_k, \text{ where}$$

$$c_k = \frac{1}{\sqrt{2}} a_k, \text{ for } k = 1, \ldots, p, \text{ and } c_k = \frac{1}{\sqrt{2}} b_{k-p}, \text{ for } k = p+1, \ldots, 2p,$$

with $c_k \in R$, for all $k = 1, \ldots, 2p$. Consequently, if $v_\lambda = \sum_{i=1}^{n} x_i^* x_i$, with $x_i \in R^*R$, $i = 1, \ldots, n$, then there are $c_j \in R$, $j = 1, \ldots, q$, such that $v_\lambda \leq \sum_{j=1}^{q} c_j^* c_j \equiv r_l$, where $q > n$ and $l = \{c_1, \ldots, c_q\}$. Let $R_l \equiv \sum_{j=1}^{q} c_j^* c_j (\frac{1}{q}1 + \sum_{j=1}^{q} c_j^* c_j)^{-1}$. Thus, as in the proof of [7, Theorem 11.5] we conclude that

$$e_\lambda = 1 - \frac{1}{n}\left(\frac{1}{n}1 + v_\lambda\right)^{-1} \leq 1 - \frac{1}{n}\left(\frac{1}{n}1 + r_l\right)^{-1} \leq 1 - \frac{1}{q}\left(\frac{1}{q}1 + r_l\right)^{-1} = R_l.$$

By the above inequality we have that $1 \geq 1 - e_\lambda \geq 1 - R_l \geq 0$. We then get $(1 - R_l)^2 \leq 1 - R_l \leq 1 - e_\lambda$, where the first inequality is due to the pro-C^*-algebras functional calculus [7, Chapter II, Section 10] for the positive element $1 - R_l$ with $1 - R_l \leq 1$. Therefore (ibid., 10.18 Corollary)

$$x^*(1 - R_l)^2 x \leq x^*(1 - e_\lambda)x, \quad \forall\, x \in R^*R,$$

which implies

$$p_n((1 - R_l)x)^2 = p_n(x^*(1 - R_l)^2 x) \leq p_n(x^*) p_n((1 - e_\lambda)x), \ \forall\, p_n \in \Gamma \text{ and } x \in R^*R.$$

Now, from the construction of l from λ and the fact that $\{e_\lambda\}$ is an approximate identity for $_AI$, together with the fact that R^*R is dense in $_AI$ and $p_n(1 - R_l) \leq 1$, for all $p_n \in \Gamma$, it follows that $\{R_l\}$ is an approximate identity for $_AI$ too.

Let now

$$(6.2)\qquad c_j=\sum_{s=1}^{m_j} {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j\in R,\quad m_j\in\mathbb{N},\, \xi_s^j\in E,\, a_s^j\in {}_AI,\, j=1,\dots,l.$$

Note that we can assume w.l.o.g. that ${}_AI$, as a pro-C^*-algebra, has a unit, so the finite sums in R of the form $\sum_{f\in F}\lambda_f\, {}_A\langle \xi_f,\xi_f\rangle^{\frac{1}{2}}$, $F\subset\mathbb{N}$ finite, $\xi_f\in E$, $\lambda_f\in\mathbb{C}$, for all $f\in F$, are "among" the above considered elements in (6.2). Let $T_j=\{1,\dots,m_j\}$, for $j=1,\dots,q$. Now, for c_j as in (6.2) we have the following calculation

$$c_j^*c_j\Big(\sum_{s=1}^{m_j}(a_s^j)^*\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}}\Big)\Big(\sum_{s=1}^{m_j} {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j\Big)=$$

$$=\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle+\sum_{s\in T_j\setminus\{1\}}(a_1^j)^*\, {}_A\langle \xi_1^j,\xi_1^j\rangle^{\frac{1}{2}}\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j+\cdots+$$

$$+\sum_{s\in T_j\setminus\{m_j\}}(a_{m_j}^j)^*\, {}_A\langle \xi_{m_j}^j,\xi_{m_j}^j\rangle^{\frac{1}{2}}\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j=$$

$$=\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle+\sum_{k=1}^{m_j-1}\Big[\sum_{s\in T_j\setminus\{1,\cdots,k\}}(a_k^j)^*\, {}_A\langle \xi_k^j,\xi_k^j\rangle^{\frac{1}{2}}\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j+$$

$$+(a_s^j)^*\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}}\, {}_A\langle \xi_k^j,\xi_k^j\rangle^{\frac{1}{2}} a_k^j\Big]=\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle+$$

$$+\sum_{k=1}^{m_j-1}\Big[\sum_{s\in T_j\setminus\{1,\cdots,k\}}2\,Re\big((a_k^j)^*\, {}_A\langle \xi_k^j,\xi_k^j\rangle^{\frac{1}{2}}\, {}_A\langle \xi_s^j,\xi_s^j\rangle^{\frac{1}{2}} a_s^j\big)\Big]\le$$

$$\le\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle+$$

$$+\sum_{k=1}^{m_j-1}\Big[\sum_{s\in T_j\setminus\{1,\cdots,k\}} {}_A\langle (a_k^j)^*\xi_k^j,(a_k^j)^*\xi_k^j\rangle+\ {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle\Big]=$$

$$=\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle+(m_j-1)\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle=$$

$$=m_j\sum_{s=1}^{m_j} {}_A\langle (a_s^j)^*\xi_s^j,(a_s^j)^*\xi_s^j\rangle=\sum_{s=1}^{m_j} {}_A\langle (\sqrt{m_j}a_s^j)^*\xi_s^j,(\sqrt{m_j}a_s^j)^*\xi_s^j\rangle.$$

In the above string of relations, the inequality is due to (6.1). Therefore,

$$\sum_{j=1}^{q} c_j^* c_j \leq \sum_{j=1}^{q} \left[\sum_{s=1}^{m_j} {}_A\langle (\sqrt{m_j} a_s^j)^* \xi_s^j, (\sqrt{m_j} a_s^j)^* \xi_s^j \rangle \right] = \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle,$$

for a finite subset T of $\mathbb{N}$ and $b_f \in {}_A I$, $\xi_f \in E$, for all $f \in T$. Reasoning as above we get that

$$\begin{aligned} u_\alpha &= \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle \left(\frac{1}{|T|} 1 + \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle \right)^{-1} \\ &= \sum_{f \in T} {}_A\left\langle \left(\frac{1}{|T|} 1 + \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle \right)^{-\frac{1}{2}} b_f \xi_f, \left(\frac{1}{|T|} 1 + \right.\right. \\ &\qquad \left.\left. + \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle \right)^{-\frac{1}{2}} b_f \xi_f \right\rangle = \sum_{f \in T} {}_A\langle \eta_f^\alpha, \eta_f^\alpha \rangle, \end{aligned}$$

is an approximate identity for ${}_A I$, where

$$\eta_f^\alpha = \left(\frac{1}{|T|} 1 + \sum_{f \in T} {}_A\langle b_f \xi_f, b_f \xi_f \rangle \right)^{-\frac{1}{2}} b_f \xi_f \quad \text{and} \quad \alpha = \{ b_f \xi_f : f \in T \}$$

ranges over all finite subsets $\{a\xi : a \in {}_A I, \ \xi \in E\}$ of E. Let Λ be the family of all such finite subsets and $\{u_\alpha\}_{\alpha \in \Lambda}$ the above approximate identity of ${}_A I$. Now, from functional calculus in the pro-C^*-algebra ${}_A I$ we have that $p_n(u_\alpha) \leq 1$, for all $p_n \in \Gamma$, $\alpha \in \Lambda$. Also, it can easily be checked that $u_\alpha \xi \to \xi$, for all $\xi \in E$. Therefore, since the Hilbert pro-C^*-bimodule E over A is by restriction a Hilbert ${}_A I$-bimodule, by a factorization result of Summers [22, Theorem 2.1], we have that for $\xi \in E$ there are $\eta \in E$, $a \in {}_A I$ such that $\xi = a\eta$ (in fact to apply the result of Summers we need only the fact that E is a left Hilbert pro-C^*-module over ${}_A I$). This fact, together with the above considerations, implies that the elements u_α of the approximate identity of ${}_A I$ are of the form $u_\alpha = \sum_{j=1}^{m} {}_A\langle \eta_j^\alpha, \eta_j^\alpha \rangle$, where $\eta_j^\alpha = \left(\frac{1}{m} 1 + \sum_{j=1}^{m} {}_A\langle \xi_j, \xi_j \rangle \right)^{-\frac{1}{2}} \xi_j$ and $\alpha = \{\xi_1, \ldots, \xi_m\}$ ranges over all finite subsets of E. $\square$

THEOREM 6.5. *Let $A[\tau_\Gamma]$ be a σ-C^*-algebra, where $\Gamma = \{p_n\}_{n \in \mathbb{N}}$. Let E be a Hilbert A-bimodule. Then ${}_A I = K_A(E)$, with respect to a topological $*$-isomorphism.*

Proof. ${}_A I$ as a closed $*$-subalgebra of $A[\tau_\Gamma]$ is a $\sigma - C^*$- algebra and hence it has an Arens-Michael decomposition, i.e., ${}_A I = \varprojlim I_n$, up to a topological $*$-isomorphism, where $I_n = {}_A I / N_n$; $N_n = \ker(p_n|_{{}_A I})$, $n \in \mathbb{N}$ (see Section 2).

Moreover, $L_A(E) = \varprojlim L_{A_n}(E_n^A)$, up to a topological $*$-isomorphism, as we noticed above. Consider the correspondence

$$\lambda^n : I_n \to L_{A_n}(E_n^A) \text{ with } \lambda^n_{a+N_n}(\xi + N_n^A) = a\xi + N_n^A, \quad a \in {}_AI,\ \xi \in E.$$

Notice that λ^n is well-defined (apply the same procedure as that for the map κ_λ in the proof of Theorem 3.1). By Lemma 6.4, ${}_AI$ has an approximate identity (u_α), with

$$u_\alpha = \sum_{i=1}^m {}_A\langle \eta_i^\alpha, \eta_i^\alpha\rangle, \qquad \eta_i^\alpha = \left(\frac{1}{m}1 + \sum_{i=1}^m {}_A\langle \xi_i, \xi_i\rangle\right)^{-\frac{1}{2}} \xi_i \in E$$

and $\alpha = \{\xi_1, \dots, \xi_m\} \subset E$ ranges over all finite subsets of E.

Now, if $u_{\alpha,n} = \Big(\sum\limits_{i=1}^m {}_A\langle \eta_i^\alpha, \eta_i^\alpha\rangle\Big) + N_n \in I_n$, then $(u_{\alpha,n})_\alpha$ is an approximate identity for I_n, for every $n \in \mathbb{N}$ [16, Lemma 1.2, p. 466]. Then if $b \in {}_AI$ with $\lambda^n_{b+N_n} = 0$, we have that

$$\begin{aligned}(b + N_n)u_{\alpha,n} &= \left(b\sum_{i=1}^m ({}_A\langle \eta_i^\alpha, \eta_i^\alpha\rangle)\right) + N_n = \sum_{i=1}^m ({}_A\langle b\eta_i^\alpha, \eta_i^\alpha\rangle) + N_n = \\ &= \sum_{i=1}^m \big({}_{A_n}\langle (b + N_n)(\eta_i^\alpha + N_n^A), \eta_i^\alpha + N_n^A\rangle\big) = \\ &= \sum_{i=1}^m \big({}_{A_n}\langle \lambda^n_{b+N_n}(\eta_i^\alpha + N_n^A), \eta_i^\alpha + N_n^A\rangle\big) = 0,\end{aligned}$$

for every α, thus $b + N_n = 0$. So λ^n is 1-1 for all n. For the last but one equality above, see discussion after Definition 3.4. For $n \leq m$, $\sigma_{mn} : E_m^A \to E_n^A$ are the connecting maps of the inverse system $\{E_m^A\}_{m\in\mathbb{N}}$ as Hilbert A_m-bimodules (Section 5). Let

$$F_{mn} : L_{A_m}(E_m^A) \to L_{A_n}(E_n^A) \text{ with } F_{mn}(T)(\xi + N_n^A) := \sigma_{mn}(T(\xi + N_m^A)),$$

$T \in L_{A_m}(E_m^A)$, $\xi \in E$, be the connecting maps of the inverse system of the C^*-algebras $\{L_{A_m}(E_m^A)\}_{m\in\mathbb{N}}$ [9, p. 44]. For $n \leq m$, consider the following diagram

$$\begin{array}{ccc} I_m & \xrightarrow{\lambda^m} & L_{A_m}(E_m^A) \\ \Big\downarrow{\scriptstyle \pi_{mn}} & & \Big\downarrow{\scriptstyle F_{mn}} \\ I_n & \xrightarrow{\lambda^n} & L_{A_n}(E_n^A) \end{array}$$

where $\pi_{mn} : I_m \to I_n$ are the connecting maps of the inverse system $\{I_m\}_{m\in\mathbb{N}}$. Then, for $a \in {}_AI$ and $\xi \in E$ we have that

$$\begin{aligned} F_{mn}\lambda^m_{a+N_m}(\xi + N^A_n) &= \sigma_{mn}(\lambda^m_{a+N_m}(\xi + N^A_m)) = \sigma_{mn}(a\xi + N^A_m) \\ &= a\xi + N^A_n = \lambda^n_{a+N_n}(\xi + N^A_n) \\ &= \lambda^n(\pi_{mn}(a + N_m))(\xi + N^A_n), \quad n \in \mathbb{N}, \end{aligned}$$

so the above diagram is commutative.

Let now $a \in {}_AI$. Define $\lambda_a : E \to E : \xi \mapsto a\xi$. Then $\lambda_a \in L_A(E)$ due to (3.4)(i). Thus, the correspondence

$$\lambda :_A I \to L_A(E) : a \mapsto \lambda_a,$$

is well-defined and it is a $*$-homomorphism between σ-C^*-algebras. Moreover, let $\pi_n :_A I \to I_n$, $\sigma_n : E \to E^A_n$ be the projective maps of the inverse systems $\{E^A_n\}_{n\in\mathbb{N}}$ and $\{I_n\}_{n\in\mathbb{N}}$ respectively and

$$F_n : L_A(E) \to L_{A_n}(E^A_n) \text{ with } F_n(S)(\xi + N^A_n) = \sigma_n(S\xi), \quad S \in L_A(E),$$

$\xi \in E$, $n \in \mathbb{N}$, the projective maps of the inverse system $\{L_{A_n}(E^A_n)\}_{n\in\mathbb{N}}$ (ibid.). Then the following diagram

$$\begin{tikzcd} {}_AI \arrow[r, "\lambda"] \arrow[d, "\pi_n"] & L_A(E) \arrow[d, "F_n"] \\ I_n \arrow[r, "\lambda^n"] & L_{A_n}(E^A_n) \end{tikzcd}$$

is commutative, since for all $a \in {}_AI$ and $\xi \in E$, we have

$$\begin{aligned} F_n(\lambda(a))(\xi + N^A_n) &= \sigma_n(\lambda_a(\xi)) = a\xi + N^A_n = \lambda^n(a + N_n)(\xi + N^A_n) \\ &= \lambda^n(\pi_n(a))(\xi + N^A_n), \quad n \in \mathbb{N}. \end{aligned}$$

Therefore, $\lambda = \varprojlim \lambda^n$ and thus λ has closed range and is a homeomorphism onto its image, according to [18, Proposition 5.3 (1)]. Then since λ maps a dense set of ${}_AI$, i.e., $span\{{}_A\langle\xi, \eta\rangle : \xi, \eta \in E\}$ onto the set $\{\theta_{\xi,\eta} : \xi, \eta \in E\}$, which is dense in $K_A(E)$, we conclude that $K_A(E) = {}_AI$, up to a topological $*$-isomorphism. □

We note that Theorem 6.5 is an extension of Proposition 1.10 in [4], in the non-normed setting. Notice that in case we have two σ-C^*-algebras A, B and a Hilbert B-A-bimodule E, then, with the same proof as above, we get that ${}_BI = K_A(E)$ and ${}_AI = K_B(E^*)$, up to topological $*$-isomorphisms, where E^* is a Hilbert A-B-bimodule defined exactly as in [4, Definition 1.4].

The work developed in this article is essentially used in a forthcoming paper, which introduces and studies the important topic of "C^*-correspondences", in the context of pro-C^*-algebras.

Further applications of the contents of the present paper will appear in a joint paper with M. Joiţa, entitled "Crossed products by Hilbert pro-C^*-bimodules" [10]. For the C^*-case of the aforementioned topics see e.g., [11, 12], respectively [4, 17].

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