A WEIGHTED PATH-FOLLOWING METHOD FOR LINEARLY CONSTRAINED CONVEX PROGRAMMING

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In this paper, we present a new approach based on weighted path-following method for a class of nonlinear problems. We are based on the notion of central path and we use Newton's method to resolve the corresponding nonlinear systems. The proposed algorithm can start with any strictly feasible point, not necessary close to the central path. The idea is to introduce weights into the barrier function. The algorithm possesses a polynomial complexity and a global convergence. Finally, we present some numerical experiments which show that the algorithm works properly.

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1. INTRODUCTION

The primal dual interior point methods are originally developed to solve linear programs [4, 5, 9, 10, 15, 21]. Their attractive theoretical and numerical properties motivated researchers to elaborate extensions for more general classes of optimization programs namely, convex quadratic programming [8, 16, 22], linear complementarity problem [6, 14, 17], semidefinite programming [2, 19], convex programming [11, 12, 18, 20] and general nonlinear programming. For the primal-dual path-following methods we have two types of algorithms: the feasible and infeasible algorithms. The idea used on the feasible algorithms is simple: draw a path of the centers defined by the perturbed KKT optimality conditions. We use Newton's method to treat the associated perturbed equations. This type of methods possesses a polynomial complexity and a rather simple implementation if the initial point is chosen in a neighborhood of the central path. This condition of centrality is very restrictive because generally, it is difficult to have at first a point close to the central path, that risks to drive to the divergence of the algorithm [4, 6, 11, 14]. To avoid this problem, some researchers suggest starting the algorithm from an infeasible strictly positive point. These methods are called infeasible central path methods [1, 13].

In this paper, we propose a primal-dual feasible path-following algorithm for solving nonlinear programming. We associate with this problem a weighted barrier function. The perturbed problem is strictly convex. Furthermore, the KKT conditions are necessary and sufficient. The new iterate is obtained by Newton's method. The algorithm can start from any strictly feasible point without supposing that it is up to the neighborhood of the central path. Such a point can be obtained by the use of the Karmarkar's algorithm [3, 10]. We demonstrate the polynomial complexity of the algorithm and its global convergence. We complete our study by numerical results.

The paper is organized as follows. We first present a description of a primal-dual method in Section 2. We next describe a weighted path-following method for convex programming in Section 3. The convergence results are presented in Section 4. For testing our approach, we give some numerical examples in Section 5. Section 6 contains a general conclusion and some suggestions for future research. We use the classical notation. In particular, R^n denotes the n-dimensional Euclidean space with $R^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ is the positive orthant and $R^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$ is the interior of R^n_+.

Given $u, v \in R^n$, $u^t v = \sum\limits_{i=1}^{n} u_i v_i$ is their inner product, $\|u\| = \sqrt{u^t u}$ is the Euclidean norm. Given a vector $x \in R^n$, $X = \operatorname{diag}(x) = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ is the $n \times n$ diagonal matrix with $X_{ii} = x_i$ for all i, whereas $\|X\| = \max_{i=1}^{n} |x_{ii}|$ is the maximum norm for a given diagonal matrix X. I is the identity matrix with order n and e is a vector in R^n such that $e_i = 1$ for $i = 1, \ldots, n$.

2. A PRIMAL-DUAL CENTRAL PATH METHOD

We consider the following nonlinear optimization problem

$$(P) \qquad \begin{cases} \min f(x) \\ Ax = b \\ x \geq 0, \end{cases}$$

where $f : R^n \to R$, is a nonlinear, convex and sufficiently differentiable function, A is a $m \times n$ matrix of a full rank row. x and b are vectors of R^n and R^m respectively.

The dual problem of (P) can be obtained as

$$(D) \qquad \begin{cases} \max(b^t y - x^t \nabla f(x) + f(x)) \\ \nabla f(x) - (A^t y + s) = 0 \\ s \geq 0. \end{cases}$$

We denote by $F = \{(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n : Ax = b,\ A^t y + s = \nabla f(x),\ (x, s) \geq 0\}$, the set of the primal-dual feasible points of (P) and (D). $F_{int} = \{(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n : Ax = b,\ A^t y + s = \nabla f(x),\ (x, s) > 0\}$, the set of the primal-dual strictly feasible points of (P) and (D). We associate to the problem (P) the following perturbed problem

$$(P)_\mu \qquad \begin{cases} \min f_\mu(x) = \left[f(x) - \mu \sum\limits_{i=1}^{n} \ln(x_i) \right] \\ Ax = b \\ x > 0, \end{cases}$$

where $\mu > 0$ is the barrier term.

The KKT conditions are necessary and sufficient, and can be obtained as follow

$$\begin{cases} A^t y + s = \nabla f(x) \\ Ax = b \\ Xs - \mu e = 0 \\ (x, s) > 0. \end{cases} \tag{2.1}$$

The set of all solutions $\{(x_\mu, y_\mu, s_\mu) : \mu > 0\}$ of the system (2.1) is called the central path. It is noted by $C = \{(x, y, s) \in F_{int} : XSe - \mu e = 0\}$.

We note that (x, y, s) is a primal-dual solution of (P) and (D) if and only if (x, y, s) satisfies the equation $\Psi_\mu(x, y, s) = 0$ with $\mu = 0$.

$$\Psi_\mu(x, y, s) = \begin{bmatrix} A^t y + s - \nabla f(x) \\ Ax - b \\ XSe - \mu e \end{bmatrix},$$

where $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ and $S = \operatorname{diag}(s_1, s_2, \ldots, s_n)$.

The Newton's direction $\triangle w = (\Delta x, \Delta y, \Delta s)$ is obtained by solving the linear system $\Psi_\mu + \nabla \Psi_\mu \Delta w = 0$ which is equivalent to the following form

$$\begin{cases} A\Delta x = 0 \\ \nabla^2 f(x)\Delta x - A^t \Delta y - \Delta s = 0 \\ X\Delta s + S\Delta x = -Xs + \mu e. \end{cases} \tag{2.2}$$

A point is neighbor of the central path if it belongs to the set $C(\theta) = \{(x, y, s) \in F_{int} : \|XSe - \mu e\| \leq \theta\mu,\ \theta \in (0, 1)\}$.

Basic Algorithm

Begin algorithm

Initialization

Let $\varepsilon > 0$ be a given precision and $\theta, \beta \in (0, 1)$; Let $(x^0, y^0, s^0) \in C(\theta)$ and $\mu^0 = \frac{(x^0)^t s^0}{n}$; $k = 0$.

- While $\mu^k > \varepsilon$ do
  - $\mu^{k+1} = \beta\mu^k = \beta\frac{(x^k)^t s^k}{n}$;
  - Determine $(\Delta x^k, \Delta y^k)$ by solving the linear system

$$\begin{bmatrix} X^k\nabla^2 f(x^k) + S^k & -X^k A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \end{bmatrix} = \begin{bmatrix} -X^k S^k e + \mu^{k+1} e \\ 0 \end{bmatrix}$$

  – Compute $(x^{k+1}, y^{k+1}) = (x^k, y^k) + (\Delta x^k, \Delta y^k)$; $s^{k+1} = \nabla f(x^{k+1}) - A^t y^{k+1}$; $k = k+1$;
- End While

End algorithm.

Remark 2.1. In the initialization, we have supposed that $(x^0, y^0, s^0) \in C(\theta)$ that means (x^0, y^0, s^0) is a strictly feasible primal-dual point and satisfies the condition $\|X^0 S^0 e - \mu^0 e\| \leq \theta\mu^0$, $0 < \theta < 1$. Anything guaranteed the convergence of this algorithm even if this point is strictly feasible. This difficulty can be surmounted by introducing weights into the barrier function.

3. A WEIGHTED CENTRAL PATH METHOD

We consider again the nonlinear problem

$$(P) \qquad \begin{cases} \min f(x) \\ Ax = b \\ x \geq 0. \end{cases}$$

We associate to (P) the perturbed problem defined as follow

$$(P)_{\mu r} \qquad \begin{cases} \min f_{\mu r}(x) = \left[f(x) - \mu \sum\limits_{i=1}^{n} r_i \ln(x_i) \right] \\ Ax = b \\ x > 0, \end{cases}$$

$\mu > 0$, $r = (r_1, r_2, \ldots, r_n)^t \in R^n_+$ is the vector of the weights associated with the barrier function.

The corresponding conditions of KKT are necessary and sufficient and can be written as

$$\text{(3.1)} \qquad \begin{cases} \nabla f(x) - \mu X^{-1} r - A^t y = 0 \\ Ax = b \\ x > 0. \end{cases}$$

Set $s = \mu X^{-1} r$ then (3.1) is equivalent to

$$\left\{\begin{array}{l} A^t y + s = \nabla f(x) \\ Ax = b \\ Xs - \mu r = 0 \\ (x, s) > 0. \end{array}\right. \tag{3.2}$$

Applying Newton's method for $(x, y, s) \in F$, we obtain the system

$$\left\{\begin{array}{l} A\Delta x = 0 \\ \nabla^2 f(x)\Delta x - A^t \Delta y - \Delta s = 0 \\ X\Delta s + S\Delta x = -Xs + \mu r. \end{array}\right.$$

(x_μ, y_μ, s_μ) is closed to the central path if it belongs to the set

$$C_r(\theta) = \left\{(x, y, s) \in F_{int} : \|XSe - \mu r\| \leq \theta\mu,\ \theta \in (0,1)\right\}.$$

Algorithm

Begin algorithm

Initialization Let $\varepsilon > 0$ be a given precision, $(x^0, y^0, s^0) \in F_{int}$,

$$\sigma = \frac{\left\|X^0 S^0 e\right\|}{\sqrt{n}}; \quad r = \frac{X^0 S^0 e}{\sigma}; \quad \eta = \min\{r_i\};$$

$$\theta = \left(1 - \frac{\sqrt{2}}{2}\right)\eta; \quad \beta = 1 - \frac{\theta}{\sqrt{n}} \quad \text{and} \quad \mu^0 = \frac{(x^0) R^t s^0}{n}; \quad k = 0.$$

- While $\mu^k > \varepsilon$ do
  - $\mu^{k+1} = \beta\mu^k = \beta\frac{(x^k)^t R s^k}{n}$;
  - Determine $(\Delta x^k, \Delta y^k)$ by solving the linear system

$$\begin{bmatrix} X^k \nabla^2 f(x^k) + S^k & -X^k A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \end{bmatrix} = \begin{bmatrix} -X^k S^k e + \mu^{k+1} r \\ 0 \end{bmatrix}$$

  - Compute

$$(x^{k+1}, y^{k+1}) = (x^k, y^k) + (\Delta x^k, \Delta y^k);$$
$$s^{k+1} = \nabla f(x^{k+1}) - A^t y^{k+1}; \quad k = k + 1;$$

- End While

End algorithm

4. CONVERGENCE OF THE ALGORITHM

Let (x, y, s) the solution of the system (3.2) being in the neighborhood of the central path, that is

$$\|XSe - \mu r\| = \left[\sum_{i=1}^{n}(x_i s_i - \mu r_i)^2\right]^{\frac{1}{2}} \leq \theta\mu,$$

where $\mu = \frac{x^t Rs}{n}$. We put $\widehat{\mu} = \beta\mu, \beta \in (0, 1)$.

Let $(\widehat{x}, \widehat{y}, \widehat{s}) = (x, y, s)+$ $(\Delta x, \triangle y, \Delta s)$, where $(\Delta x, \triangle y, \Delta s)$ is the solution of the system

$$\text{(4.1)} \qquad \begin{cases} A\Delta x = 0 & \text{(a)} \\ A^t\Delta y + \Delta s = \nabla^2 f(x)\Delta x & \text{(b)} \\ X\Delta s + S\Delta x = -Xs + \widehat{\mu}r & \text{(c).} \end{cases}$$

Taking $D = (XS^{-1})^{\frac{1}{2}}$ and $\Delta X = \operatorname{diag}\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n)$. To prove the convergence of our algorithm we need the following lemmas:

Lemma 4.1. *For $\eta = \min r_i$ and $(x, y, s) \in C_r(\theta)$ such that $0 < \theta < 1$, we have $\left\|(XS)^{-\frac{1}{2}}\right\| \leq \frac{1}{\sqrt{(\eta-\theta)\mu}}$ for $\eta > \theta$.*

Proof. (x, y, s) satisfy the condition $\|XSe - \mu r\| \leq \theta\mu$, then

$$|x_i s_i - \mu r_i| \leq \left[\sum_{i=1}^{n}(x_i s_i - \mu r_i)^2\right]^{\frac{1}{2}} = \|XSe - \mu r\| \leq \theta\mu$$

$$\Rightarrow (r_i - \theta)\mu \leq x_i s_i \leq (r_i + \theta)\mu$$

$$\Rightarrow (x_i s_i)^{-1/2} \leq (r_i - \theta)^{-1/2}\mu^{-1/2} \leq (\eta - \theta)^{-1/2}\mu^{-1/2}$$

$$\Rightarrow \left\|(XS)^{-\frac{1}{2}}\right\| = \max\left\{\frac{1}{\sqrt{x_i s_i}}\right\} \leq \frac{1}{\sqrt{(\eta-\theta)\mu}}. \quad \square$$

Lemma 4.2. *Let $z = (XS)^{-\frac{1}{2}}(-Xs + \widehat{\mu}r)$, $\delta = (1 - \beta)\sqrt{n}$, $0 < \beta < 1$ and $\mu = \frac{x^t Rs}{n}$, then*

$$\|z\| \leq \sqrt{\frac{(\theta^2 + \delta^2)\mu}{(\eta - \theta)}}.$$

Proof. $\|z\|^2 = \left\|(XS)^{-\frac{1}{2}}(\widehat{\mu}r - XSe)\right\|^2 = \left\|(XS)^{-\frac{1}{2}}[(\mu r - XSe) + (\widehat{\mu} - \mu)r]\right\|^2 \leq \frac{1}{(\eta-\theta)\mu}\left[\theta^2\mu^2 + (1-\beta)^2\mu^2 n\right]$ (by Lemma 4.1); $\|z\| \leq \sqrt{\frac{(\theta^2+\delta^2)}{(\eta-\theta)}\mu}$. $\square$

LEMMA 4.3. *For* $(\widehat{x},\widehat{y},\widehat{s}) \in C_r(\theta)$, *we have*
1) $\|\widehat{X}\widehat{S}e - \widehat{\mu}r\| \leq \frac{(\theta^2+\delta^2)}{2(\eta-\theta)}\mu$;
2) $\left|\frac{\widehat{x}^t R\widehat{s}}{n} - \widehat{\mu}\right| \leq \frac{(\theta^2+\delta^2)}{2\sqrt{n}(\eta-\theta)}\mu$.

Proof. 1) By equation (4.1.c), we have $D\Delta s + D^{-1}\Delta x = z$, that is $\sqrt{\frac{x_i}{s_i}}\Delta s_i + \sqrt{\frac{s_i}{x_i}}\Delta x_i = \frac{\widehat{\mu}r_i - x_i s_i}{\sqrt{x_i s_i}}$.

We put $u_i = \sqrt{\frac{x_i}{s_i}}\Delta s_i$ then $u = D\Delta s$, $v_i = \sqrt{\frac{s_i}{x_i}}\Delta x_i$ then $v = D^{-1}\Delta x$.

We have $u_i + v_i = z_i$, $\widehat{x}_i\widehat{s}_i - \widehat{\mu}r_i = \Delta x_i \Delta s_i$, then $\widehat{X}\widehat{S}e - \widehat{\mu}r = \Delta X \Delta s$,

$$\left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| = \left[\sum_{i=1}^{n}(\widehat{x}_i\widehat{s}_i - \widehat{\mu}r_i)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n}(\Delta x_i \Delta s_i)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n}(u_i v_i)^2\right]^{\frac{1}{2}}.$$

We have $\sum\limits_{i=1}^{n}(u_i v_i)^2 \leq \left(\sum\limits_{i=1}^{n} u_i v_i\right)^2 \Rightarrow \left[\sum\limits_{i=1}^{n}(u_i v_i)^2\right]^{\frac{1}{2}} \leq \sum\limits_{i=1}^{n} u_i v_i$,

$$\left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| \leq \sum_{i=1}^{n} u_i v_i \leq \frac{1}{2}\sum_{i=1}^{n}(u_i + v_i)^2 = \frac{1}{2}\sum_{i=1}^{n} z_i^2 = \frac{1}{2}\|z\|^2 \leq \frac{\theta^2+\delta^2}{2(\eta-\theta)}\mu.$$

2) $\left|\frac{\widehat{x}^t R\widehat{s}}{n} - \widehat{\mu}\right| = \left|\frac{\widehat{x}^t R\widehat{s}}{n} - \widehat{\mu}\frac{r^t r}{n}\right| = \frac{1}{n}\left|\widehat{x}^t R\widehat{s} - \widehat{\mu}r^t r\right| = \frac{1}{n}\left|\sum\limits_{i=1}^{n}\widehat{x}_i r_i \widehat{s}_i - \widehat{\mu}r^t r\right| = \frac{1}{n}\left|r^t(\widehat{X}\widehat{S}e - \widehat{\mu}r)\right| \leq \frac{1}{n}\|r\|\left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| \leq \frac{(\theta^2+\delta^2)}{2\sqrt{n}(\eta-\theta)}\mu$. □

Remark 4.1. By Lemma (4.3), we have $\left|\frac{\widehat{x}^t R\widehat{s}}{n} - \widehat{\mu}\right| \leq \frac{(\theta^2+\delta^2)}{2\sqrt{n}(\eta-\theta)}\mu$ then

$$\left(1 - \frac{(\theta-\delta)^2 + 2\delta\eta}{2\sqrt{n}(\eta-\theta)}\right)\mu \leq \frac{\widehat{x}^t R\widehat{s}}{n} \leq \left(1 + \frac{(\theta+\delta)^2 - 2\delta\eta}{2\sqrt{n}(\eta-\theta)}\right)\mu. \tag{4.2}$$

Conditions on the choice of the parameters

1) In order to stay in the neighborhood of the central path we have from Lemma (4.3)

$$\left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| \leq \frac{(\theta^2+\delta^2)}{2(\eta-\theta)}\mu,$$

so it is necessary that the following condition is satisfied

$$\frac{\theta^2+\delta^2}{2(\eta-\theta)} \leq \beta\theta.$$

2) We must have $\frac{\widehat{x}^t R\widehat{s}}{n} \leq C\frac{x^t Rs}{n}$ with $C < 1$.

From (4.2), we have $\frac{\widehat{x}^t R\widehat{s}}{n} \leq \left(1 + \frac{(\theta+\delta)^2 - 2\delta\eta}{2\sqrt{n}(\eta-\theta)}\right)\mu$, then

$$\frac{(\theta+\delta)^2 - 2\delta\eta}{2\sqrt{n}(\eta-\theta)} < 0.$$

We have the following conditions

$$\begin{cases} \eta > \theta \\ \dfrac{\theta^2+\delta^2}{2(\eta-\theta)} \leq \left(1 - \dfrac{\delta}{\sqrt{n}}\right)\theta \\ \dfrac{(\theta+\delta)^2 - 2\delta\eta}{2\sqrt{n}(\eta-\theta)} < 0. \end{cases} \tag{4.3}$$

We only give a simple solution of the system (4.3), let $\theta = \delta$, thus we obtain

$$\begin{cases} \theta^2 - (2\sqrt{n}+\eta)\theta + \eta\sqrt{n} \geq 0 \\ \theta < \dfrac{\eta}{2}. \end{cases} \tag{4.4}$$

We can verify easily that for $0 < \theta < \frac{\eta}{3}$, both conditions of the system (4.4) are satisfied.

Now, we want to find the optimal value of θ in the sense that the quantity $\frac{x^t Rs}{n}$ decreases most rapidly. In other words, under conditions (4.4), we make the term $1 + \frac{(\theta+\delta)^2 - 2\delta\eta}{2\sqrt{n}(\eta-\theta)}$ as small as possible. For this, we define the function $g(\theta) = \frac{2\theta^2 - \eta\theta}{\eta-\theta}$. We have $g'(\theta) = \frac{-2\theta^2 + 4\eta\theta - \eta^2}{(\eta-\theta)^2}$.

The unique critical point of g in the interval $(0,\eta)$ is $\left(1 - \frac{\sqrt{2}}{2}\right)\eta$, furthermore, we have $g''(\theta) = \frac{2\eta^2}{(\eta-\theta)^3} > 0$, then $\theta^* = \left(1 - \frac{\sqrt{2}}{2}\right)\eta$ is a global minimum of g in $(0,\eta)$.

We present now the results of convergence of our algorithm in the following theorems.

Theorem 4.1. *Let* $\theta = \delta = \left(1 - \frac{\sqrt{2}}{2}\right)\eta$, *let us suppose that* $(x, y, s) \in C_r(\theta)$ *and* $\widehat{\mu} = \beta\mu$ *such that* $\beta = \left(1 - \frac{\delta}{\sqrt{n}}\right)$, $\mu = \frac{x^t Rs}{n}$ *then*

1) $(\widehat{x}, \widehat{y}, \widehat{s}) \in C_r(\theta)$;

2) $\frac{\widehat{x}^t R\widehat{s}}{n} \leq \left(1 - \frac{\eta}{6\sqrt{n}}\right)\frac{x^t Rs}{n}$.

Proof. 1) Under the previous conditions, we have $\left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| \leq \theta\widehat{\mu}$.

It remains to satisfy the condition $(\widehat{x}, \widehat{s}) > 0$. We have

$$|\widehat{x}_i\widehat{s}_i - \widehat{\mu}r_i| \leq \left\|\widehat{X}\widehat{S}e - \widehat{\mu}r\right\| \leq \theta\widehat{\mu}$$
$$-\theta\widehat{\mu} \leq \widehat{x}_i\widehat{s}_i - \widehat{\mu}r_i \leq \theta\widehat{\mu} \Rightarrow (r_i - \theta)\widehat{\mu} \leq \widehat{x}_i\widehat{s}_i \leq (r_i + \theta)\widehat{\mu}$$
$$\widehat{x}_i\widehat{s}_i \geq (r_i - \theta)\widehat{\mu} \geq (\eta - \theta)\widehat{\mu} > 0 \text{ since } \eta > \theta,$$

then $\widehat{x}_i$ and $\widehat{s}_i$ have the same sign. We suppose that $\widehat{x}_i < 0$, $\widehat{s}_i < 0$

$$\widehat{x}_i = x_i + \Delta x_i \Rightarrow \Delta x_i = \widehat{x}_i - x_i < 0$$
$$|\Delta x_i| = -\Delta x_i = x_i - \widehat{x}_i > -\widehat{x}_i = |-\widehat{x}_i| \Rightarrow |\Delta x_i| > |-\widehat{x}_i|\,.$$

In the same way, we have $|\Delta s_i| > |-\widehat{s}_i|$.

Then $\Delta x_i\Delta s_i = (-|\Delta x_i|)(-|\Delta s_i|) = |\Delta x_i|\;|\Delta s_i| > |-\widehat{x}_i|\,|-\widehat{s}_i| = |\widehat{x}_i\widehat{s}_i| = \widehat{x}_i\widehat{s}_i(\widehat{x}_i\widehat{s}_i > 0)$, $\Delta x_i\Delta s_i > \widehat{x}_i\widehat{s}_i$.

We have $\widehat{x}_i\widehat{s}_i - \widehat{\mu}r_i = \Delta x_i\Delta s_i < \widehat{x}_i\widehat{s}_i$ contradiction, then $\widehat{x}_i > 0$, $\widehat{s}_i > 0$.

2) We have $g(\theta^*) = -(\sqrt{2}-1)^2\eta < -\frac{\eta}{6}$,

$$C = 1 + \frac{1}{\sqrt{n}}g(\theta^*) < 1 - \frac{\eta}{6\sqrt{n}}. \quad \square$$

Theorem 4.2. *Let $\varepsilon > 0$ be a given precision. Suppose that the algorithm generates a sequence of iterates (x^k, y^k, s^k) that satisfies $(x^{k+1})^t Rs^{k+1} < \left(1 - \frac{\eta}{6\sqrt{n}}\right)(x^k)^t Rs^k$ for $\eta > 0$, then there exist K with $K = O\left[\sqrt{n}\ln\left(\frac{(x^0)^t Rs^0}{\varepsilon}\right)\right]$ such that $(x^k)^t Rs^k \leq \varepsilon$ for all $k \geq K$.*

Proof. For such k we have $(x^k)^t Rs^k \leq \left(1 - \frac{\eta}{6\sqrt{n}}\right)(x^{k-1})^t Rs^{k-1}$.

We have $\ln\left((x^k)^t Rs^k\right) \leq k\ln\left(1 - \frac{\eta}{6\sqrt{n}}\right) + \ln\left((x^0)^t Rs^0\right)$,

$$\ln\left((x^k)^t Rs^k\right) \leq k\left(-\frac{\eta}{6\sqrt{n}}\right) + \ln\left((x^0)^t Rs^0\right) \quad (\ln(1+\beta) \leq \beta \text{ for } \beta \geq -1)$$
$$k\left(-\frac{\eta}{6\sqrt{n}}\right) + \ln\left((x^0)^t Rs^0\right) \leq \ln(\varepsilon),$$
$$k \geq K = \left[\frac{6\sqrt{n}}{\eta}\ln\left(\frac{(x^0)^t Rs^0}{\varepsilon}\right)\right]. \quad \square$$

5. NUMERICALS RESULTS

Recall that the examples treated are of the following form

$$(P) \qquad \begin{cases} \min f(x) \\ Ax = b \\ x \geq 0, \end{cases}$$

where f is a nonlinear, convex and differentiable function.

Example with fixed size

The below examples are taken from literature [7].

Example	Total number of iterations	Optimal value
Example 1	48	-6.534002
Example 2	50	-3.051993
Example 3	44	2.184359
Example 4	139	0.800000

Remark 5.1. Total number of iterations concerns the number of iterations of strictly feasible primal solution, strictly feasible dual solution and primal-dual optimal solution.

Example with variable size (Erikson's problem 1980). We consider the following convex program

$$\begin{cases} \min \left\{ f(x) = \sum\limits_{i=1}^{n} x_i \ln\left(\frac{x_i}{a_i}\right) \right\} & \\ x_i + x_{i+m} = b_i & i = 1, \dots, m, \ n = 2m \\ x \geq 0, & \end{cases}$$

where $a_i \in R_+^*$ and $b_i \in R$ are fixed. We have tested this example for different value of n, a_i and b_i.

Case 1 ($a_i = 1,\ b_i = 6$)

Size (n)	Total number of iterations	Optimal value
10	32	32.958355
20	31	65.916847
30	31	98.874969
48	31	158.200394

Case 2 ($a_i = 2,\ b_i = 4$)

Size (n)	Total number of iterations	Optimal value
10	28	0.000008
20	25	0.000003
30	23	-0.000009
48	23	0.000001

Case 3 ($a_i = 10,\ b_i = 1$)

Size (n)	Total number of iterations	Optimal value
10	32	-14.978663
20	31	-29.957319
30	31	-44.935986
48	32	-71.897545

6. CONCLUSION

In this paper, we have proposed a primal-dual interior point method for a class of nonlinear convex programming. These methods are efficient for solving large size problems. Unfortunately, its inconvenient is the initialization concerned by the determination of an initial strictly feasible point nearby the central path. Theoretically, we suppose known this point, but, numerically, to obtain this point we propose to use Karmarkar's algorithm.

The introduction of weights has entrained a flexibility in the choice of the initial solution and thus, in the choice of the path to be followed. Furthermore, the performances of the algorithm are preserved specially the arithmetical complexity. Numerically, we have presented several examples to appreciate the theoretical results. In the future, our efforts will be concentrated on:

– The choise of the weight which has an influence on the global complexity of our algorithm.

– How to choose an appropriate step length in the Newton iteration to acheive good practical performance.

– The introduction of new search directions to improve the complexity of the algorithm.

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