PRODUCTS AND SUMS OF BOUNDED AND UNBOUNDED NORMAL OPERATORS: FUGLEDE-PUTNAM VERSUS EMBRY

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In this survey, we go through some important results about products and sums of normal (bounded and unbounded) operators that are based upon the Fuglede-Putnam theorem. We also examine the same results using a result by M.R. Embry in 1970. It turns out that both Fuglede-Putnam and Embry theorems are powerful enough to prove similar results with slightly different hypotheses. We will also be discussing throughout this paper what the pros and cons of each theorem are.

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1. INTRODUCTION

We start by recalling some notions, definitions and basic results about linear bounded and unbounded operators. All Hilbert spaces are over $\mathbb{C}$ and are separable. All operators considered are linear. If an operator is bounded, then it will be assumed to be so on the whole Hilbert space H. If it is unbounded, then it will be assumed to have a dense domain and in such case, we say that it is densely defined. The notions of (bounded) self-adjoint, normal and unitary operators are defined in their usual fashion. If A is bounded, then the numerical range of A, denoted by $W(A)$, is defined as

$$W(A) = \{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}.$$

If A is an unbounded operator with domain $D(A)$, then A is said to be closed if its graph, i.e., the set

$$G(A) = \{(x, Ax) :\ x \in D(A)\}$$

is closed in $H \times H$.

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B is called an extension of A if $G(A) \subset G(B)$, i.e., if

$$D(A) \subset D(B) \text{ and } \forall x \in D(A): \ Ax = Bx.$$

The product AB of two unbounded operators is defined by

$$(AB)x = A(Bx) \text{ on } D(AB) = \{x \in D(B): \ Bx \in D(A)\}.$$

A bounded B and an unbounded A are said to commute if $BA \subset AB$. We say that A is symmetric if $A \subset A^*$ (for the definition of A^* see [6] or [38]). The adjoint A^* is always closed. The reader should be aware that $B^*A^* \subset (AB)^*$ with equality if A is bounded.

We call A self-adjoint if $A = A^*$ (hence a self-adjoint operator is automatically closed). A is said to be normal, if it is closed and $AA^* = A^*A$. It is known that if A is closed, then AA^* is always self-adjoint. It is also known that self-adjoint operators are maximally symmetric, that is if B is symmetric and A is self-adjoint, then $B \supset A \Rightarrow B = A$ (we shall refer to this result as the MS-property). Thus, e.g., if A is closed and $AA^* \subset A^*A$, then A must be normal (the MS-property and the previous observation have a tremendous role in proving the main results in [25, 27, 23, 28]).

As introduced by Devinatz-Nussbaum in [8], we say that the unbounded operators T, N and M have the property P if they are normal and if $T = NM = MN$. For other notions and results about bounded and unbounded operators, the reader may consult [6, 7, 12, 14, 16, 34, 38, 42].

We also recall the following result:

Theorem 1 ([25]). *Let A be an unbounded normal operator with domain $D(A)$. Let B be a bounded normal operator. Assume further that $B^*A \subset AB^*$. Then $A + B$ is normal on $D(A)$.*

The celebrated Fuglede-Putnam ([10] and [32]) in its classical form is as follows:

Theorem 2. *If A is a bounded operator and if M and N are normal operators, then*

$$AN \subset MA \Rightarrow AN^* \subset M^*A.$$

Fuglede [10] proved the foregoing theorem in 1950 and in the case $N = M$. One year later, Putnam [32] proved it as it stands. Then Rosenblum [37] in 1958 gave an elegant proof of the theorem in two steps (one for bounded operators and the other for unbounded ones). His proof is usually the one considered by many authors in many textbooks. Another proof of Theorem 2 in the case that M and N are bounded may be found in [33]. For more on the Fuglede-Putnam theorem, see [3] and [4].

There have been several attempts to prove the Fuglede-Putnam theorem for non-normal (bounded) operators and they are known in the literature (see the references in [18]). Paliogiannis [30] gave a different proof of the Fuglede theorem using bounding sequences. In [17], another version may be found. For an unbounded non-normal version of the Fuglede-Putnam theorem, see [40]. Lastly, and in [18], an attempt was even made to prove a 4-operator-version (with all operators bounded) of the Fuglede-Putnam theorem.

The Fuglede-Putnam theorem is a very powerful tool when dealing with products (and even sums) involving normal operators, and also for equations involving normal operators. As basic applications for instance, two similar normal operators are automatically unitary equivalent. Also, the product (and the sum) of two commuting normal remains normal. For more "bounded and unbounded" applications, see [1, 5, 11, 13, 17, 19, 21, 25, 27, 28, 36, 41].

Very recently, the following all-unbounded-operator-version was proved:

Theorem 3 ([26]). *Let A be a closed operator with domain $D(A)$. Let M and N be two unbounded normal operators with domains $D(N)$ and $D(M)$ respectively. If $D(N) \subset D(AN) \subset D(A)$, then*

$$AN \subset MA \Rightarrow AN^* \subset M^*A.$$

Embry [9] proved the following interesting theorem:

Theorem 4. *Let N and M be two bounded normal operators that commute. If A is bounded such that $0 \notin W(A)$, then*

$$AN = MA \Rightarrow N = M.$$

The following is an interesting and practical consequence:

Corollary 1. *Let A be a bounded operator such that $0 \notin W(A)$. If H is a bounded normal operator such that $AH = H^*A$, then H is self-adjoint.*

The proof of Theorem 4 was based on the spectral theorem. Embry drew interesting consequences of her theorem. The unbounded version of this theorem can be found in [20] and is recalled here for the reader's convenience:

Theorem 5. *Assume N and M are unbounded operators having the property such that $NM = MN$ is normal. Also assume that $D(N) \subset D(M)$. Assume further that A is a bounded operator for which $0 \notin W(A)$ and such that $AN \subset MA$. Then $N = M$.*

As an interesting corollary (cf. [29]) we have:

Corollary 2. *Assume that A is a bounded operator such that $0 \notin W(A)$. If H is an unbounded normal operator such that $AH \subset H^*A$, then H is self-adjoint.*

Embry's paper has been cited several times but we feel that it did not have the publicity it deserved. We have observed that although the Fuglede-Putnam and Embry's theorems look somehow different, it is amazing how they can actually solve the same problems with only minor differences of hypotheses, i.e., they are powerful enough to produce the same conclusions with slightly different assumptions. This is the main purpose of this survey. So what we will be doing next is to gather as many results as possible that exploit the Fuglede-Putnam theorem and we will adapt Embry's theorem to each one of them and see how far we can go with each question.

2. FUGLEDE-PUTNAM *Vs.* EMBRY

2.1. Normal products of self-adjoint operators

We start by the following theorems (the whole question and its answers have appeared in [1, 17, 19, 36], see also [20]) (for normal products of normal operators see e.g. [11, 13, 15, 27, 28, 31]).

THEOREM 6. *If A and B are self-adjoint bounded operators, one of them is positive, then the normality of AB implies its self-adjointness.*

Proof. Take A to be positive. We have by the Fuglede-Putnam theorem

$$A(BA) = (AB)A \Rightarrow A(BA)^* = (AB)^*A \Rightarrow A^2B = BA^2$$

and hence B commutes with the square root of A^2, i.e., with A. □

Embry's theorem can be applied to the following:

THEOREM 7. *If A and B are self-adjoint bounded operators, one of them is strictly positive, then the normality of AB implies its self-adjointness.*

Proof. Assume A is strictly positive, then $0 \notin W(A)$. Now writing

$$A(BA) = (AB)A = (BA)^*A.$$

By Embry, we have $BA = (BA)^*$. □

The same idea works for unbounded operators. We have

THEOREM 8. *If A and B are self-adjoint operators (one of them is unbounded), one of them is positive, then the normality of AB implies its self-adjointness.*

Proof. Just write

$$A(BA) = (AB)A$$

and the rest follows as in the bounded case. □

THEOREM 9. *If A and B are self-adjoint operators (only B is bounded), where B is also strictly positive, then the normality of AB implies its self-adjointness.*

Proof. Just write

$$B(AB) = (BA)B$$

and the rest follows as in the bounded case. □

If both A and B are unbounded, then by another version of Fuglede-Putnam established in [17] the following was proved:

THEOREM 10. *If A and B are two unbounded self-adjoint operator (B is positive), then the normality of AB implies its self-adjointness.*

In [19], it was shown by a counterexample that

THEOREM 11. *If A and B are two unbounded self-adjoint operator (A is positive), then the normality of AB does not necessarily imply its self-adjointness.*

Since no all-unbounded-operator version of Embry's theorem exists under any natural generalization (see [20]), we must say that we cannot discuss the case of two unbounded operators using Embry (this is the first weak point of it).

2.2. **Commutativity up to a factor**

Broadly speaking, the commutativity up to a factor is something like $AB = \lambda BA$ where λ is complex. The question some authors have been interested in, is what is the allowed set of the λ if A and B are normal or self-adjoint. See [5] and [41]. For more results and the unbounded case see [21]. For the same problems on Banach algebras see [39].

Yang-Du [41] improved some results in [5] and using the Fuglede-Putnam theorem they obtained:

THEOREM 12. *Let A, B be bounded operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}^*$. Then*

(1) *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*

(2) *if either A or B is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$; and*

(3) *if A and B are both normal, then $|\lambda| = 1$.*

Proof. The proof makes use of the Fuglede-Putnam theorem on several occasions. □

By using Embry's theorem and its generalization to the unbounded case, the following results were proved in [21]:

THEOREM 13. *Assume that A and B are two bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. If A or B is normal and the other does not have 0 in its numerical range, then $\lambda = 1$.*

Proof. Since B is normal, λB is normal and it obviously commutes with B. As $0 \notin W(A)$, then Embry gives $B = \lambda B$ and hence $\lambda = 1$.

The proof is very similar if one assumes that A is normal and that $0 \notin W(B)$. □

COROLLARY 3. *Let A and B be two bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. If A or B is normal and the other is strictly positive, then $\lambda = 1$.*

THEOREM 14 ([21]). *Let A be an unbounded operator and let B be a bounded one. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then*

(1) *λ is real if A is self-adjoint.*

(2) *$\lambda = 1$ if $0 \notin W(B)$ (the numerical range of B) and if A is normal; hence $\lambda = 1$ if B is strictly positive and A is normal.*

(3) *$\lambda \in \{-1, 1\}$ if A is normal and B is self-adjoint.*

Proof. The proof uses both the Fuglede-Putnam and the Embry theorems.

(1) Since $BA \subset \lambda AB$ and since A is self-adjoint (and hence A and λA are normal), the Fuglede-Putnam theorem yields $BA \subset \overline{\lambda} AB$. Now for $f \in D(A) = D(BA) \subset D(\lambda AB) = D(\overline{\lambda} AB)$, one has

$$\lambda ABf = \overline{\lambda} ABf.$$

Hence λ is real as $AB \neq 0$.

(2) Let us prove the first part of the assertion. Since A is normal, so is λA. Besides $\lambda AA = A\lambda A = \lambda A^2$. Since $0 \notin W(B)$, Theorem 5 yields $\lambda = 1$.

Now we prove the second assertion. We note that B cannot have 0 in its numerical range as B is strictly positive. Since A is self-adjoint, λA is normal and hence Theorem 5 gives $A = \lambda A$ which, in its turn, gives $\lambda = 1$.

(3) One has

$$BA \subset \lambda AB \Rightarrow B^2 A \subset \lambda BAB \subset \lambda^2 AB^2.$$

Since B is self-adjoint, B^2 is positive and by (2) of this theorem we obtain that $\lambda^2 = 1$. Thus $\lambda = 1$ or $\lambda = -1$. □

2.3. **Products related to sums of self-adjoint operators**

Let A and B be two operators (bounded or not) on a Hilbert space. A priori, there seems to be no reason why the normality of AB would imply that of $A + B$. One can construct many counterexamples (for instance 2 by 2 matrices).

But, Rehder proved using the Fuglede-Putnam theorem in [35] (which was a generalization of a work by Beck-Putnam [2], cf. [22]) the following result:

PROPOSITION 1. *Let $T = A + iB$ with A and B two bounded self-adjoint operators. If A or B is positive and if AB is normal, then so is $A + iB$.*

We may also use Embry's theorem to prove a very similar result, that is:

PROPOSITION 2. *Let $T = A + iB$ with A and B two bounded self-adjoint operators. If either A or B is strictly positive and if AB is normal, then $T = A + iB$ is also normal.*

Proof. If A is strictly positive, then $0 \notin W(A)$. Since AB is normal and A and B are self-adjoint, BA is normal. Besides since $A(BA) = (AB)A$, Embry' theorem yields $AB = BA$. Thus $T = A + iB$ is normal. □

In the same spirit, if B is self-adjoint, then iB will be normal. Hence it is natural to give the following result.

THEOREM 15. *Let A and B be two bounded normal operators. If AB and AB^* are normal such that A is positive, then $A + B$ is normal.*

Proof. Just write

$$BAB = (AB^*)^*B$$

and apply Fuglede-Putnam. □

But this time Embry's theorem does not help much and one has to impose stronger hypotheses to use Embry.

A very similar result is obtained if B is unbounded. We have

THEOREM 16. *Let A be a positive bounded operator. Let B be an unbounded self-adjoint operator. If AB is normal, then $A+iB$ is normal on $D(B)$.*

Proof. Since AB is normal, the other hypotheses combined with a result in [17] implies that AB must be self-adjoint, i.e., A and B commute. Theorem 1 does the remaining job. □

Quite similarly, we have

THEOREM 17. *Let A be a strictly positive bounded operator. Let B be an unbounded self-adjoint operator. If AB is normal, then $A + iB$ is normal on $D(B)$.*

Proof. The same routine, but apply the corollary to Theorem 5. □

The case of two unbounded operators does not hold in general.

Example. Let A be an unbounded self-adjoint operator. Set $B = -A$. Then B is also self-adjoint with domain $D(B) = D(A)$. Hence $AB = -A^2$ is self-adjoint on $D(A^2)$ while

$$A + B = A - A \subset 0$$

and hence $A + B$ is not closed and thus it cannot even be normal.

Another example which shows the failure of the generalization if the product is self-adjoint and the sum is required to be self-adjoint too, is the following:

Example. Take A to be any densely defined closed operator. Then it is known that A^*A is self-adjoint (see [38]) while $A + A^*$ is not self-adjoint (see [24]). Observe that for closed A, $A + A^*$ is only symmetric.

2.4. **Implicit division of linear operators**

The problem considered here is the following: given two linear operators (bounded or not) A and B such that BA satisfies some property "P" such that A (or B) also satisfies it, then when does B (or A) satisfy the property "P"? of course one will have to assume that B satisfies some property "Q" which is either independent of "P" or weaker than it, e.g. if BA and A are self-adjoint, then when is B self-adjoint? Similarly, if both BA and A are nonnegative, then when is B nonnegative? ... etc.

We refer the interested reader to [23] where an answer to this type of questions may be found. Yet again, the theorems of Fuglede-Putnam and Embry are the tools by excellence for the proofs.

3. **CONCLUSION**

In the end, we must admit that the Fuglede-Putnam theorem excels a lot more than Embry does and in many situations. For example it has allowed us to prove that the sum of two commuting normal operators A and B (only of them is bounded) stays normal (see [25]). The converse is not always true even for bounded operators. This was observed by Patel-Ramanujan [31] who gave a counterexample but we can give many counterexamples using 2 by 2 matrices. For the converse to be true, they assumed further that AB^* and B^*A had to be self-adjoint. We digress a little bit to say that the same hypothesis works for unbounded operators (and keeping B bounded, say) and this constitutes the converse of Theorem 1 which was not done in [25]. Indeed, we have

Theorem 18. *If A and B are two normal operators (only B is bounded) and if AB^* and B^*A are self-adjoint, then the normality of $A+B$ implies that A and B commute.*

Proof. Since A, B and $A+B$ are normal, we have

$$A^*B + B^*A = BA^* + AB^*.$$

By the self-adjointness of AB^* and B^*A, we have

$$BA^* \subset (AB^*)^* = AB^* \quad \text{and} \quad (B^*A)^* = B^*A = A^*B.$$

Hence

$$B^*A \subset AB^*$$

and Fuglede yields the commutativity of A^* and B^* or that of A and B. □

The product of a unitary and an unbounded normal operator which commute (regardless of the order of the operator in the product) is always normal, i.e., we have

Theorem 19 ([27]). (1) *Assume that B is a unitary operator. Let A be an unbounded normal operator. If B and A commute (i.e., $BA \subset AB$), then BA is normal.*

(2) *Assume that A is a unitary operator. Let B be an unbounded normal operator. If A and B commute (i.e., $AB \subset BA$), then BA is normal.*

The proof relies again on the theorem of Fuglede-Putnam. Some interesting generalizations of Kaplansky's results (see [13]) have been also obtained in [28] thanks to the theorem of Fuglede-Putnam and to the non-normal version of it by Stochel (see [40]).

For the previous results, and for others in the literature, we just cannot see how we can apply Embry's theorem to these matters. Nonetheless, we believe that Embry should have been given much more credit for her work. We may also ask whether we can prove some different versions of Embry's theorem for non-normal operators (this will certainly be achieved if we can prove Embry's theorem by algebraic techniques only, i.e., without using the spectral theorem so that we may apply this proof to non-normal operators). If this is realizable, then some important consequences will be added to Embry's theorem assets.

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