SOME SUBCLASSES OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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By using a generalized differential operator, defined by means of the Hadamard product, we introduce some new subclasses of multivalent analytic functions in the open unit disk and investigate their inclusion relationships. Some integral preserving properties of these subclasses are also discussed.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\}, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. For functions f given by (1) and g given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n},$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{p+n}.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ with

$$w(0) = 0, \quad |w(z)| < 1, \quad z \in \mathbb{U},$$

such that
$$f(z) = g(w(z)), \quad z \in \mathbb{U}.$$
We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$, $z \in \mathbb{U} \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex univalent in $\mathbb{U}$ and for which $\Re\{h(z)\} > 0$, $z \in \mathbb{U}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ ($q, s \in \mathbb{N} \cup \{0\}$, $q \leq s+1$) be complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by
$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D},$$
where $(x)_n$ denotes the Pochhammer symbol defined by
$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \text{ for } n \in \mathbb{N} \text{ and } (x)_0 = 1.$$

Corresponding to a function $\mathcal{G}^p_{q,s}(\widetilde{\alpha_1}; \widetilde{\beta_1}; z)$ defined by
$$\mathcal{G}^p_{q,s}(\widetilde{\alpha_1}, \widetilde{\beta_1}; z) := z^p\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{2}$$
where $\widetilde{\alpha_1} = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ and $\widetilde{\beta_1} = (\beta_1, \beta_2, \ldots, \beta_s)$, C. Selvaraj and K.R. Karthikeyan [11] recently defined the following generalized differential operator $D^{p,m}_{\lambda}(\alpha_1, \beta_1)f : \mathcal{A}_p \to \mathcal{A}_p$ by
$$D^{p,0}_{\lambda}(\alpha_1, \beta_1)f(z) = f(z) * \mathcal{G}^p_{q,s}(\widetilde{\alpha_1}, \widetilde{\beta_1}; z),$$
$$D^{p,1}_{\lambda}(\alpha_1, \beta_1)f(z) = (1-\lambda)(f(z) * \mathcal{G}^p_{q,s}(\widetilde{\alpha_1}, \widetilde{\beta_1}; z)) + \frac{\lambda}{p} z(f(z) * \mathcal{G}^p_{q,s}(\widetilde{\alpha_1}, \widetilde{\beta_1}; z))',$$
$$D^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z) = D^{p,1}_{\lambda}(D^{p,m-1}_{\lambda}(\alpha_1, \beta_1)f(z)), \tag{3}$$
where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$.

If $f(z) \in \mathcal{A}_p$, then we have
$$D^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p + \lambda n}{p}\right)^m \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} a_{p+n} \frac{z^{p+n}}{n!}. \tag{4}$$
It can be seen that, by specializing the parameters the operator $D^{p,m}_{\lambda}(\alpha_1, \beta_1)$ reduces to many known and new integral and differential operators. In particular, when $m = 0$ and $p = 1$ the operator $D^{p,m}_{\lambda}(\alpha_1, \beta_1)$ reduces to the well known Dziok-Srivastava operator [3] and for $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, it reduces to the operator introduced by F. Al-Oboudi [1]. Further we remark that, when $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$ and $\lambda = 1$ the operator $D^{p,m}_{\lambda}(\alpha_1, \beta_1)$ reduces to the operator introduced by G.S. Sălăgean [10].

It can be easily verified from (4) that
$$\lambda z(D^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z))' = pD^{p,m+1}_{\lambda}(\alpha_1, \beta_1)f(z) - p(1-\lambda)D^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z) \tag{5}$$

and

(6) $z(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))' = \alpha_1 D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z) - (\alpha_1-p)D_\lambda^{p,m}(\alpha_1,\beta_1)f(z).$

Throughout this paper, we assume that $p, k \in \mathbb{N}$, $\varepsilon_k = \exp(\frac{2\pi i}{k})$ and

$$f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-jp}(D_\lambda^{p,m}(\alpha_1,\beta_1)f(\varepsilon_k^j z)) = z^p + \cdots, \quad f \in \mathcal{A}_p. \tag{7}$$

Clearly, for $k = 1$, we have

$$f_{\lambda,1}^{p,m}(\alpha_1,\beta_1;z) = D_\lambda^{p,m}(\alpha_1,\beta_1)f(z).$$

We now introduce the following subclasses of analytic functions.

Definition 1.1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$, if it satisfies

$$\frac{z(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))'}{p f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)} \prec h(z), \quad z \in \mathbb{U}, \tag{8}$$

where $h \in \mathcal{P}$ and $f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z) \neq 0$, $z \in \mathbb{U}$.

Remark 1.1. If we let $m = 0$, then $\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$ reduces to the function class $\mathcal{S}_{p,k}^{q,s}(\alpha_1;h)$ introduced and investigated by Z.-G. Wang, Y.-P. Jiang and H.M. Srivastava [14].

Remark 1.2. If we let $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$, then $\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$ reduces to the function class $T_{p,k}(a,c;h)$ introduced and investigated by N.-E. Xu and D.-G. Yang [15].

Remark 1.3. If we let $m = 0$, $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, then $\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$ reduces to the known subclass $\mathcal{S}_s^{(k)}(\phi)$ of close-to-convex functions with respect to k-symmetric points, introduced and studied recently by Z.-G. Wang, C.-Y. Gao and S.-M. Yuan [13].

Remark 1.4. Let $h(z) = \frac{1+z}{1-z}$ and let $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$. Then $\mathcal{S}_{\lambda,2}^{1,0}(\alpha_1,\beta_1;h) = S_s^*$. The class S_s^* of functions starlike with respect to symmetric points has been studied by several authors (see [8], [9], [16]).

Definition 1.2. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$, if it satisfies

$$\frac{z(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))'}{p\, g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)} \prec h(z), \quad z \in \mathbb{U}, \tag{9}$$

for some $g(z) \in \mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$, where $h \in \mathcal{P}$ and $g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z) \neq 0$ is defined as in (7).

Remark 1.5. If we let $m = 0$, $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, then $\mathcal{K}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$ reduces to the known subclass $\mathcal{C}^{(k)}_s(\phi)$ of quasi-convex functions with respect to k-symmetric points, introduced and studied recently by Z.-G. Wang, C.-Y. Gao and S.-M. Yuan [13].

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{QC}^{p,m}_{\lambda,k}(\delta; \alpha_1, \beta_1; h)$, if it satisfies

$$(10)\quad (1-\delta)\frac{z(D^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z))'}{p\, g^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)} + \delta\frac{z(D^{p,m}_{\lambda}(\alpha_1+1, \beta_1)f(z))'}{p\, g^{p,m}_{\lambda,k}(\alpha_1+1, \beta_1; z)} \prec h(z), \quad z \in \mathbb{U},$$

for some δ, $\delta \geq 0$ and $g(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$, where $h \in \mathcal{P}$ and $g^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z) \neq 0$.

Remark 1.6. If we let $m = 0$, $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, then $\mathcal{QC}^{p,m}_{\lambda,k}(\delta; \alpha_1, \beta_1; h)$ reduces to the known subclass $\mathcal{QC}^{(k)}_s(\lambda, \phi)$ of δ-quasi-convex functions with respect to k-symmetric points, introduced and studied recently by S.-M. Yuan and Z.-M. Liu [17]. Further if we set,

$$m = 0,\ p = k = 1,\ q = 2,\ s = 1,\ \alpha_1 = \beta_1,\ \alpha_2 = 1,\ \delta = 1 \text{ and } h(z) = \frac{1+z}{1-z},$$

then $\mathcal{QC}^{p,m}_{\lambda,k}(\delta; \alpha_1, \beta_1; h)$ reduces to the familiar class of quasi-convex functions which was introduced and studied earlier by K.I. Noor [7].

We need the following lemmas to derive our results.

Lemma 1.7 ([5]). *Let β and γ be complex numbers and let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $\Re\{\beta h(z) + \gamma\} > 0$, $z \in \mathbb{U}$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then the subordination*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z), \quad z \in \mathbb{U}$$

implies that

$$q(z) \prec h(z), \quad z \in \mathbb{U}.$$

Lemma 1.8 ([6]). *Let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ and let $w(z)$ be analytic in $\mathbb{U}$ with $\Re\{w(z)\} \geq 0$, $z \in \mathbb{U}$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then the subordination*

$$q(z) + w(z)zq'(z) \prec h(z), \quad z \in \mathbb{U}$$

implies that

$$q(z) \prec h(z), \quad z \in \mathbb{U}.$$

Lemma 1.9. *Let $f(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$. Then*

$$(11)\qquad \frac{z(f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z))'}{pf^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)} \prec h(z), \quad z \in \mathbb{U}.$$

Proof. For $f(z) \in \mathcal{A}_p$, we have from (7) that

$$\begin{aligned} f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; \varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{-np} D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^{n+j} z) \\ &= \frac{\varepsilon_k^{jp}}{k}\sum_{n=0}^{k-1} \varepsilon_k^{-(n+j)p} D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^{n+j} z) \\ &= \varepsilon_k^{jp} f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z), \quad j \in \{0, 1, \dots, k-1\} \end{aligned}$$

and

$$(f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z))' = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(1-p)} (D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^j z))'.$$

Hence

$$\begin{aligned} \frac{z(f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z))'}{p f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)} &= \frac{1}{k}\sum_{j=0}^{k-1} \frac{\varepsilon_k^{j(1-p)} z (D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^j z))'}{p f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)} \\ &= \frac{1}{k}\sum_{j=0}^{k-1} \frac{\varepsilon_k^{j} z (D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^j z))'}{p f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)}, \quad z \in \mathbb{U}. \end{aligned} \tag{12}$$

Since $f(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$, we have

$$\frac{\varepsilon_k^{j} z (D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(\varepsilon_k^j z))'}{p f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; \varepsilon_k^j z)} \prec h(z), \quad \text{for } j \in \{0, 1, \dots, k-1\}. \tag{13}$$

Noting that $h(z)$ is convex univalent in $\mathbb{U}$, from (12) and (13) we conclude that (11) holds true. □

2. **A SET OF INCLUSION RELATIONSHIPS**

THEOREM 2.1. *Let* $h(z) \in \mathcal{P}$ *with*

$$\Re\{h(z)\} > \max\left\{1 - \frac{1}{\lambda}, 1 - \frac{\alpha_1}{p}\right\}, \quad z \in \mathbb{U},\ \lambda > 1. \tag{14}$$

Then

$$\mathcal{S}^{p,m+1}_{\lambda,k}(\alpha_1 + 1, \beta_1; h) \subset \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1 + 1, \beta_1; h) \subset \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h).$$

Proof. From (5) and (7), we have

(15)
$$(1-\lambda)f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)+\frac{\lambda z}{p}(f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z))' =$$
$$=\frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(\varepsilon_k^j z))=f_{\lambda,k}^{p,m+1}(\alpha_1,\beta_1;z).$$

Let $f(z)\in\mathcal{S}_{\lambda,k}^{p,m+1}(\alpha_1+1,\beta_1;h)$ and

(16)
$$w(z)=\frac{z(f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z))'}{pf_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}.$$

Then $w(z)$ is analytic in $\mathbb{U}$, with $w(0)=1$ and from (15) (with α_1 replaced by α_1+1) and (16) we have

(17)
$$1-\lambda+\lambda w(z)=\frac{f_{\lambda,k}^{p,m+1}(\alpha_1+1,\beta_1;z)}{f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}.$$

Differentiating (17) with respect to z and using (16), we get

(18)
$$w(z)+\frac{zw'(z)}{\frac{p}{\lambda}(1-\lambda)+pw(z)}=\frac{z(f_{\lambda,k}^{p,m+1}(\alpha_1+1,\beta_1;z))'}{p\,f_{\lambda,k}^{p,m+1}(\alpha_1+1,\beta_1;z)}.$$

From (18) and Lemma 1.9 (with α_1 replaced by α_1+1) we note that

(19)
$$w(z)+\frac{zw'(z)}{\frac{p}{\lambda}(1-\lambda)+pw(z)}\prec h(z),\quad z\in\mathbb{U}.$$

In view of (14) and (19), we deduce from Lemma 1.7 that

(20)
$$w(z)\prec h(z),\quad z\in\mathbb{U}.$$

Suppose that
$$q(z)=\frac{z(D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z))'}{pf_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}.$$

Then $q(z)$ is analytic in $\mathbb{U}$, with $q(0)=1$ and we obtain from (5) (with α_1 replaced by α_1+1) that

(21)
$$f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)q(z)=\frac{1}{\lambda}D_\lambda^{m+1}(\alpha_1+1,\beta_1)f(z)+\left(1-\frac{1}{\lambda}\right)D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z).$$

Differentiating both sides of (21) with respect to z, we get

(22)
$$zq'(z)+\left(p\Big(\frac{1}{\lambda}-1\Big)+\frac{z(f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z))'}{f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}\right)q(z)=\frac{z(D_\lambda^{p,m+1}(\alpha_1+1,\beta_1)f(z))'}{\lambda f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}.$$

Since $f(z) \in \mathcal{S}^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;h)$, we find from (15) (with α_1 replaced by α_1+1), (16) and (22) that

$$(23)\quad q(z)+\frac{zq'(z)}{\frac{p}{\lambda}(1-\lambda)+pw(z)}=\frac{z(D^{p,m+1}_{\lambda}(\alpha_1+1,\beta_1)f(z))'}{pf^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;z)}\prec h(z),\quad z\in\mathbb{U}.$$

From (14) and (20) we observe that

$$\Re\left\{\frac{p}{\lambda}(1-\lambda)+pw(z)\right\}>0.$$

Therefore, from (23) and Lemma 1.8 we conclude that

$$q(z)\prec h(z),\quad z\in\mathbb{U}$$

which shows that $f(z)\in\mathcal{S}^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;h)$.

To prove the second inclusion relationship, we now let $f(z)\in\mathcal{S}^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;h)$ and set

$$s(z)=\frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z))'}{pf^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}.$$

Then, by using a similar argument as detailed above, it follows from (6) and Lemma 1.8 that

$$s(z)\prec h(z),\quad z\in\mathbb{U}.$$

This implies that $f(z)\in\mathcal{S}^{p,m}_{\lambda,k}(\alpha_1,\beta_1;h)$. □

THEOREM 2.2. *Let $h(z)\in\mathcal{P}$ with*

$$(24)\qquad \Re\{h(z)\}>\max\left\{1-\frac{1}{\lambda},\,1-\frac{\alpha_1}{p}\right\},\quad z\in\mathbb{U},\ \lambda>1.$$

Then

$$(25)\qquad \mathcal{K}^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;h)\subset\mathcal{K}^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;h)\subset\mathcal{K}^{p,m}_{\lambda,k}(\alpha_1,\beta_1;h).$$

Proof. We only prove the first inclusion relationship in (25), since the other inclusion relationship can be justified using similar arguments.

Let $f(z)\in\mathcal{K}^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;h)$. Then there exists a function $g(z)\in\mathcal{S}^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;h)$ such that

$$(26)\qquad \frac{z(D^{p,m+1}_{\lambda}(\alpha_1+1,\beta_1))'}{p\,g^{p,m+1}_{\lambda,k}(\alpha_1+1,\beta_1;z)}\prec h(z),\quad z\in\mathbb{U}.$$

An application of Theorem 2.1 yields $g(z)\in\mathcal{S}^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;h)$ and Lemma 1.9 leads to

$$(27)\qquad \psi(z)=\frac{z(g^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;z))'}{p\,g^{p,m}_{\lambda,k}(\alpha_1+1,\beta_1;z)}\prec h(z),\quad z\in\mathbb{U}.$$

Let

$$\phi(z)=\frac{z(D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}.$$

By using (5) (with α_1 replaced by α_1+1), $q(z)$ can be written as follows

$$g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)\phi(z)=\frac{1}{\lambda}D_\lambda^{p,m+1}(\alpha_1+1,\beta_1)f(z)+ \\ +\left(1-\frac{1}{\lambda}\right)D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z). \tag{28}$$

Differentiating both sides of (28) with respect to z and using (15) (with f replaced by g and α_1 replaced by α_1+1), we get

$$\phi(z)+\frac{z\phi'(z)}{\frac{p}{\lambda}(1-\lambda)+p\psi(z)}=\frac{z(D_\lambda^{p,m+1}(\alpha_1+1,\beta_1 f(z))'}{p\,g_{\lambda,k}^{p,m+1}(\alpha_1+1,\beta_1;z)}. \tag{29}$$

Now, from (26) and (29) we find that

$$\phi(z)+\frac{z\phi'(z)}{\frac{p}{\lambda}(1-\lambda)+p\psi(z)}\prec h(z),\quad z\in\mathbb{U}. \tag{30}$$

Combining (24), (27) and (30), we deduce from Lemma 1.8 that

$$\phi(z)\prec h(z),\quad z\in\mathbb{U}$$

which shows that $f(z)\in\mathcal{K}_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;h)$ with respect to $g(z)\in\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;h))$. $\square$

THEOREM 2.3. *Let $h(z)\in\mathcal{P}$ with $\Re\{h(z)\}>\{1-\frac{\alpha_1}{p}\}$, $z\in\mathbb{U}$. Then*

$$\mathcal{QC}_{\lambda,k}^{p,m}(\delta_2;\alpha_1,\beta_1;h)\subset\mathcal{QC}_{\lambda,k}^{p,m}(\delta_1;\alpha_1,\beta_1;h),\quad 0\le\delta_1<\delta_2.$$

Proof. From (6) and (7), we have

$$(\alpha_1-p)f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)+z(f_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z))'= \\ =\alpha_1\frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}(D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(\varepsilon_k^j z))=\alpha_1 f_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z). \tag{31}$$

Let $f(z)\in\mathcal{QC}_{\lambda,k}^{p,m}(\delta_2;\alpha_1,\beta_1;h)$. Then there exists a function $g(z)$ in the class $\mathcal{S}_{\lambda,k}^{p,m}(\alpha_1,\beta_1;h)$ such that

$$(1-\delta_2)\frac{z(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}+\delta_2\frac{z(D_\lambda^{p,m}(\alpha_1+1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}\prec h(z),\ z\in\mathbb{U}. \tag{32}$$

Furthermore, it follows from Lemma 1.9 that

$$\nu(z)=\frac{z(g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}\prec h(z),\quad z\in\mathbb{U}. \tag{33}$$

We now set

$$\varphi(z)=\frac{z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}. \tag{34}$$

Then $\varphi(z)$ is analytic in $\mathbb{U}$, with $\varphi(0)=1$. By using (6), $q(z)$ can be written as

$$g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)\varphi(z)=\frac{\alpha_1}{p}D_{\lambda}^{p,m}(\alpha_1+1,\beta_1)f(z)- \\ -\left(\frac{1-\alpha_1}{p}\right)D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z). \tag{35}$$

Differentiating both sides of (35) with respect to z and using (31) (with f replaced by g), we find

$$\varphi(z)+\frac{z\varphi'(z)}{\alpha_1-p+p\psi(z)}=\frac{z(D_{\lambda}^{p,m}(\alpha_1+1,\beta_1 f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}. \tag{36}$$

Equivalently,

$$\varphi(z)+\frac{\delta_2 z\varphi'(z)}{\alpha_1-p+p\psi(z)}=(1-\delta_2)\frac{z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}+ \\ +\delta_2\frac{z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}\prec h(z). \tag{37}$$

Since $\nu(z)\prec h(z)$ and $\frac{1}{\delta_2}\Re\{p\,h(z)+\alpha_1-p\}>0$, $z\in\mathbb{U}$, it follows from (37) and Lemma 1.8 that $\varphi(z)\prec h(z)$, $z\in\mathbb{U}$.

Since $h(z)$ is convex univalent in $\mathbb{U}$ and $0\le\frac{\delta_1}{\delta_2}<1$, we deduce from (32) and (34) that

$$(1-\delta_1)\frac{z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}+\delta_1\frac{z(D_{\lambda}^{p,m}(\alpha_1+1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}= \\ =\frac{\delta_1}{\delta_2}\left((1-\delta_2)\frac{z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1,\beta_1;z)}+\delta_2\frac{z(D_{\lambda}^{p,m}(\alpha_1+1,\beta_1)f(z))'}{p\,g_{\lambda,k}^{p,m}(\alpha_1+1,\beta_1;z)}\right)+ \\ +\left(1-\frac{\delta_1}{\delta_2}\right)\varphi(z)\prec h(z).$$

Thus $f(z)\in\mathcal{QC}_{\lambda,k}^{p,m}(\delta_1;\alpha_1,\beta_1;h)$ which completes the proof of Theorem 2.3. □

3. INTEGRAL OPERATOR

In this section, we consider the generalized Bernardi–Libera–Livingston integral operator $F(z)$ defined by (cf. [2] and [4])

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\mathrm{d}t, \quad f \in \mathcal{A}_p,\ c > -p. \tag{38}$$

It is easy to verify that, $F(z) \in \mathcal{A}_p$, p-valent (cf. [12]) and

$$\begin{aligned} F(z) &= \left(z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} z^{p+n}\right) * f(z) \\ &= \left(z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n}\, a_n z^{p+n}\right), \quad f \in \mathcal{A}_p,\ c > -p. \end{aligned} \tag{39}$$

We first prove

Theorem 3.1. *Let $h(z) \in \mathcal{P}$ and*

$$\Re\{h(z)\} > \max\left\{0, -\frac{\Re(c)}{p}\right\}, \quad z \in \mathbb{U},$$

where c is a complex number such that $\Re(c) > -p$. If $f(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$, then the function $F(z)$ defined by (38) *is also in the class $\mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$.*

Proof. Let $f(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$. Then from (38), (39) and $\Re(c) > -p$, we note that $F(z) \in \mathcal{A}_p$, p-valent and

$$(c+p) D^{p,m}_{\lambda}(\alpha_1, \beta_1) f(z) = c D^{p,m}_{\lambda} F(z) + z(D^{p,m}_{\lambda}(\alpha_1, \beta_1) F(z))'. \tag{40}$$

Also, from the above, we have

$$\begin{aligned} &(c+p) f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z) = \\ &= \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-jp}\big(c D^{p,m}_{\lambda}(\alpha_1, \beta_1) F(\varepsilon_k^j z) + \varepsilon_k^j z (D^{p,m}_{\lambda}(\alpha_1, \beta_1) F(\varepsilon_k^j z))'\big) = \\ &= c F^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z) + z\big(F^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)\big)'. \end{aligned} \tag{41}$$

Let

$$\vartheta(z) = \frac{z(F^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z))'}{p\, F^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)}.$$

Then $\vartheta(z)$ is analytic in $\mathbb{U}$, with $\vartheta(0) = 1$ and from (41) we observe that

$$p\,\vartheta(z) + c = (c+p)\frac{f^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)}{F^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z)}. \tag{42}$$

Differentiating both sides of (42) with respect to z and using Lemma 1.9, we obtain

$$\vartheta(z) + \frac{z\,\vartheta'(z)}{p\,\vartheta(z)+c} = \frac{z(f^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z))'}{p\,f^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)} \prec h(z). \tag{43}$$

In view of (43), Lemma 1.7 leads to $\vartheta(z) \prec h(z)$.

If we let

$$\Phi(z) = \frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)F(z))'}{p\,F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}$$

then $\Phi(z)$ is analytic in $\mathbb{U}$, with $\Phi(0)=1$ and it follows from (40) that

$$F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)\Phi(z) = \frac{c+p}{p}\,D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z) - \frac{c}{p}\,D^{p,m}_{\lambda}(\alpha_1,\beta_1)F(z). \tag{44}$$

Differentiating both sides of (44), we get

$$z\Phi'(z) + \frac{z(F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z))'}{F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}\,\Phi(z) =$$
$$= (c+p)\frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z))'}{p\,F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)} - c\frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)F(z))'}{p\,F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}.$$

Equivalently,

$$z\Phi'(z) + \big(p\vartheta(z)+c\big)\Phi(z) = (c+p)\frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z))'}{p\,F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}. \tag{45}$$

Now, from (42) and (45) we deduce that

$$\begin{aligned}\Phi(z) + \frac{z\Phi'(z)}{p\vartheta(z)+c} &= \frac{c+p}{p\vartheta(z)+c}\;\frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z))'}{p\,F^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)}\\ &= \frac{z(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z))'}{p\,f^{p,m}_{\lambda,k}(\alpha_1,\beta_1;z)} \prec h(z).\end{aligned} \tag{46}$$

Combining, $\Re\{h(z)\} > \max\{0, -\frac{\Re(c)}{p}\}$ and $\vartheta(z) \prec h(z)$ we have $\Re\{p\,\vartheta(z) + c\} > 0$, $z \in \mathbb{U}$. Therefore, from (46) and Lemma 1.8 we find that $\Phi(z) \prec h(z)$, which shows that $F(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1,\beta_1;h)$. □

By applying similar method as in Theorem 3.1, we have

Theorem 3.2. *Let $h(z) \in \mathcal{P}$ and*

$$\Re\{h(z)\} > \max\left\{0, -\frac{\Re(c)}{p}\right\}, \quad z \in \mathbb{U},\ \Re(c) > -p.$$

If $f(z) \in \mathcal{K}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$ *with respect to* $g(z) \in \mathcal{S}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$*, then the function*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) \mathrm{d}t$$

belongs to the class $\mathcal{K}^{p,m}_{\lambda,k}(\alpha_1, \beta_1; h)$ *with respect to*

$$G(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} g(t) \mathrm{d}t,$$

provided that $G^{p,m}_{\lambda,k}(\alpha_1, \beta_1; z) \neq 0$*,* $z \in \mathbb{U}$*.*

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