MODULI IN MODERN MAPPING THEORY

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The paper is a short survey of our monograph with the same title, see [25]. The purpose of this book is to present modern developments and applications of moduli techniques in new classes of mappings in $\mathbb{R}^n$, $n \geq 2$, and in metric spaces.

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1. INTRODUCTION

The modulus method was initiated by Lars Ahlfors and Arne Beurling in [1]. Later on this method was extended and enhanced by several other authors. The techniques are geometric and have turned out to be an indispensable tool in the study of quasiconformal and quasiregular mappings. The book [25] is based on rather recent research papers and extends the modulus method beyond its classical applications presented in many monographs, see, e.g., [6], [24], [36], [37], [48]–[50]. It has also been employed in metric measure spaces, now called Loewner spaces, see, e.g., [13] and [15].

Mapping theory started in the 18th century. Beltrami, Carathéodory, Christoffel, Gauss, Hilbert, Liouville, Poincaré, Riemann, Schwarz, Stoilow and so on all left their marks in this theory. Conformal mappings and their applications to potential theory, mathematical physics, Riemann surfaces, and technology played a key role in this development. During the late 1920s and early 1930s, Grötzsch, Lavrentiev, and Morrey introduced a more general and less rigid class of mappings that were later named quasiconformal, were later defined in higher dimensions (Lavrentiev, Gehring, Väisälä), and were further extended to quasiregular mappings (Reshetnyak, Martio, Rickman, and Väisälä).

Recently, various generalizations of quasiconformal mappings have been studied intensively, see, e.g., [4], [7], [10]–[12] [17], [18], [21], [22], [25]–[31], [38]–[44], [45]–[47] and further references on the mappings with finite distortion in the monograph [19]. However, the moduli method has not been used very much

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to study mappings of finite distortion and related mappings. The reason is that extremal metrics are more difficult to find and the estimates for the modulus of a path family become more complicated than in the quasiconformal case. In spite of these drawbacks, the modulus method has certain advantages since it is naturally connected to the metric and geometric behavior of mappings.

In the monograph [25] the modulus method is applied to the generalizations of quasiconformal mappings. The main goal is to study the classes of mappings with distortion of moduli dominated by a given measurable function Q. Functions Q like BMO (bounded mean oscillation), FMO (finite mean oscillation), $L^1_{\rm loc}$, etc. are included and the principal tool is the modulus method. We follow the traditional research directions of the quasiconformal theory like differentiability, absolute continuity, local and boundary behavior, removability of singularities, normal families, convergence and many other and everywhere we demonstrate a strong efficiency of the modulus method.

Recall definitions and basic facts. Let Γ be a path family in $\mathbb{R}^n$, $n \geq 2$. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, $\rho \in \operatorname{adm}\Gamma$ for short, if

$$\int_\gamma \rho \, \mathrm{d}s \geq 1 \tag{1}$$

for each $\gamma \in \Gamma$. The (*conformal*) *modulus* of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \rho^n(x) \, \mathrm{d}m(x), \tag{2}$$

where $\mathrm{d}m(x)$ stands for the Lebesgue measure in $\mathbb{R}^n$.

By the classical geometric definition of Väisälä, see, e.g., 13.1 in [48], a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geq 2$, is K-*quasiconformal*, K-*qc mapping* for short, if

$$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma) \tag{3}$$

for every path family Γ in D. A homeomorphism $f : D \to D'$ is called *quasiconformal*, *qc* for short, if f is K-quasiconformal for some $K \in [1, \infty)$, i.e., if the distortion of the moduli of path families under the mapping f is bounded.

By Theorem 34.3 in [48], a homeomorphism $f : D \to D'$ is quasiconformal if and only if

$$M(f\Gamma) \leq KM(\Gamma) \tag{4}$$

for some $K \in [1, \infty)$ and for every path family Γ in D. In other words, it is sufficient to verify that

$$\sup \frac{M(f\Gamma)}{M(\Gamma)} < \infty, \tag{5}$$

where the supremum is taken over all path families Γ in D for which $M(\Gamma)$ and $M(f\Gamma)$ are not simultaneously 0 or ∞. Then we also have

$$\sup \frac{M(\Gamma)}{M(f\Gamma)} < \infty. \tag{6}$$

Gehring was the first to note that the suprema in (5) and (6) remain the same if we restrict ourselves to families of paths connecting the boundary components of rings in D; see [8] or Theorem 36.1 in [48]. Thus, the geometric definition of a K-quasiconformal mapping by Väisälä is equivalent to Gehring's ring definition.

Moreover, condition (6) has been shown to be equivalent to the statement that f is ACL (absolutely continuous on lines), a.e., differentiable, and

$$\operatorname{ess\,sup} \frac{\|f'(x)\|^n}{J(x,f)} < \infty, \tag{7}$$

where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix $f'(x)$ of the mapping f, i.e., $\max\{|f'(x)h| : h \in \mathbb{R}^n,\ |h| = 1\}$, and $J(x,f)$ its determinant at a point $x \in D$ [here the ratio is equal to 1 if $f'(x) = 0$]. Furthermore, it turns out that the suprema in (6) and (7) coincide; see Theorem 32.3 in [48]. The three given properties of f form the analytic definition for a quasiconformal mapping that is equivalent to the above geometric definition; see Theorem 34.6 in [48].

In the light of the interconnection between conditions (3) and (4), the following concept proposed by Olli Martio is a natural extension of the geometric definition of quasiconformality, see, e.g., [26]–[31]. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1,\infty]$ be a measurable function. We say that a homeomorphism $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x)\, \mathrm{d}m(x) \tag{8}$$

for every family Γ of paths in D and every admissible function ρ for Γ. This concept is related in a natural way to the theory of the so-called moduli with weights, see, e.g., [2, 3]. The quasiconformal mappings give the simplest examples of Q-homeomorphisms with inner dilatation $K_I(x,f)$ as $Q(x)$, see, e.g., Lemma 2.1 in [5] and [24], p. 221. Theorem 12 and Corollary 12 below give more advanced examples.

The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. The subject of Q-homeomorphisms is interesting in its own right and has applications to many classes of mappings that we also investigate. For example, the theory of Q-homeomorphisms can be applied to mappings in local Sobolev's classes and to the mappings with finite length distortion.

Except a general introduction, the book [25] contains detailed introductions in Moduli and Capacity in Metric Spaces and Moduli and Domains, Chapters 2 and 3, written by Olli Martio. Moreover, the book includes a survey on Mappings with Finite Mean Dilatations, Chapter 12, written by Anatoly Golberg. In the Appendix of our book one can find classical articles of Fuglede, Gehring, Hesse, Ziemer and others in the moduli theory. We think that it would be useful for everybody to have a both handbook and research monograph.

Chapter 4 is devoted to the basic theory of space Q-homeomorphisms f for $Q \in L^1_{\rm loc}$. Differentiability a.e., absolute continuity, estimates from below for distortion, extension to the boundary and other properties are considered.

Chapter 5 includes estimates of distortion, removability of isolated singularities, theorems on continuous and homeomorphic extension to boundaries, and other results on Q-homeomorphisms for Q in BMO, where BMO refers to functions with bounded mean oscillation introduced by John and Nirenberg.

Results on Q-homeomorphisms for Q in FMO (finite mean oscillation) and in more general classes are given in Chapter 6. Analogies of the Painleve theorem on removability of singularities of length zero and applications to mappings in the Sobolev class $W^{1,n}_{\rm loc}$ are presented.

Extensions of the quasiconformal theory to ring and lower Q-homeomorphisms and applications to mappings with finite length and area distortion are found in Chapters 7–10. Existence theorems of ring Q-homeomorphisms are given in Chapter 11. Some results on mappings quasiconformal in the mean related to moduli techniques are contained in Chapter 12.

Chapter 13 contains the theory of Q-homeomorphisms in general metric spaces with measures. In particular, here we study properties of weakly flat spaces which are a far-reaching generalization of the recently introduced Loewner spaces, including as special cases Riemannian manifolds and the well-known groups of Carnot and Heisenberg. On this basis, we create the theory of the boundary behavior and removal singularities for quasiconformal mappings and their generalizations, which can be applied to any of the mentioned classes of spaces. In particular, we prove a generalization and strengthening of the known Gehring–Martio theorem on homeomorphic extension to the boundary of quasiconformal mappings between quasiextremal distance domains in $\mathbb{R}^n$, $n \geq 2$, see [9].

We use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the *spherical metric* $h(x,y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in

$\mathbb{R}^{n+1}$, that is,

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad x \neq \infty \neq y, \quad h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}.$$

Recall that a (continuous) mapping $f : D \to \mathbb{R}^n$ is *absolutely continuous on lines*, $f \in ACL$ for short, if, for every closed parallelepiped P in D whose faces are perpendicular to the coordinate axes, each coordinate function of $f|P$ is absolutely continuous on almost every line segment in P that is parallel to the coordinate axes. In particular, f is ACL if f is in the Sobolev class $W^{1,1}_{\rm loc}$.

We shall next formulate some of our research results from Chapters 4–6 in [25].

2. Q-HOMEOMORPHISMS WITH ARBITRARY $Q \in L^1_{\rm loc}$

Chapter 4 in [25] contains results for Q-homeomorphisms with locally integrable Q, see, e.g., [26]–[31] and [45].

THEOREM 1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\rm loc}$. Then f is differentiable a.e.*

Remark 1. Note also that f^{-1} has Lusin's (N)-property and $J(x,f) \neq 0$ a.e. for every Q-homeomorphism f with $Q \in L^1_{\rm loc}$.

THEOREM 2. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\rm loc}$. Then $f \in$ ACL.*

COROLLARY 1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to D'$ be a Q-homeomorphism with $Q \in L^1_{\rm loc}$. Then f belongs to $W^{1,1}_{\rm loc}$.*

THEOREM 3. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ be a Q-homeomorphism with $Q \in L^1(\mathbb{B}^n)$, $f(0) = 0$, $h(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n)) \geq \delta > 0$, and $h(f(x_0), f(0)) \geq \delta$ for some $x_0 \in \mathbb{B}^n$. Then for all $|x| < r = \min(|x_0|/2, 1-|x_0|)$ we have*

$$|f(x)| \geq \psi(|x|), \tag{9}$$

where $\psi(t)$ is a strictly increasing function with $\psi(0) = 0$ that depends only on the L^1-norm of Q in $\mathbb{B}^n$, n, and δ.

Later on, for given sets A, B, and C in $\mathbb{R}^n$, $\Delta(A,B;C)$ denotes a collection of all paths $\gamma : [0,1] \to \mathbb{R}^n$ joining A and B in C, i.e., $\gamma(0) \in A$, $\gamma(1) \in B$, and $\gamma(t) \in C$ for all $t \in (0,1)$. Following [9], we say that a domain D in $\mathbb{R}^n$, $n \geq 2$, is a *quasiextremal distance domain*, *QED domain* for short, if

$$M(\Delta(E,F;\mathbb{R}^n)) \leq K\, M(\Delta(E,F;D)) \tag{10}$$

for a finite number $K \geq 1$ and all continua E and F in D.

Now, introduce a new class of domains in $\mathbb{R}^n$, $n \geq 2$, which are wider than the class of QED domains described earlier. The significance of such a type of domains is that conformal and quasiconformal mappings as well as many of their generalizations between them admit homeomorphic extensions to their boundary.

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$ defined below are localizations and generalizations of the corresponding notions introduced in [30, 31]; compare with the properties P_1 and P_2 by Väisälä [48] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki [35]. Lemma 1 below establishes the relation of weak flatness formulated in terms of moduli of path families with the general topological notion of local connectedness on the boundary, see [21].

We say that ∂D is *weakly flat at a point* $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Delta(E, F; D)) \geq P \tag{11}$$

for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *weakly flat* if it is weakly flat at every point in ∂D.

We also say that a point $x_0 \in \partial D$ is *strongly accessible* if, for every neighborhood U of the point x_0, there exist a compactum E in D, a neighborhood $V \subset U$ of x_0, and a number $\delta > 0$ such that

$$M(\Delta(E, F; D)) \geq \delta \tag{12}$$

for all continua F in D intersecting ∂U and ∂V. We say that the boundary ∂D is *strongly accessible* if every point $x_0 \in \partial D$ is strongly accessible.

Remark 2. Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods U and V of a point x_0 only balls (closed or open) centered at x_0 or only neighborhoods of x_0 in another fundamental system of its neighborhoods. These concepts can also be extended in a natural way to the case of $\overline{\mathbb{R}^n}$, $n \geq 2$, and $x_0 = \infty$.

PROPOSITION 1. *If a domain D in $\mathbb{R}^n$, $n \geq 2$, is weakly flat at a point $x_0 \in \partial D$, then the point x_0 is strongly strongly accessible from D.*

COROLLARY 2. *Weakly flat boundaries of domains in $\mathbb{R}^n$, $n \geq 2$, are strongly accessible.*

Recall that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be *locally connected at a point* $x_0 \in \partial D$ if, for every neighborhood U of the point x_0, there is a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is connected. Note that every

Jordan domain D in $\mathbb{R}^n$ is locally connected at each point of ∂D, see, e.g., [51], p. 66.

LEMMA 1. *If a domain D in $\mathbb{R}^n$, $n \geq 2$, is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0.*

COROLLARY 3. *A domain D in $\mathbb{R}^n$, $n \geq 2$, with a weakly flat boundary is locally connected at every boundary point.*

Remark 3. As is well known, see, e.g., 10.12 in [48], we have

$$M(\Delta(E, F; \mathbb{R}^n)) \geq c_n \log \frac{R}{r} \tag{13}$$

for all sets E and F in $\mathbb{R}^n$, $n \geq 2$, intersecting all the spheres $S(x_0, \varrho)$, $\varrho \in (r, R)$. Hence, it follows directly from the definitions that a QED domain has a weakly flat boundary.

COROLLARY 4. *Every* QED *domain D in $\mathbb{R}^n$, $n \geq 2$, is locally connected at each boundary point and ∂D is strongly accessible.*

THEOREM 4. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let f be a Q-homeomorphism of D onto D' with $Q \in L^1(D)$. If D is locally connected at ∂D and $\partial D'$ is weakly flat, then f^{-1} has a continuous extension to $\overline{D'}$.*

It is necessary to stress here that the extension problem for the direct mappings f is much more complicated, see Proposition 3.

In particular, by Theorem 4, we obtain the following conclusion.

THEOREM 5. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$. If D' is locally connected at $\partial D'$ and ∂D is weakly flat, then any quasiconformal mapping $f : D \to D'$ admits a continuous extension to the boundary $\overline{f} : \overline{D} \to \overline{D'}$.*

COROLLARY 5. *If D and D' are domains with weakly flat boundaries, then any quasiconformal mapping $f : D \to D'$ admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Note that these results on the extension to weakly flat boundaries are new even for the class of conformal mappings in the plane. Here, we do not assume that the domains are simply connected.

3. Q-HOMEOMORPHISMS WITH SPECIAL Q

Chapter 5 in [25] demonstrates that the main part of the quasiconformal theory can be extended to Q-homeomorphisms with Q $\in$ BMO, see also [26]–[31]. Let us give some examples of the corresponding results.

THEOREM 6. *Let* $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ *be a* Q*-homeomorphism with* $Q \in \mathrm{BMO}(\mathbb{B}^n)$*. If* $h(\overline{\mathbb{R}^n} \setminus f(B^n(1/e))) \geq \delta > 0$*, then*

$$h(f(x), f(0)) \leq \frac{C}{(\log 1/|x|)^{\alpha}} \tag{14}$$

for all $|x| < \mathrm{e}^{-2}$*, where* C *and* α *are positive constants that depend only on* n, δ*, the* BMO *norm* $\|Q\|_*$ *of* Q*, and the average* Q_1 *of* Q *over the ball* $|x| < 1/\mathrm{e}$*.*

COROLLARY 6. *Let* $\mathcal{F}$ *be a family of* Q*-homeomorphisms* $f : D \to \mathbb{R}^n$ *with* $Q \in \mathrm{BMO}(D)$ *and let* $\delta > 0$*. If every* $f \in \mathcal{F}$ *omits two points* a_f *and* b_f *in* $\mathbb{R}^n$ *with* $h(a_f, b_f) \geq \delta$*, then* $\mathcal{F}$ *is equicontinuous.*

THEOREM 7. *Let* $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ *be a* Q*-homeomorphism with* $Q \in \mathrm{BMO}(\mathbb{B}^n \setminus \{0\})$*. Then* f *has a* $Q(x)$*-homeomorphic extension to* $\mathbb{B}^n$*.*

LEMMA 2. *Let* D *and* D' *be domains in* $\mathbb{R}^n$*,* $n \geq 2$*, and let* $f : D \to D'$ *be a* Q*-homeomorphism with* $Q \in \mathrm{BMO}(\overline{D})$*. If* D *is locally connected at* ∂D *and* $\partial D'$ *is strongly accessible, then* f *has a continuous extension* $\tilde{f} : \overline{D} \to \overline{D}'$*.*

Combining Lemma 2 and Theorem 4, we obtain

COROLLARY 7. *Let* $f : D \to D' \subset \mathbb{R}^n$ *be a* Q*-homeomorphism onto* D' *with* $Q \in \mathrm{BMO}(\overline{D})$*. If* D *locally connected at* ∂D *and* $\partial D'$ *is weakly flat, then* f *has a homeomorphic extension* $\tilde{f} : \overline{D} \to \overline{D'}$*.*

COROLLARY 8. *Let* $f : D \to D' \subset \mathbb{R}^n$ *be a* Q*-homeomorphism onto* D' *with* $Q \in \mathrm{BMO}(\overline{D})$*. If* ∂D *and* $\partial D'$ *are weakly flat, then* f *has a homeomorphic extension* $\tilde{f} : \overline{D} \to \overline{D'}$*.*

The next theorem extends the known Gehring–Martio–Vuorinen results from qc mappings to Q-homeomorphisms with $Q \in \mathrm{BMO}(\overline{D})$, cf. [9], p. 196, and [32], p. 36.

THEOREM 8. *Let* $f : D \to D'$ *be a* Q*-homeomorphism between* QED *domains* D *and* D' *with* $Q \in \mathrm{BMO}(\overline{D})$*. Then* f *has a homeomorphic extension* $\tilde{f} : \overline{D} \to \overline{D'}$*.*

More advanced results on Q-homeomorphisms for the case of $Q \in \mathrm{FMO}$ and more general situations are proved in Chapter 6. For this goal, we develop here a general method of singular functional parameters, see, e.g., [17, 18].

Our study concerns isolated boundary points, thin parts of the boundary in terms of Hausdorff measures, and domains with regular boundaries such as the quasiextremal distance domains of Gehring–Martio, uniform, convex, smooth, etc. In particular, we show that an isolated singularity is removable for Q-homeomorphisms provided that $Q(x)$ has finite mean oscillation at this point. An analogue of the well-known Painlevé theorem for analytic functions also follows if $Q(x)$ has finite mean oscillation at each point of a singular set of the length zero.

Let D be a domain in $\mathbb{R}^n$, $n \geq 1$. Following [17], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation at a point* $x_0 \in \overline{D}$ if

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, \mathrm{d}m(x) < \infty, \tag{15}$$

where

$$\overline{\varphi}_\varepsilon = \fint_{D(x_0,\varepsilon)} \varphi(x) \, \mathrm{d}m(x) = \frac{1}{|D(x_0,\varepsilon)|} \int_{D(x_0,\varepsilon)} \varphi(x) \, \mathrm{d}m(x) \tag{16}$$

is the mean value of the function $\varphi(x)$ over $D(x_0,\varepsilon) = D \cap B(x_0,\varepsilon)$, $\varepsilon > 0$. Here $B(x_0,\varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$.

Note that under (15) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$ and that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$ and not a subset of L^p_{loc} for any $p > 1$.

Proposition 2. *If, for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \mathrm{d}m(x) < \infty, \tag{17}$$

then φ has finite mean oscillation at x_0.

Corollary 9. *If, for a point $x_0 \in \overline{D}$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(x_0,\varepsilon)} |\varphi(x)| \mathrm{d}m(x) < \infty, \tag{18}$$

then φ has finite mean oscillation at x_0.

It is well known that isolated singularities are removable for conformal as well as quasiconformal mappings. The following statement shows that any power of integrability of $Q(x)$ cannot guarantee the removability of isolated singularities of Q-homeomorphisms. This is a new phenomenon.

Proposition 3. *For any $p \in [1, \infty)$, there is a Q-homeomorphism $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$, $n \geq 2$, with $Q \in L^p(\mathbb{B}^n)$ that has no continuous extension to $\mathbb{B}^n$.*

However, as the next lemma shows, it is sufficient for the removability of isolated singularities of Q-homeomorphisms to require that $Q(x)$ is integrable with suitable weights.

Lemma 3. *Let $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$, $n \geq 2$, be a Q-homeomorphism. If*

$$\int_{\varepsilon<|x|<1} Q(x) \cdot \psi^n(|x|) \, \mathrm{d}m(x) = o(I(\varepsilon)^n) \tag{19}$$

as $\varepsilon \to 0$*, where* $\psi(t)$ *is a nonnegative measurable function on* $(0,\infty)$ *such that*

$$0 < I(\varepsilon) :\, = \int_{\varepsilon}^{1} \psi(t)\,\mathrm{d}t < \infty, \quad \varepsilon \in (0,1), \tag{20}$$

then f *has a continuous extension to* $\mathbb{B}^n$ *that is a* Q*-homeomorphism.*

Remark 4. Note also that (19) holds, in particular, if

$$\int_{\mathbb{B}^n} Q(x) \cdot \psi^n(|x|)\,\mathrm{d}m(x) < \infty \tag{21}$$

and $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the removability of a singularity at $x = 0$, it is sufficient that integral (21) converges for some nonnegative function $\psi(t)$ that is locally integrable over $(0,1)$ but has a nonintegrable singularity at 0. The functions $Q(x) = \log^{\lambda}(\mathrm{e}/|x|)$, $\lambda \in (0,1)$, $x \in \mathbb{B}^n$, $n \geq 2$, and $\psi(t) = 1/(t\ \log(\mathrm{e}/t))$, $t \in (0,1)$, show that the condition (21) is compatible with the condition $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In particular, we have shown that functions $Q \in \mathrm{FMO}$ satisfy (19) and (21) with the given ψ for $n \geq 2$ and $n \geq 3$, respectively.

THEOREM 9. *Let* $f : D\backslash\{x_0\} \to \mathbb{R}^n$*,* $n \geq 2$*, be a* Q*-homeomorphism where* $Q(x)$ *has finite mean oscillation at a point* $x_0 \in D$*. Then* f *has a* Q*-homeomorphic extension to* D*.*

In other words, an isolated singularity of a Q-homeomorphism is removable if $Q(x)$ has finite mean oscillation at the point. As consequences of Theorem 9 and Corollary 9, we also obtain the following statements.

COROLLARY 10. *If* $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{R}^n$*,* $n \geq 2$*, is a* Q*-homeomorphism,*

$$\overline{\lim_{\varepsilon \to 0}} \;⨍_{B(\varepsilon)} Q(x)\,\mathrm{d}m(x) < \infty, \tag{22}$$

then f *has a* Q*-homeomorphic extension to* $\mathbb{B}^n$*.*

Choosing in Lemma 3 the function $\psi(t) = 1/t$ as a weight, we come to the following more general statement.

THEOREM 10. *Let* $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{R}^n$*,* $n \geq 2$*, be a* Q*-homeomorphism. If*

$$\int_{\varepsilon<|x|<1} Q(x)\frac{\mathrm{d}m(x)}{|x|^n} = o\left(\left[\log \tfrac{1}{\varepsilon}\right]^n\right) \quad \text{as } x \to 0, \tag{23}$$

then f *has a* Q*-homeomorphic extension to* $\mathbb{B}^n$*.*

COROLLARY 11. *In particular, the conclusion of Theorem* 10 *holds if*

$$Q(x) = o\left(\left[\log \tfrac{1}{|x|}\right]^{n-1}\right) \quad \text{as } x \to 0. \tag{24}$$

Let us give conditions of other types that are often met in the mapping theory, see, e.g., [23] and [34].

THEOREM 11. *Let* $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n$, $n \geq 2$, *be a* Q *homeomorphism and assume that*

$$\int_0^{\varepsilon_0} \frac{\mathrm{d}r}{rq^{\beta}(r)} = \infty, \quad \beta \geq 1/(n-1), \tag{25}$$

where $q(r)$ *is the mean integral value of the function* $Q(x)$ *over the sphere* $|x| = r$. *Then* f *has a* Q*-homeomorphic extension to* $\mathbb{B}^n$.

We also studied *super* Q*-homeomorphisms*, i.e., such Q-homeomorphisms $f : D \to \mathbb{R}^n$, $n \geq 2$, that inequality (8) holds not only for all families Γ of continuous paths $\gamma : (0,1) \to D$ but also for *dashed lines* $\gamma : \Delta \to D$, i.e., continuous mappings γ of open subsets Δ of the real axis $\mathbb{R}$ into D. Recall that every open set Δ in $\mathbb{R}$ consists of a countable collection of mutually disjoint intervals $\Delta_i \subset \mathbb{R}$, $i = 1, 2, \dots$. This fact gives reasons for the term "dashed line".

THEOREM 12. *Let* $f : D \to \mathbb{R}^n$ *be a homeomorphism in the class* $W^{1,n}_{\mathrm{loc}}$ *with* $f^{-1} \in W^{1,n}_{\mathrm{loc}}$. *Then* f *is a super* Q*-homeomorphism with* $Q(x) = K_I(x, f)$.

As above, $K_I(x, f)$ is the inner dilatation of f at x. It is known that homeomorphisms of the class $W^{1,n}_{\mathrm{loc}}$ with $K_I \in L^1_{\mathrm{loc}}$ have f^{-1} in the same class; see Corollary 2.3 in [20]. Thus, we have the next assertion.

COROLLARY 12. *Let* $f : D \to \mathbb{R}^n$ *be a homeomorphism in the class* $W^{1,n}_{\mathrm{loc}}$ *with* $K_I \in L^1_{\mathrm{loc}}$. *Then* f *is a super* Q*-homeomorphism with* $Q(x) = K_I(x, f)$.

Thus, Theorem 12 shows that super Q-homeomorphisms form a wide subclass of Q-homeomorphisms including many mappings with finite distortion.

Consider the problem of removability of singularities of length zero for super Q-homeomorphisms. A set X in $\mathbb{R}^n$ is called a *set of length zero* if X can be covered by a sequence of balls in $\mathbb{R}^n$ with an arbitrary small sum of diameters. As known, such sets have (Lebesgue) measure zero,

$$\dim X = 0, \tag{26}$$

hence they are totally disconnected, see, e.g., [16], pp. 22 and 104. By the theorem of Menger and Urysohn, see, e.g., [16], condition (26) guarantees that X does not disconnect a domain D in $\mathbb{R}^n$, $n \geq 2$, and, thus, if X is closed in D, then $D^* = D \setminus X$ is also a domain. Classical examples of such sets are sets C of the Cantor type. Note that C is *perfect*, i.e., it is closed and without isolated points. Hence each neighborhood of a point in C contains a subset of C of the continuum cardinality, see [52].

THEOREM 13. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let X be a closed subset of D of length zero, and let $f : D \backslash X \to \mathbb{R}^n$ be a super Q-homeomorphism. If the function $Q(x)$ has finite mean oscillation at every point $x_0 \in X$, then f has a homeomorphic extension to D.*

COROLLARY 13. *Let X be a closed subset of length zero in D and let*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0,\varepsilon)} Q(x)\, \mathrm{d}m(x) < \infty \tag{27}$$

for every $x_0 \in X$. Then every super Q-homeomorphism $f : D \backslash X \to \mathbb{R}^n$ has a homeomorphic extension to D.

Other results on singular sets of length zero under Q-homeomorphisms are formulated in terms on Q which is similar to the case of isolated singular points. We also consider the case of singular sets of the Hausdorff $(n-1)$-dimensional measure zero.

Our results on continuous and homeomorphic extensions of Q homeomorphisms to boundary points are also formulated in similar terms on the majorant $Q(x)$, e.g., if $Q(x)$ has finite mean oscillation at the corresponding points. In particular, the well-known Gehring–Martio theorem on the homeomorphic extension to the boundary of quasiconformal mappings is also generalized to Q-homeomorphisms with $Q \in \mathrm{FMO}$. All these results can be applied to homeomorphisms of Sobolev's classes.

Chapter 13 in [25] extends many of these results to arbitrary metric spaces with measures, see also [38].

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