PRIMALITY, IRREDUCIBILITY, AND COMPLETE IRREDUCIBILITY IN MODULES OVER COMMUTATIVE RINGS

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The aim of this paper is to extend from ideals to modules over commutative rings some results of Fuchs, Heinzer, and Olberding [8], [9] concerning primal and completely irreducible ideals. In particular, it is shown that if a proper submodule N of a module M is an irredundant intersection of completely irreducible submodules of M then the maximal ideals occurring as adjoint ideals of these submodules are independent of the intersection. In addition, it is proved for a module M, that every primal submodule of M is irreducible if and only if M is arithmetical.

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1. INTRODUCTION

In this paper we extend from ideals to modules over commutative rings some results of Fuchs, Heinzer and Olberding ([8], [9]) concerning primal and completely irreducible ideals. Two methods are used to do this. Firstly, we prove the results for modules directly and secondly we use trivial extensions of modules.

In Section 2 we introduce the basic terminology and notation we will be using throughout the paper. Section 3 introduces the concept of a primal submodule of a module and various equivalent characterizations of completely irreducible submodules are given (Proposition 3.4). Trivial extensions are discussed in Section 4. We show that irreducibility, complete irreducibility, and primality are nicely transferred via trivial extensions, which allow us to prove a module version of a theorem of Fuchs, Heinzer and Olberding [8, Theorem 3.2] concerning irredundant intersections of completely irreducible submodules (Theorem 4.6). The final section is concerned with arithmetical modules. It is shown that, for any commutative ring R, an R-module M is arithmetical if and only if every primal submodule is irreducible (Theorem 5.3);

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this extends from rings to modules a result of Fuchs, Heinzer and Olberding [8, Theorem 1.8] characterizing arithmetical rings in terms of primal ideals.

2. TERMINOLOGY AND NOTATION

Throughout this paper R will always denote a commutative ring with a non-zero identity and M a unital R-module. The lattice of all submodules of M will be denoted by $\mathcal{L}(M)$. The notation $N \leqslant M$ (resp. $N < M$) means that N is a submodule (resp. proper submodule) of M. Whenever we want to indicate that X is merely a subset (resp. proper subset) of Y, then we shall write $X \subseteq Y$ (resp. $X \subset Y$). We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of all natural numbers, and by $\mathbb{Z}$ the ring of rational integers.

The ideals of R will be denoted by small Gothic letters $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{m}$, and the submodules of M by X, L, N, P, Q. By $\mathrm{Spec}(R)$ we denote the set of all *prime ideals* of R, and by $\mathrm{Max}(R)$ the set of all *maximal ideals* of R.

For any subsets X, Y of M and any subset I of R we set

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq X\},$$
$$(X : Y) := \{a \in R \mid aY \subseteq X\}, \quad (X : I) := \{z \in M \mid Iz \subseteq X\}.$$

If $x \in M$ then we denote $(0 : \{x\})$ by $\mathrm{Ann}_R(x)$. The annihilator of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(0 : M)$ of R. Note that if $N \leqslant M$, then $(N : M) = \mathrm{Ann}_R(M/N)$.

As in Bourbaki [5], for any module M we denote by $\mathrm{Ass}(M)$ the *assassin* of M, by $\mathrm{Ass}_f(M)$ the *weak assassin* of M ("f" for the French *faible* = *weak*), and by $Z(M)$ the set of all *zero divisors* on M. That is,

$$\mathrm{Ass}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists\, x \in M \text{ with } \mathfrak{p} = \mathrm{Ann}_R(x)\},$$
$$\mathrm{Ass}_f(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists\, x \in M \text{ with } \mathfrak{p} \text{ minimal in } V(\mathrm{Ann}_R(x))\},$$

and

$$Z(M) := \{a \in R \mid \exists\, x \in M,\ x \neq 0, \text{ with } ax = 0\}.$$

It is well-known (see, e.g., Bourbaki [5]) that for any R-module M one has

$$Z(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_f(M)} \mathfrak{p}.$$

Let M be a module. A submodule X of M is said to be *irreducible*, if $X \neq M$ and whenever $X = N \cap P$ for N, $P \in \mathcal{L}(M)$, then $X = N$ or $X = P$ and in this case the module M/X is called *uniform*. Next, a submodule X of M is said to be *completely irreducible*, abbreviated *CI*, if $X \neq M$ and whenever $X = \bigcap_{i \in I} X_i$ for a family $(X_i)_{i \in I}$ of submodules of M, then $X = X_j$ for some $j \in I$ and in this case the module M/X is called *completely uniform*.

In the literature, completely uniform modules are usually known as *subdirectly irreducible* modules and we will use this terminology throughout this paper.

Note that the concepts above can be also defined for modules over not necessarily commutative rings, and more generally, for complete lattices (see Albu [1]). Moreover, one can define the very general concept of a *subdirectly irreducible poset* as in Albu, Iosif and Teply [2, Definition 0.1]: a poset P with least element 0 is said to be *subdirectly irreducible* if $P \neq \{0\}$ and the set $P \setminus \{0\}$ has a least element. Note that a module M is subdirectly irreducible if and only the lattice $\mathcal{L}(M)$ of all its submodule is subdirectly irreducible if and only $M \neq 0$ and M has a simple essential socle (see Albu [1, Proposition 0.5]).

Recall that a module M_R is said to be a *chain module* if the lattice $\mathcal{L}(M)$ is a chain, or equivalently, if every proper submodule of M is irreducible. One may ask what are the modules M for which every proper submodule of M is CI. It is easy to see that this happens if and only if every nonempty set of proper submodules of M has a unique least submodule (i.e., the lattice $\mathcal{L}(M)$ is a well-ordered poset), and that this occurs if and only if M is an Artinian chain module.

Finally, recall that a module M is called *semi-Artinian* if every nonzero homomorphic image contains a simple submodule. Observe that if M is a semi-Artinian module, then every irreducible submodule of M is CI; this follows from a more generally latticial result of Albu [1, Corollary 0.6] applied to the lattice $L = \mathcal{L}(M)$.

3. PRIMAL AND COMPLETELY IRREDUCIBLE SUBMODULES

The concept of *primal ideal* of a commutative ring has been introduced by Fuchs [7] and extended to modules over rings which are not necessarily commutative by Dauns [6] using the term of "not right prime element to a submodule". The definition below, for commutative rings, is a reformulation in terms of "zero divisor on a module" of Dauns' definition.

Definition 3.1. A module M is said to be *coprimal* if $M \neq 0$ and $Z(M)$ is an ideal of R. A submodule N of a module M is called *primal* if the quotient module M/N is coprimal, and in this case $Z(M/N)$ is called the adjoint ideal of N and will be denoted by adj N. □

The following result is elementary but is included for completeness.

LEMMA 3.2. *If M is a coprimal module, then $Z(M)$ is a prime ideal of R.*

Proof. We have $M \neq 0$ since M is coprimal, so $Z(M) \neq R$. Now let $a, b \in R$ with $ab \in Z(M)$. Then $(ab)z = 0$ for some $0 \neq z \in M$. If $bz = 0$, then $b \in Z(M)$. If $bz \neq 0$, then $a(bz) = 0$, so that $a \in Z(M)$. This shows that the ideal $Z(M)$ of R is prime. $\square$

For an R-module M we denote by $\mathcal{P}_R(M)$ the set of all primal submodules of M. If $N \in \mathcal{P}_R(M)$ and if the adjoint ideal adj N of N is the prime ideal $\mathfrak{p}$, then N is said to be $\mathfrak{p}$-*primal*. We also denote by $\mathcal{I}_R(M)$ the set of all irreducible submodules of M and by $\mathcal{I}_R^c(M)$ the set of all completely irreducible submodules of M. The subscript R is deleted if there is no danger of ambiguity.

The next results extend, from ideals to modules, some results of Section 1 of Fuchs Heinzer and Olberding [9]. The proofs are straightforward but are included for completeness.

LEMMA 3.3. *For any module M we have $\mathcal{I}^c(M) \subseteq \mathcal{I}(M) \subseteq \mathcal{P}(M)$, and*

$$\operatorname{adj} N \in \operatorname{Ass}(M/N) \cap \operatorname{Max}(R)$$

for any $N \in \mathcal{I}^c(M)$.

Proof. The inclusion $\mathcal{I}^c(M) \subseteq \mathcal{I}(M)$ is clear. For the inclusion $\mathcal{I}(M) \subseteq \mathcal{P}(M)$ let $N \in \mathcal{I}(M)$. Let a, $b \in Z(M/N)$. There exist $x, y \in M \setminus N$ such that $ax \in N$ and $by \in N$. Because N is irreducible, there exists $z \in (Rx + N) \cap (Ry + N) \setminus N$. Note that $(a + b)z \in N$ so that $a + b \in Z(M/N)$. It follows that $Z(M/N)$ is an ideal of R, as required.

For the last part of lemma, it is sufficient to prove that

$$Z(M) \in \operatorname{Ass}(M) \cap \operatorname{Max}(R)$$

for any subdirectly irreducible module M. By Albu [1, Proposition 0.5] applied to the lattice $L = \mathcal{L}(M/N)$, M is an essential extension of a simple module S, so $\{\mathfrak{m}\} = \operatorname{Ass}(S) = \operatorname{Ass}(M)$, where $\mathfrak{m} \in \operatorname{Max}(R)$ is such that $S \simeq R/\mathfrak{m}$. To complete the proof, we have to show that $Z(M) = \mathfrak{m}$. We have $\mathfrak{m} \subseteq Z(M)$ since $\mathfrak{m} \in \operatorname{Ass}(M)$, so $\mathfrak{m} = \operatorname{Ann}_R(x)$ for some $0 \neq x \in M$. Hence, $\mathfrak{m} = Z(M)$ since $\mathfrak{m} \in \operatorname{Max}(R)$ and $Z(M) \neq R$. $\square$

Note that the set inclusions in Lemma 3.3 are, in general, strict as we show below. Now, if $\mathfrak{p} \in \operatorname{Spec}(R)$ and $N \leqslant M$, then $N_{(\mathfrak{p})}$ will denote the $(R \setminus \mathfrak{p})$-*saturation* of N in M, i.e.,

$$N_{(\mathfrak{p})} := \{x \in M \mid \exists\, s \in R \setminus \mathfrak{p},\, sx \in N\}.$$

Note that $N_{(\mathfrak{p})}$ is a submodule of M such that $N \subseteq N_{(\mathfrak{p})}$. Recall that N is said to be $(R \setminus \mathfrak{p})$-*saturated* if $N = N_{(\mathfrak{p})}$.

PROPOSITION 3.4. *The following statements are equivalent for a proper submodule N of a module M.*

(1) $N \in \mathcal{I}^c(M)$.

(2) $\bigcap\limits_{N<P\leqslant M} P \neq N$.

(3) *M/N has a simple essential socle.*

(4) $N \in \mathcal{I}(M)$ *and* $\mathrm{Soc}(M/N) \neq 0$.

(5) $N \in \mathcal{I}(M)$ *and* $N < (N : \mathfrak{m})$ *for some* $\mathfrak{m} \in \mathrm{Max}(R)$.

(6) $N \in \mathcal{I}(M)$ *and* $\mathrm{adj}\, N \in \mathrm{Ass}(M/N) \cap \mathrm{Max}(R)$.

(7) $N \in \mathcal{I}(M)$, $\mathrm{adj}\, N \in \mathrm{Max}(R)$, *and* $\mathrm{adj}\, N = (N : x)$ *for some* $x \in M \setminus N$.

(8) $N = N_{(\mathfrak{m})}$ *for some* $\mathfrak{m} \in \mathrm{Max}(R)$, *and* $N_{\mathfrak{m}} \in \mathcal{I}^c_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow at once from Albu [1, Proposition 0.5] applied to the lattice $L = \mathcal{L}(M/N)$.

(1) $\Rightarrow$ (6) follows from Lemma 3.3.

(6) $\Rightarrow$ (5): Let $\mathfrak{m} = \mathrm{adj}\, N$, and let $x \in M \setminus N$ be such that $\mathfrak{m} = \mathrm{Ann}_R(\widehat{x}) = (N : x)$, where $\widehat{x}$ is the coset of $x + N$ of x modulo N. Then $x \in (N : \mathfrak{m}) \setminus N$, as desired.

(5) $\Rightarrow$ (4): If we set $X := (N : \mathfrak{m})$, then $N < X$, and $\mathfrak{m}(X/N) = 0$, so X/N is a non-zero $(R/\mathfrak{m})$-module, and it surely has a simple R-submodule isomorphic to $R/\mathfrak{m}$.

(6) $\Leftrightarrow$ (7) Clear.

(7) $\Rightarrow$ (8): Let $\mathrm{adj}\, N = \mathfrak{m} \in \mathrm{Max}(R)$. First, we are going to show that N is $(R \setminus \mathfrak{m})$-saturated, i.e., if $s \in R \setminus \mathfrak{m}$, $y \in M$, and $sy \in N$, then necessarily $y \in N$. Assume that $y \notin N$. Then $\widehat{0} \neq \widehat{y}$ in M/N and $s\widehat{y} = \widehat{0}$, so $s \in Z(M/N) = \mathfrak{m}$, which is a contradiction.

In order to prove that $N_{\mathfrak{m}}$ is a completely irreducible $R_{\mathfrak{m}}$-module, using the equivalence (1) $\Leftrightarrow$ (7) already proved, it is sufficient to show that $\mathfrak{m}R_{\mathfrak{m}} = (N_{\mathfrak{m}} : (x/1))$ for some $x \in M \setminus N$, and $N_{\mathfrak{m}}$ is an irreducible $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$.

The equality $\mathfrak{m}R_{\mathfrak{m}} = (N_{\mathfrak{m}} : (x/1))$ follows immediately from $(N : x)_{\mathfrak{m}} = (N_{\mathfrak{m}} : (x/1))$. It is known that there is a lattice isomorphism between the lattice of all $R_{\mathfrak{m}}$-submodules of $M_{\mathfrak{m}}$ and the lattice of all $(R \setminus \mathfrak{m})$-saturated R-submodules of M (see, e.g., Bourbaki [5, Proposition 10, Chapter 2]); this implies that $N_{(\mathfrak{m})}$ is an irreducible R-submodule of M if and only if $N_{\mathfrak{m}}$ is an irreducible $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$.

(8) $\Rightarrow$ (7): Use the same arguments as in implication (7) $\Rightarrow$ (8). $\square$

Let R be a ring and M an R-module. A submodule K of M is called *prime* provided $K \neq M$ and whenever $r \in R$ and $m \in M$ such that $rm \in K$ then $m \in K$ or $rM \subseteq K$. Suppose that K is a prime CI submodule of M. There exists a submodule L of M such that $K \subseteq L$ and L/K is simple. If $\mathfrak{p} = (K : L)$ then $\mathfrak{p}$ is a maximal ideal of R and $\mathfrak{p}M \subseteq K$. Because M/K is uniform, M/K is simple. We have thus proved

Corollary 3.5. *A prime submodule P of a module M is CI if and only if it is maximal.* $\square$

Remarks 3.6. (1) The inclusions $\mathcal{I}^c(M) \subseteq \mathcal{I}(M) \subseteq \mathcal{P}(M)$ from Lemma 3.3 are in general strict. For example, $0 \in \mathcal{I}_{\mathbb{Z}}(\mathbb{Z}) \setminus \mathcal{I}^c_{\mathbb{Z}}(\mathbb{Z})$ by Corollary 3.5.

Consider now the ring $F[X,Y]$ of polynomials in the indeterminates X and Y over a field F. Then the ideal $\mathfrak{q} := (X^2, XY)$ of the ring $F[X,Y]$ is primal with adjoint prime ideal (X,Y) (see, e.g., Fuchs [7]), but it is not irreducible since $\mathfrak{q} = (X^2, Y) \cap (X)$, $\mathfrak{q} \subset (X^2, Y)$, $\mathfrak{q} \subset (X)$. However, more spectacular examples can be given. Let R be a ring with a maximal ideal $\mathfrak{m}$ such that there exists an infinite family of prime ideals $\mathfrak{p}_i\,(i \in I)$ with $\mathfrak{p}_i \subseteq \mathfrak{m}$ for all $i \in I$. Let M denote the R-module $(R/\mathfrak{m}) \oplus (\bigoplus_{i\in I}(R/\mathfrak{p}_i))$. Clearly, M is an infinite direct sum of uniform submodules but it is easy to check that the zero submodule of M is $\mathfrak{m}$-primal.

(2) As is well-known, any prime ideal of a commutative ring is irreducible. This result does not hold for modules, i.e., a prime submodule of a module M is not necessarily irreducible. Indeed, if F is any field, then for any non-zero F-module M, any proper submodule N of M is prime, but N is irreducible if and only if M/N is cyclic. $\square$

4. TRANSFERRING PROPERTIES VIA TRIVIAL EXTENSIONS

For any (commutative) ring R (with identity element) and any (unital) R-module M one defines the *trivial extension* of M by R or the *idealization* of M, denoted by $R \rtimes M$ or by $R(+)M$, to be the commutative ring whose elements are of the form (r,m), where $r \in R$ and $m \in M$, with addition and multiplication defined as

$$(r,m) + (r',m') = (r+r', m+m')$$

and

$$(r,m)(r',m') = (rr', rm' + r'm)$$

for all $r, r' \in R$ and $m, m' \in M$ (see, e.g., Huckaba [10, p. 161] or Anderson and Winders [4]). Note that the ring $R \rtimes M$ has identity element (1,0).

Let N be a submodule of M and let $\mathfrak{a} = (N : M)$. Now, we define $N^{\#}$ to be the set of elements (a,x) in $R \rtimes M$ such that $a \in \mathfrak{a}$ and $x \in N$, i.e., $N^{\#} = \mathfrak{a} \rtimes N$. It is easy to check that $N^{\#}$ is an ideal of $R \rtimes M$. The assignment $N \mapsto 0 \rtimes N$ determines a lattice isomorphism between the lattice $\mathcal{L}(M)$ of all submodules of M and the lattice of all ideals of $R \rtimes M$ that are contained in $0 \rtimes M$. Observe that the module M, identified with $0 \rtimes M$, becomes an ideal of $R \rtimes M$, which also explains the term *idealization*.

Lemma 4.1. *Let U be an R-module, and set $S := R \ltimes U$. Then ${}_SS$ is a uniform module if and only if ${}_RU$ is a faithful uniform module.*

Proof. Assume that the module ${}_RU$ is faithful and uniform. Let s_1 and s_2 be any non-zero elements of S. Then $s_1 = (r_1, m_1)$ and $s_2 = (r_2, m_2)$ for some $r_1, r_2 \in R$ and $m_1, m_2 \in U$. Suppose that $r_1 \neq 0$. Then $r_1U \neq 0$, so that $r_1x \neq 0$ for some $x \in U$. Note that

$$0 \neq (0, r_1x) = (0, x)(r_1, m_1) \in Ss_1.$$

On the other hand, if $r_1 = 0$ then

$$0 \neq (0, m_1) = s_1 \in Ss_1.$$

Thus we can suppose without loss of generality that $r_1 = r_2 = 0$. There exist elements a_1 and a_2 in R such that $0 \neq a_1m_1 = a_2m_2$. Then

$$0 \neq (0, a_1m_1) = (a_1, 0)(0, m_1) = (a_2, 0)(0, m_2) \in Ss_1 \cap Ss_2.$$

It follows that S is a uniform S-module.

Conversely, suppose that ${}_RU$ is not faithful, i.e., there exists $0 \neq r \in R$ such that $rU = 0$. If we set $s := (r, 0)$ and $X := 0 \ltimes U$, then X is a non-zero ideal of S and $Ss \cap X = 0$, so that ${}_SS$ is not uniform. Now suppose that ${}_RU$ is not uniform. Then there exist non-zero elements u and v in U such that $Ru \cap Rv = 0$. It follows that $S(0, u) \cap S(0, v) = 0$, so that again ${}_SS$ is not uniform. $\square$

Corollary 4.2. *Let ${}_RM$ be a non-zero module with annihilator $\mathfrak{a}$ in R. Then $\mathfrak{a} \ltimes 0$ is an irreducible ideal of $R \ltimes M$ if and only if M is a uniform module.*

Proof. Set $S := R \ltimes M$ and $B := \mathfrak{a} \ltimes 0$. Observe that the map

$$R \ltimes M \to (R/\mathfrak{a}) \ltimes M, \quad (r, x) \mapsto (r + \mathfrak{a}, x),$$

is a surjective ring morphism with kernel B, so it induces a ring isomorphism $S/B \simeq (R/\mathfrak{a}) \ltimes M$.

Assume that ${}_RM$ is uniform. Because M is clearly a faithful uniform $(R/\mathfrak{a})$-module, the S/B-module S/B is uniform by Lemma 4.1, and so, the S-module S/B is also uniform, which says exactly that B is an irreducible ideal of S, as desired.

Conversely assume that B is an irreducible ideal of S, and set $R' := R/\mathfrak{a}$ and $S' := S/B$. Then M is an R'-module, and $S' \simeq R' \ltimes M$. Since B is an irreducible ideal of S, S' is uniform as an S'-module, so $R' \ltimes M$ is uniform as an $R' \ltimes M$-module. Let m_1, m_2 be non-zero elements of M. Then there exist (r_1, x_1), $(r_2, x_2) \in R' \ltimes M$ such that

$$(0, 0) \neq (r_1, x_1)(0, m_1) = (r_2, x_2)(0, m_2) = (0, r_1m_1) = (0, r_2m_2),$$

and so, $0 \neq r_1 m_1 = r_2 m_2$. This shows that ${}_R M$ is uniform. $\square$

LEMMA 4.3. *Let $N = \bigcap_{i \in I} N_i$ be an irredundant decomposition of a proper submodule N of an R-module M. Then $N^{\#} = \bigcap_{i \in I} N_i^{\#}$ is an irredundant decomposition of the ideal $N^{\#}$ of S.*

Proof. Note that $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{a}_i$. It can easily be seen that $N^{\#} = \bigcap_{i \in I} N_i^{\#}$ and that $N^{\#} \neq \bigcap_{i \in J} N_i^{\#}$, for every proper subset J of I. This proves the result. $\square$

We introduce one further piece of notation. Let $\mathfrak{a}$ be any ideal of R. Then $\mathfrak{a}^+$ will denote the set of elements of $R \rtimes M$ of the form (a, m) with $a \in \mathfrak{a}$ and $m \in M$, i.e., $\mathfrak{a}^+ = \mathfrak{a} \rtimes M$. Note that if $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}^+$ is a prime ideal of $R \rtimes M$ and that, moreover, every prime ideal of S is of the form $\mathfrak{p}^+$ for some prime ideal $\mathfrak{p}$ of R (see, e.g., Anderson and Winders [4, Theorem 3.2]).

PROPOSITION 4.4. *Let M be an R-module, $N \leqslant M$, and $\mathfrak{a} := (N : M)$. If $S := R \rtimes M$ and $N^{\#} := \mathfrak{a} \rtimes N$, then the following assertions hold.*

(1) *N is an irreducible submodule of $M \Leftrightarrow N^{\#}$ is an irreducible ideal of S.*

(2) *N is a CI submodule of $M \Leftrightarrow N^{\#}$ is a CI ideal of S.*

(3) *N is a primal submodule of M with adjoint prime ideal $\mathfrak{p} \Leftrightarrow N^{\#}$ is a primal ideal of S with adjoint prime ideal $\mathfrak{p}^+$.*

Proof. (1) Clearly, M/N becomes an $(R/\mathfrak{a})$-module by defining

$$(r + \mathfrak{a})(m + N) = rm + N, \quad \forall\, r \in R,\, m \in M.$$

Note that M/N is a faithful $(R/\mathfrak{a})$-module. Set $T := (R/\mathfrak{a}) \rtimes M/N$. Then the mapping $S \to T$, $(r, m) \mapsto (r + \mathfrak{a}, m + N)$, $r \in R$, $m \in M$, is a surjective ring morphism with kernel $N^{\#}$, which induces a ring isomorphism $S/N^{\#} \simeq T$. Now apply Lemma 4.1.

(2) Suppose that N is a CI submodule of M. Then there exists a submodule L of M, $N < L$ and L/N is a simple essential submodule of the R-module M/N. Let $K = \mathfrak{a} \rtimes L$. Then $N^{\#} \leqslant K$. Let $r \in R$, $m \in M$ such that $s = (r, m) \notin N^{\#}$. Suppose that $r \in \mathfrak{a}$. Then $m \notin N$ so that $L \subseteq N + Rm$. Thus, for any $x \in L$ there exist $u \in N$ and $c \in R$ such that $x = u + cm$. Now, $(0, x) = (-cr, u) + (c, 0)(r, m) \in N^{\#} + Ss$. It follows that $K \subseteq N^{\#} + Ss$. Now, suppose that $r \notin \mathfrak{a}$, hence rM is not contained in N. It follows that $L \subseteq N + rM$. For any $y \in L$ there exist $v \in N$ and $w \in M$ such that $y = v + rw$ and, in this case, $(0, y) = (0, v) + (0, w)(r, m) \in N^{\#} + Ss$. Again, it follows that $K \subseteq N^{\#} + Ss$. Thus, $K/N^{\#}$ is a simple essential submodule of the S-module $S/N^{\#}$, hence $N^{\#}$ is a CI ideal of S. Conversely, suppose that $N^{\#}$ is a CI ideal of S. Let $L_i\, (i \in I)$ be any collection of submodules of M

such that $N = \bigcap_{i\in I} L_i$. By Lemma 4.3, $N^{\#} = \bigcap_{i\in I} L_i^{\#}$, hence $N^{\#} = L_i^{\#}$ for some i in I. It follows that $N = L_i$. Hence N is CI.

(3) Suppose that N is a $\mathfrak{p}$-primal submodule of M,

$$\mathfrak{p} = \{r \in R \mid \exists\, u \in M \setminus N,\, ru \in N\}.$$

We are going to show that $\mathfrak{p}^+$ coincides with the set $Z(S/N^{\#})$ of all zero divisors of the S-module $S/N^{\#}$, which means precisely that $N^{\#}$ is a primal ideal of S with adjoint prime ideal $\mathfrak{p}^+$. Let $(p,m) \in \mathfrak{p}^+$, where $p \in \mathfrak{p}$ and $m \in M$. There exists $v \in M \setminus N$ such that $pv \in N$. Then $(p,m)(0,v) = (0,pv) \in N^{\#}$ but $(0,v) \notin N^{\#}$. Thus, $\mathfrak{p}^+ \subseteq Z(S/N^{\#})$. Now, suppose that there exists $(d,y) \in Z(S/N^{\#}) \setminus \mathfrak{p}^+$. There exist elements $r' \in R$ and $m' \in M$ such that $(d,y)(r',m') \in N^{\#}$ but $(r',m') \notin N^{\#}$. Now, $dr' \in \mathfrak{a}$ which implies that $dr'M \subseteq N$. But $d \notin \mathfrak{p}$ then gives $r'M \subseteq N$, hence $r' \in \mathfrak{a}$. Moreover, $dm' + r'y \in N$. But $r' \in \mathfrak{a}$, so that $r'y \in N$ and hence $dm' \in N$. Since $d \notin \mathfrak{p}$, we have $m' \in N$. We have proved that $(r',m') \in N^{\#}$, which is a contradiction. It follows that $Z(S/N^{\#}) \subseteq \mathfrak{p}^+$, as desired.

Conversely, suppose that $N^{\#}$ is a primal ideal of S with adjoint prime ideal $\mathfrak{p}^+$. Let $p \in P$. Then $(p,0)(r,m) \in N^{\#}$ for some $r \in R, m \in M$ such that $(r,m) \notin N^{\#}$. Note that $pr \in \mathfrak{a}$ and $pm \in N$. Suppose that $m \in N$. Then $r \notin \mathfrak{a}$, so that $rM \nsubseteq N$ but $prM \subseteq N$. It is now clear that $p \in \operatorname{adj} N$. Hence $P \subseteq \operatorname{adj} N$. Now, suppose that $a \in \operatorname{adj} N$. There exists an element $x \in M$ such that $ax \in N$ but $x \notin N$. This implies that $(a,0)(0,x) \in N^{\#}$, hence $(a,0) \in \mathfrak{p}^+$. In other words, $a \in P$. Thus $\operatorname{adj} N = \mathfrak{p}$ and N is $\mathfrak{p}$-primal. □

Remark 4.5. If N is a prime submodule of M then $N^{\#}$ is never a prime ideal of $R \ltimes M$ because every prime ideal of $R \ltimes M$ contains the ideal $0 \ltimes M$ with square 0. □

The next result generalizes Fuchs, Heinzer and Olberding [9, Theorem 3.2].

THEOREM 4.6. *Let M be an R-module and N a proper submodule of M such that*

$$N = \bigcap_{i\in I} K_i = \bigcap_{j\in J} L_j$$

are irredundant intersection representations of N in terms of CI submodules $K_i\,(i \in I)$ and $L_j\,(j \in J)$ of M. Then

(i) *Each maximal ideal of R occurring as the adjoint prime ideal of some submodule K_i occurs as the adjoint prime ideal of some submodule L_j. Moreover, if a maximal ideal occurs a finite number of times in one intersection then it occurs the same number of times in the other intersection.*

(ii) *For each $i \in I$ there exists $j \in J$ such that replacing K_i by L_j in the first intersection gives another irredundant intersection representation of N.*

Proof. Let $S := R \ltimes M$ be the trivial extension of M by R. By Lemma 4.3 and Proposition 4.4,

$$N^{\#} = \bigcap_{I \in I} K_i^{\#} = \bigcap_{j \in J} L_j^{\#}$$

are irredundant intersections of the ideal $N^{\#}$ of S in terms of CI ideals $K_i^{\#}$ $(i \in I)$ and $L_j^{\#}$ $(j \in J)$ of S. Moreover, if $\mathfrak{p}_i$ is the adjoint prime ideal of K_i then $\mathfrak{p}_i^{+}$ is the adjoint prime ideal of $K_i^{\#}$ for each $i \in I$, and there is a similar description for the adjoint prime ideals of $L_j^{\#}$ for each $j \in J$. The result now follows by Fuchs, Heinzer and Olberding [9, Theorem 3.2]. $\square$

5. ARITHMETICAL MODULES

In this section we extend from rings to modules Fuchs, Heinzer and Olberding [8, Theorem 1.8] characterizing arithmetical rings in terms of primal ideals.

Recall that a (commutative) ring R is called *arithmetical* provided the lattice of all ideals of R is distributive, or equivalently, if the local ring $R_{\mathfrak{p}}$ is a chained ring for every maximal ideal $\mathfrak{p}$ of R. If R is an arbitrary ring then, as in Albu and Năstăsescu [3], we say that an R-module M is *arithmetical* provided the lattice $\mathcal{L}(M)$ of all submodules of M is distributive, or equivalently, if the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is chained for every maximal ideal $\mathfrak{p}$ of R (see Albu and Năstăsescu [3, Proposition 1.3 and Théorème 1.6]). The arithmetical modules are also known as *distributive* modules (see Stephenson [11]).

Lemma 5.1 (Stephenson [11, Theorem 1.6]). *Given any ring R, an R-module M is arithmetical if and only if*

$$(Rx : Ry) + (Ry : Rx) = R$$

for all elements x and y of M. $\square$

Lemma 5.2. *Let $\mathfrak{p}$ be a maximal ideal of a ring R and let N be a finitely generated submodule of an R-module M. Let $L = \{m \in M \mid (1-p)m \in \mathfrak{p}N$ for some $p \in \mathfrak{p}\}$. Then L is a $\mathfrak{p}$-primal submodule of M or $(1-q)N = 0$ for some $q \in \mathfrak{p}$.*

Proof. Suppose first that $N \subseteq L$. For each $x \in N$ there exists $a \in \mathfrak{p}$ such that $(1-a)x \in \mathfrak{p}N$ and hence $x \in \mathfrak{p}N$. Thus, $N = \mathfrak{p}N$. Because N is finitely generated, the usual determinant argument gives that $(1-b)N = 0$ for some $b \in \mathfrak{p}$. Now, suppose that $N \nsubseteq L$. Then $\mathfrak{p}N \subseteq L$ implies that $\mathfrak{p} \subseteq Z(M/L)$. To prove now the other inclusion, let $r \in Z(M/L)$. There exists $m \in M \backslash L$ such that $rm \in L$. Hence $(1-p)rm \in \mathfrak{p}N$ for some $p \in \mathfrak{p}$. Suppose that $r \notin \mathfrak{p}$. Then $rs = 1-q$ for some $s \in R$ and $q \in \mathfrak{p}$. But this implies $(1-p)(1-q)m \in \mathfrak{p}N$

so that $m \in L$, a contradiction. It follows that $r \in \mathfrak{p}$. Thus $Z(M/L) \subseteq \mathfrak{p}$ and we conclude $Z(M/L) = \mathfrak{p}$. Thus, $\mathfrak{p} = Z(M/L)$, hence L is $\mathfrak{p}$-primal. $\square$

THEOREM 5.3. *Let R be any commutative ring. Then an R-module M is arithmetical if and only if every primal submodule of M is irreducible.*

Proof. Suppose first that M is arithmetical. Let K be any primal submodule of M. Then K is a proper submodule of M. Let $Z = Z(M/K)$ and note that Z is a proper ideal of R. There exists a maximal ideal $\mathfrak{p}$ of R such that $Z \subseteq \mathfrak{p}$. Suppose that K is not irreducible. Then there exist submodules G and H of M, both properly containing K such that $K = G \cap H$. Let $g \in G \backslash H$ and $h \in H \backslash G$. By Lemma 5.1, $R = (Rg : Rh) + (Rh : Rg)$. Thus, without loss of generality we can suppose that there exists $c \in \mathfrak{p}$ such that $(1 - c)g \in Rh$. Then $(1 - c)g \in G \cap H = K$, hence $1 - c \in Z \subseteq \mathfrak{p}$, a contradiction. Thus K is irreducible.

Conversely, suppose that M is not arithmetical. By Lemma 5.1 there exist $u, v \in M$ such that $(Ru : Rv) + (Rv : Ru) \subseteq \mathfrak{q}$, for some maximal ideal $\mathfrak{q}$ of R. Let $N = Ru + Rv$. Suppose that $(1 - q)N = 0$ for some $q \in \mathfrak{q}$. Then $(1 - q)u = 0 \in Rv$ and hence $1 - q \in \mathfrak{q}$, a contradiction. Thus, $(1 - q)N \neq 0$ for all $q \in \mathfrak{q}$. It follows that $N \neq \mathfrak{q}N$ so that, without loss of generality, $(1 - p)u \notin \mathfrak{q}N$ for all $p \in \mathfrak{q}$. By Lemma 5.2 the submodule $L = \{m \in M : (1 - d)m \in \mathfrak{q}N \text{ for some } d \in \mathfrak{q}\}$ is $\mathfrak{q}$-primal. Note that $\mathfrak{q}N \subseteq L$, hence $(N + L)/L$ is semisimple. Suppose that L is irreducible. Then $(N + L)/L$ is simple. Since $u \notin L$ we have $N \subseteq Ru + L$. There exist $s \in R$ and $z \in L$ such that $v = su + z$. But $(1 - e)z = q_1u + q_2v$ for some $q_1, q_2 \in \mathfrak{q}$, hence $(1 - e - q_2)v \in Ru$, a contradiction. Thus L is a primal submodule which is not irreducible. $\square$

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