# A NOTE ON GENERALIZED PATH ALGEBRAS

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We develop the theory of generalized path algebras as defined by Coelho and Xiu [4]. In particular, we focus on the relation between a set of algebras and its associated generalized path algebra for a given quiver. Explicitly, we describe the modules over a generalized path algebra by means of generalized linear representation of the generalized quiver in a similar way as stated for standard path algebras. Last, in the finite dimensional case, we find the Gabriel quiver (in the usual sense) of a generalized path algebra.

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*Key words:* generalized path algebra, Gabriel quiver.

## 1. INTRODUCTION AND PRELIMINARIES

The Representation Theory of Algebras has provided many worthwhile results, and is nowadays considered a classic and fruitful theory. For that reason, in the literature, there are different efforts trying to extend it to a wider context and generalize methods already known for finite dimensional algebras to a broader framework. Among these tools, the quiver-theoretical techniques developed by Gabriel and his school is mostly accepted as one of the most powerful of them, see for example [2], [3] and [6]. In this work we deal with a generalization of the well known notion of path algebra of a quiver: generalized path algebras.

Generalized path algebras were defined by Coelho and Xiu [4]. The idea of such algebras is to focus on the vertices of a quiver and endow each of them with a structure of algebra which is not necessary the ground field (as usually done for the ordinary path algebras). This way gives us a method to obtain more examples of algebras starting from a given set of them. In this note, we attend to the relation between this new algebra and the set of original ones. In particular, in Section 2, we study the modules over a generalized path algebra and we get a relation between the category of modules over a generalized path algebra and the category of generalized linear representation of the generalized quiver finding a similar result as stated for standard path algebras (Theorem 2.4), see [2]. Last, in Section 3, we describe the ordinary

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quiver of a finite dimensional generalized path algebra by means of the ordinary quivers of the algebras attached to the set of vertices (Theorem 3.3).

Throughout $K$ will be an algebraically closed field. Following Gabriel [5], by a quiver, $Q$, we mean a quadruple $(Q_0, Q_1, s, e)$ where $Q_0$ is the set of vertices (points), $Q_1$ is the set of arrows and for each arrow $\alpha \in Q_1$, the vertices $s(\alpha)$ and $e(\alpha)$ are the source (or start point) and the sink (or end point) of $\alpha$, respectively (see [2], [3] and [6]).

If $i$ and $j$ are vertices, an (oriented) path in $Q$ of length $m$ from $i$ to $j$ is a formal composition $p = \alpha_m \cdots \alpha_2 \alpha_1$ of arrows, where $s(\alpha_1) = i$, $e(\alpha_m) = j$ and $e(\alpha_{k-1}) = s(\alpha_k)$, for $k = 2, \ldots, m$. To any vertex $i \in Q_0$ we attach a trivial path of length 0, say $e_i$, starting and ending at $i$ such that $\alpha e_i = \alpha$ (resp. $e_j \beta = \beta$) for any arrow $\alpha$ (resp. $\beta$) with $s(\alpha) = i$ (resp. $e(\beta) = i$). We identify the set of vertices and the set of trivial paths. A cycle is a path which starts and ends at the same vertex.

Let $KQ$ be the $K$-vector space generated by the set of all paths in $Q$. Then $KQ$ can be endowed with the structure of a (non necessarily unitary) $K$-algebra with multiplication induced by concatenation of paths, that is, if $\alpha = \alpha_m \cdots \alpha_2 \alpha_1$ and $\beta = \beta_n \cdots \beta_2 \beta_1$ then

$$\alpha\beta = \begin{cases} \alpha_m \cdots \alpha_2 \alpha_1 \beta_n \cdots \beta_2 \beta_1 & \text{if } e(\beta_n) = s(\alpha_1), \\ 0 & \text{otherwise;} \end{cases}$$

$KQ$ is the path algebra of the quiver $Q$. The algebra $KQ$ can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m \oplus \cdots ,$$

where $Q_m$ is the set of all paths of length $m$ and $Q_0$ is a complete set of primitive orthogonal idempotents of $KQ$. If $Q_0$ is finite then $KQ$ is unitary, and it is clear that $KQ$ has finite dimension if and only if $Q$ is finite and has no cycles. For each $n \in \mathbb{N}$, we denote by $KQ_{\geq n}$ the ideal of the path algebra $KQ$ generated by the paths in $Q$ of length greater or equal than $n$.

We denote by ${}_R\mathcal{M}^f$ and ${}_R\mathcal{M}$ the category of finitely generated and all left modules over the ring $R$, respectively.

For completeness, we remind the famous Gabriel theorem for finite dimensional algebras, see [2], [3] and [6] for details. We recall that the Gabriel quiver, $Q_A$, of a finite dimensional algebra $A$ may be obtained considering as vertices the complete set of primitive orthogonal idempotent elements, say $\{e_1, e_2, \ldots, e_n\}$, and whose number of arrows from a vertex $e_i$ to a vertex $e_j$ is given by the dimension of the vector space $e_j(J/J^2)e_i$, where $J$ denotes the Jacobson radical of $A$.

Theorem 1.1 (Gabriel Theorem). *Let $K$ be an algebraically closed field. Then every basic finite dimensional algebra $A$ is isomorphic to a quotient*

*$KQ_A/\Omega$, where $\Omega$ is an ideal of $KQ_A$ such that*

$$K(Q_A)_{\geq n} \subseteq \Omega \subseteq K(Q_A)_{\geq 2}$$

*for some integer $n \geq 2$.*

*Moreover, there exists a $K$-linear equivalence of categories*

$$F :_A \mathcal{M} \to \mathrm{Rep}_K(Q, \Omega)$$

*between the category of left $A$-modules and linear representations of the quiver with relations $(Q, \Omega)$. This equivalence restricts to an equivalence*

$$F :_A \mathcal{M}^f \to \mathrm{rep}_K(Q, \Omega)$$

*between the category of finitely generated left $A$-modules and finite dimensional linear representations of $(Q, \Omega)$.*

*Remark* 1.2. In the literature, such an ideal $\Omega$ is usually called an admissible ideal of the path algebra $KQ$.

Let now $Q = (Q_0, Q_1)$ be an acyclic and finite quiver, i.e., it has no cycle and the sets $Q_0$ and $Q_1$ are finite. Let $\mathcal{A} = \{A_i\}_{i\in Q_0}$ be a set of finite dimensional $K$-algebras indexed by the set of vertices. We call the pair $(Q, \mathcal{A})$ a generalized quiver. Following [4], an $\mathcal{A}$-path of length $n$ from $x \in Q_0$ to $y \in Q_0$ is a formal expression

$$a_n\beta_n a_{n-1}\beta_{n-1}\cdots a_1\beta_1 a_0,$$

where $\beta_n \cdots \beta_1$ is an (ordinary) path in $Q$ of length $n$ from $x$ to $y$, $a_i \in A_{e(\beta_i)}$ for all $i = 1, \ldots, n$ and $a_0 \in A_{s(\beta_1)}$. The elements of the set $\bigcup_{i=1}^n A_i$ are called the zero-length $\mathcal{A}$-paths. Let us consider the $K$-vector space generated by the set of all $\mathcal{A}$-paths modulo the subspace of all expressions of the form

$$a_{n+1}\beta_n\cdots\beta_{j+1}(a_j^1+\cdots+a_j^m)\beta_j\cdots\beta_1 a_0 - \sum_{l=1}^{m}(a_{n+1}\beta_n\cdots\beta_{j+1}a_j^l\beta_j\cdots\beta_1 a_0)$$

This quotient vector space is denoted by $K(Q, \mathcal{A})$. We may endow $K(Q, \mathcal{A})$ with a structure of $K$-algebra given by the following multiplication. For each two elements $a = a_{n+1}\beta_n\cdots a_1\beta_1 a_0$ and $b = b_{m+1}\gamma_m\cdots b_1\gamma_1 b_0$ in $K(Q, \mathcal{A})$, define

$$ab = \begin{cases} a_{n+1}\beta_n\cdots a_1\beta_1(a_0 b_{m+1})\gamma_m\cdots b_1\gamma_1 b_0 & \text{if } s(\beta_1) = e(\gamma_m), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $K(Q, \mathcal{A})$ has unit if and only if $Q_0$ is finite and the algebra $A_i$ has unit, say $1_{A_i}$, for all $i \in Q_0$. In such a case the unit element is given by $1 = 1_{A_1} + \cdots + 1_{A_n}$. $K(Q, \mathcal{A})$ is finite dimensional if and only if $Q$ is finite and acyclic and the algebra $A_i$ is finite dimensional for all $i \in Q_0$.

Throughout, for any algebra $A$, we denote by $J(A)$ the Jacobson radical of $A$, see [1] for basic facts and properties of the Jacobson radical of a ring. Let now $Q = (Q_0, Q_1)$ be a quiver and $\mathcal{A} = \{A_i\}_{i \in Q_0}$ be a family of algebras indexed by the set of vertices. We say that an $\mathcal{A}$-path $p$ is regular if it is either a zero-length path in $J(A_i)$ for some vertex $i \in Q_0$ which does not belong to a cycle in $Q$, or $p = a_{n+1}\beta_n \cdots a_1\beta_1 a_0$, where $\beta_n \cdots \beta_1$ is not a subpath of a cycle in $Q$ (that is, it is a regular path in $Q$ in the usual sense), see [4].

We may calculate the Jacobson radical of $K(Q, \mathcal{A})$ by means of the regular $\mathcal{A}$-paths. The next result is proved in [4].

Proposition 1.3. *Let $Q = (Q_0, Q_1)$ be a quiver and $\mathcal{A} = \{A_i\}_{i \in Q_0}$ be a family of algebras indexed by the set of vertices. Then, the Jacobson radical of $K(Q, \mathcal{A})$ is generated by the set of all regular $\mathcal{A}$-paths.*

## 2. GENERALIZED LINEAR REPRESENTATIONS

Let $Q = (Q_0, Q_1)$ be a finite and acyclic quiver and $\mathcal{A} = \{A_i\}_{i \in Q_0}$ be a set of algebras indexed by the set of vertices of $Q$. A generalized $K$-linear representation of $(Q, \mathcal{A})$ is a system

$$(X_i, \varphi_\alpha)_{i \in Q_0,\, \alpha \in Q_1},$$

where $X_i$ is an $A_i$-module for each $i \in Q_0$, and $\varphi_\alpha : X_i \to X_j$ is a morphism of $K$-vector spaces for each arrow $\alpha : i \to j$ in $Q_1$. The generalized linear representation $(X_i, \varphi_\alpha)_{i \in Q_0,\, \alpha \in Q_1}$ is said to be finitely generated if $X_i$ is finitely generated as $A_i$-module for all $i \in Q_0$. Given two representations $(X_i, \varphi_\alpha)_{i \in Q_0,\, \alpha \in Q_1}$ and $(Y_i, \psi_\alpha)_{i \in Q_0,\, \alpha \in Q_1}$ of $(Q, \mathcal{A})$, a morphism of representations is a system $f = (f_i)_{i \in Q_0}$, where $f_i : X_i \to Y_i$ is a morphism of $A_i$-modules such that the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_\alpha} & X_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ Y_i & \xrightarrow{\psi_\alpha} & Y_j \end{array}$$

is commutative if $\alpha : i \to j$ is an arrow in $Q_1$. It is clear that the (finitely generated) generalized linear representations of $(Q, \mathcal{A})$ form a category which we denote by $(\mathrm{rep}_K(Q, \mathcal{A}))$ $\mathrm{Rep}_K(Q, \mathcal{A})$. Let us recall the following well known result (see [3] for the definition of the tensor algebra):

Lemma 2.1. *Let $\Sigma$ and $\Delta$ be two rings and $V$ a $\Sigma$-bimodule. Let $f : \Sigma \oplus V \to \Delta$ be a map satisfying the conditions below.*

(a) *The restriction $f_{|\Sigma} : \Sigma \to \Delta$ is a morphism of rings.*

(b) *The restriction* $f_{|V} : V \to \Delta$ *is a morphism of* $\Sigma$*-bimodules* (*viewing* $\Delta$ *as a* $\Sigma$*-bimodule via* $f_{|\Sigma}$).

*Then there exists a unique morphism* $\widetilde{f} : T(\Sigma, V) \to \Delta$ *of rings between the tensor algebra* $T(\Sigma, V)$ *and the ring* $\Delta$ *such that* $\widetilde{f}_{|\Sigma \oplus V} = f$.

Using the previous lemma we may consider $K(Q, \mathcal{A})$ as a tensor algebra: let $K(Q, \mathcal{A})_0$ be the vector space generated by the zero-length $\mathcal{A}$-paths and $K(Q, \mathcal{A})_1$ the vector space generated by the one-length $\mathcal{A}$-paths. Then we have the inclusion

$$i : K(Q, \mathcal{A})_0 \oplus K(Q, \mathcal{A})_1 \to K(Q, \mathcal{A}),$$

hence, by Lemma 2.1, there exists a unique ring morphism

$$f : T(K(Q, \mathcal{A})_0, K(Q, \mathcal{A})_1) \to K(Q, \mathcal{A}).$$

It is easy to see that $f$ is an isomorphism of algebras.

Our aim now is to give an equivalence between module categories and categories of representations as stated in Gabriel's theorem for standard path algebras. For that purpose we define the functor

$$F : \mathrm{Rep}_K(Q, \mathcal{A}) \to_{K(Q,\mathcal{A})} \mathcal{M}$$

between the category of generalized linear representations of $(Q, \mathcal{A})$ and the category of left $K(Q, \mathcal{A})$-modules, as

$$F(X) = \bigoplus_{i=0}^{n} X_i$$

for each generalized linear representation $X = (X_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Let us show that $F(X)$ is a left $K(Q, \mathcal{A})$-module. To do that, it is enough to have a ring morphism

$$\phi : K(Q, \mathcal{A}) \to \mathrm{End}\,(F(X))\,.$$

Consider the maps

$$\phi_0 : K(Q, \mathcal{A})_0 \to \mathrm{End}(F(X))$$

and

$$\phi_1 : K(Q, \mathcal{A})_1 \to \mathrm{End}(F(X))$$

defined as follows.

(a) For any $a_i \in A_i$ and $(x_1, \ldots, x_n) \in F(X)$, set

$$\phi_0(a_i)(x_1, \ldots, x_n) = (0, \ldots, a_i x_i, \ldots, 0),$$

where $a_i x_i$ is placed in the $i$-th coordinate. This is clearly a morphism of rings.

(b) For each $a_j \alpha a_i \in K(Q, \mathcal{A})_1$ and $(x_1, \ldots, x_n) \in F(X)$, set

$$\phi_1(a_j \alpha a_i)(x_1, \ldots, x_n) = (0, \ldots, a_j \varphi_\alpha(a_i x_i), \ldots, 0),$$

where $a_j\varphi_\alpha(a_ix_i)$ is placed in the $j$-th coordinate. It is easy to see that $\phi_1$ is a morphism of $K(Q,\mathcal{A})_0$-bimodules. Note that since $K(Q,\mathcal{A})$ is a $K$-algebra, $k\alpha(x_1,\dots,x_n) = \alpha k(x_1,\dots,x_n)$, so

$$(0,\dots,k\varphi_\alpha(x_i),\dots,0) = (0,\dots,\varphi_\alpha(kx_i),\dots,0)$$

and then $k\varphi_\alpha(x_i) = \varphi_\alpha(kx_i)$. That is, $\varphi_\alpha$ is a $K$-linear map.

Therefore, by Lemma 2.1, we obtain the desired ring morphism $\phi$ while $F(X)$ is endowed with a structure of left $K(Q,\mathcal{A})$-module. That is, the functor $F$ is well defined.

*Remark* 2.2. Note that the functor $F$ restricts to a functor

$$F : \mathrm{rep}_K(Q,\mathcal{A}) \to_{K(Q,\mathcal{A})} \mathcal{M}^f$$

between the category of finitely generated generalized linear representations of $(Q,\mathcal{A})$ and the category of finitely generated left $K(Q,\mathcal{A})$-modules.

For simplicity, we will write $1_i$ instead of $1_{A_i}$ for any $i = 1,\dots,n$. We remind that

$$M = 1_{K(Q,A)}M = (1_1 + \cdots + 1_n)M = 1_1M \oplus \cdots \oplus 1_nM$$

for each left $K(Q,\mathcal{A})$-module $M$. Then we may define the functor

$$G :_{K(Q,\mathcal{A})} \mathcal{M} \to \mathrm{Rep}_K(Q,\mathcal{A})$$

by

$$G(M) = (X_i,\varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1},$$

where $X_i = 1_iM$ for any $i \in Q_0$ and $\varphi_\alpha : 1_iM \to 1_jM$ is given by $\varphi_\alpha(x) = \alpha\cdot x$ for each $\alpha \in Q_1$.

On the other hand, if $f : M \to N$ is a morphism of $K(Q,\mathcal{A})$-modules then we set $G(f) = (f_{|1_iM})_{i\in Q_0}$, where $f_{|T}$ denotes the restriction of $f$ to a submodule $T \subseteq M$.

The functor $G$ is well defined. Indeed,

(a) If $x \in 1_iM$ then $x = 1_im$ for some $m \in M$. Therefore

$$\varphi_\alpha(x) = (\alpha\cdot 1_i)\cdot m = 1_j\cdot(\alpha\cdot m) \in 1_jM.$$

Thus $\varphi_\alpha$ is well defined.

(b) Since $\varphi_\alpha(\lambda x) = \alpha\cdot\lambda x = \lambda(\alpha\cdot x) = \lambda\varphi_\alpha(x)$ for any $\lambda \in K$, any $x \in 1_iM$ and any arrow $\alpha : i \to j$ in $Q$, the map $\varphi_\alpha$ is $K$-linear.

c) We have

$$(f_{|1_jM}\circ\varphi_\alpha)(m) = f_{|1_jM}(\alpha\cdot m) = \alpha\cdot(f_{|1_iM}(m)) = (\psi_\alpha\circ f_{|1_iM})(m)$$

for any $m \in 1_iM$. Then $f_{|1_jM}\circ\varphi_\alpha = \psi_\alpha\circ f_{|1_iM}$ for all $\alpha : i \to j$. Thus $f$ is a morphism of generalized linear representations.

*Remark* 2.3. It is obvious that $G$ also restricts to a functor between ${}_{K(Q,\mathcal{A})}\mathcal{M}^f$ and $\text{rep}_K(Q,\mathcal{A})$.

Now, we are able to prove the main result of this section.

THEOREM 2.4. *The functor*

$$F : \text{Rep}_K(Q,\mathcal{A}) \to_{K(Q,\mathcal{A})} \mathcal{M}$$

*is a $K$-linear equivalence of categories between the category of generalized $K$-linear representations of the generalized quiver $(Q,\mathcal{A})$ and the category of left $K(Q,\mathcal{A})$-modules.*

*Moreover, $F$ restricts to an equivalence*

$$F : \text{rep}_K(Q,\mathcal{A}) \to_{K(Q,\mathcal{A})} \mathcal{M}^f$$

*between the category of finitely generated generalized linear representations of $(Q,\mathcal{A})$ and finitely generated left $K(Q,\mathcal{A})$-modules.*

*Proof.* By the above discussion it only remains to prove that $F$ and $G$ are inverse to each other. Let $(X_i,\varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$ be a generalized linear representation. Then

$$GF(X_i,\varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1} = G\left(\bigoplus_{i=1}^{n} X_i\right) = (\varepsilon_i(X_i),\varphi'_\alpha)_{i\in Q_0,\,\alpha\in Q_1},$$

where $\varepsilon_i$ is given by $\varepsilon_i(X_i) = (0,\ldots,X_i,\ldots,0)$ for each $i \in Q_0$, and

$$\varphi'_\alpha(0,\ldots,x_i,\ldots,0) = (0,\ldots,\varphi_\alpha(x_i),\ldots,0)$$

for each arrow $\alpha : i \to j$ in $Q_1$. Taking $\varepsilon = (\varepsilon_i)_{i\in Q_0}$, it is clear that

$$\varepsilon : (X_i,\varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1} \to (\varepsilon_i(X_i),\varphi'_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$$

is an isomorphism of generalized linear representations. Thus, $GF \cong 1_{\text{Rep}_K(Q,A)}$.

Let now $M$ be a right $K(Q,\mathcal{A})$-module. Then

$$FG(M) = F(1_i M,\varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1} = \bigoplus_{i=1}^{n} 1_i M \cong M.$$

This completes the proof. $\square$

## 3. FINITE DIMENSIONAL GENERALIZED PATH ALGEBRAS

In this section we deal with the relation between the generalized quiver of a finite-dimensional generalized path algebra and its standard Gabriel quiver. For simplicity, we introduce the following notation. Let $Q$ be any quiver and

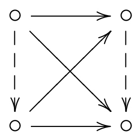
$\mathfrak{Q} = \{Q_i\}_{i \in Q_0}$ a family of quivers indexed by the set of vertices of $Q$. Denote by $Q_{\mathfrak{Q}}$ the quiver described as follows:

- the set of vertices $(Q_{\mathfrak{Q}})_0 = \bigcup_{i \in Q_0} (Q_i)_0$;
- for each pair of vertices $a \in Q_i$ and $b \in Q_j$ with $i, j \in Q_0$, if $i = j$ then the number of arrows from $a$ to $b$ is the number of arrows from $a$ to $b$ in $Q_i$ while if $i \neq j$ then the number of arrows from $a$ to $b$ is the number of arrows from $i$ to $j$ in $Q$.

*Example* 3.1. Let us consider the quiver $Q$ : ① ⟶ ② and the set of quivers $\mathfrak{Q} = \{Q_1, Q_2\}$, where

$$Q_1 : \circ \longrightarrow \circ \qquad \text{and} \qquad Q_2 : \circ \longrightarrow \circ$$

Then the quiver $Q_{\mathfrak{Q}}$ is

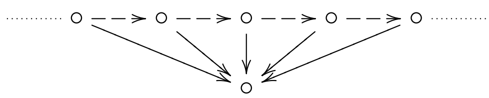


where the dashed arrows correspond to the arrows of the quivers $Q_1$ and $Q_2$.

*Example* 3.2. Observe that there is no condition on any quiver $Q_i \in \mathfrak{Q}$ in the above definition. For instance, let us consider the quiver $Q$ of the previous example, the quiver $Q_2$ formed by only one vertex and without loops and the infinite quiver $Q_1$ below:



Then $Q_{\mathfrak{Q}}$ is the quiver



THEOREM 3.3. *Let $Q$ be a finite and acyclic quiver with $Q_0 = \{1, \ldots, n\}$, $K$ an algebraically closed field and $\mathcal{A} = \{A_1, \ldots, A_n\}$ a set of finite dimensional basic $K$-algebras. Let us suppose that $\mathfrak{Q} = \{Q_1, \ldots, Q_n\}$ is a set of quivers such that $A_i \cong KQ_i/(\Omega_i)$ as algebras for all $i = 1, \ldots, n$, where $\Omega_i$ is an admissible ideal of $KQ_i$. Then $K(Q, \mathcal{A}) \cong KQ_{\mathfrak{Q}}/(\Omega_1, \ldots, \Omega_n)$.*

*Proof.* Clearly, $K(Q, \mathcal{A})$ is a finite dimensional algebra. Let us denote by $J_i$ the Jacobson radical of $A_i$ for all $i = 1, \ldots, n$, and by $J$ the Jacobson radical of $K(Q, A)$. By Proposition 1.3, $J$ is generated by the $\mathcal{A}$-paths of length greater that zero and the set $\{J_1, J_2, \ldots, J_n\}$. Then $A_1, \ldots, A_n$ are basic if and only if $A_i/J_i \cong \bigoplus_{j=1}^{k_i} D_j^i$ for all $i = 1, \ldots, n$, where $D_j^i$ ia a division rings for any $i$ and $j$. Therefore, $K(Q, A)/J \cong \bigoplus_{i=1}^{n} A_i/J_i \cong \bigoplus_{i,j} D_j^i$ and then $K(Q, A)$ is also basic. Therefore, by Gabriel's theorem (Theorem 1.1), there exists a finite quiver $Q'$ and an admissible ideal $\Omega$ in $KQ'$ such that $K(Q,\mathcal{A}) \cong KQ'/\Omega$.

Let us consider $E_i = \{e^i_1, e^i_2, \dots, e^i_{k_i}\}$ a complete set of primitive orthogonal idempotent elements of $A_i$ for any $i = 1, \dots, n$. Then

$$1_{K(Q,A)} = 1_1 + \cdots + 1_n = \sum_{i=1}^{n} \sum_{j=1}^{k_i} e^i_j,$$

thus $E = \{e^i_j\}^{i=1,\dots,n}_{j=1,\dots,k_i}$ is a complete set of primitive orthogonal idempotent elements of $K(Q,A)$. Hence the quiver $Q'$ has $\sum_{i=1}^n k_i$ vertices which are in one-to-one correspondence with the elements of the set $E$.

Let us now calculate the arrows of $Q'$. For this purpose we recall the facts below.

(0) The zero-length $\mathcal{A}$-paths in $J^2$ are the elements in $J_i^2$ for all $i = 1, \dots, n$. Consequently, the classes of the zero-length $\mathcal{A}$-paths in $J/J^2$ are the elements in $J_i/J_i^2$ for all $i = 1, \dots, n$.

(1) The one-length $\mathcal{A}$-paths in $J^2$ are the one-length $\mathcal{A}$-paths $a\alpha b$ such that $\alpha : i \to j$ is an arrow in $Q$ and either $b \in J_i$ or $a \in J_j$ for some $i, j \in Q_0$. Consequently, the classes of the one-length $\mathcal{A}$-paths in $J/J^2$ are the $\mathcal{A}$-paths $a\alpha b$ such that $\alpha : i \to j$ is an arrow in $Q$, $a \in A_j/J_j$ and $b \in A_i/J_i$ for some $i, j \in Q_0$. For simplicity, we denote this space by $\mathcal{J}_1$.

(+1) Every $\mathcal{A}$-path of length greater than one is contained in $J^2$. Consequently, the classes of such elements in $J/J^2$ are zero.

Summarizing, $J/J^2$ is generated by the elements in $\bigcup_{i\in Q_0} J_i/J_i^2$ and the one-length $\mathcal{A}$-paths $a\alpha b$ such that $\alpha : i \to j$ is an arrow in $Q$, $a \in A_j/J_j$ and $b \in A_i/J_i$ for some $i, j \in Q_0$.

For the reader's convenience, we shall denote by $p(e^i_j, e^l_m)$ the number of arrows in $Q'$ from $e^i_j$ to $e^l_m$ for any $j \in \{1, \dots, k_i\}$, $m \in \{1, \dots, k_l\}$ and $i, l \in \{1, \dots, n\}$.

We shall distinguish two cases:

(a) Two vertices $e^i_j$ and $e^i_m$ are associated with the same quiver $Q_i$ for some $i \in \{1, 2, \dots, n\}$. Since $Q$ has no cycle, we have

$$p(e^i_j, e^i_m) = \dim_K \left(e^i_m (J/J^2) e^i_j\right) = \dim_K \left(e^i_m (J_i/J_i^2) e^i_j\right).$$

Hence $p(e^i_j, e^i_m)$ is the number of arrows from $e^i_j$ to $e^i_m$ in $Q_i$.

(b) Two vertices $e^i_j$ and $e^l_m$ are in $Q_{\mathfrak{Q}}$ with $i \neq l$. Then

$$\begin{aligned} p(e^i_j, e^l_m) &= \dim_K\big(e^l_m(J/J^2)e^i_j\big) = \dim_K\big(e^l_m(\mathcal{J}_1)e^i_j\big) \\ &= \sum_{\alpha: i\to l} \dim_K\big(e^l_m(A_l/J_l)\alpha(A_i/J_i)e^i_j\big) \\ &= \sum_{\alpha: i\to l} \dim_K\big(e^l_m(A_l/J_l)\big)\cdot \dim_K\big((A_i/J_i)e^i_j\big) \\ &= \sum_{\alpha: i\to l}\sum_{s=1}^{k_l}\sum_{t=1}^{k_i} \dim_K\big(e^l_m(A_l/J_l)e^l_s\big)\cdot \dim_K\big(e^i_t(A_i/J_i)e^i_j\big) \\ &= \sum_{\alpha: i\to l}\sum_{s=1}^{k_l}\sum_{t=1}^{k_i} \delta_{m,s}\cdot\delta_{j,t} = \sum_{\alpha: i\to l} 1. \end{aligned}$$

Therefore, $p(e^i_j, e^l_m)$ is the number of arrows from $i$ to $l$ in $Q$. This proves that the standard Gabriel quiver of $K(Q,\mathcal{A})$ is the quiver $Q_{\mathfrak{Q}}$.

Let us now consider the isomorphisms of algebras

$$f_i : KQ_i/(\Omega_i) \to A_i$$

for any $i = 1,\ldots,n$, as obtained from the method of the proof of the Gabriel theorem, see for instance [3]. Then we can get a surjective morphism of algebras

$$g : KQ_{\mathfrak{Q}} \to K(Q,A)$$

as follows. Consider the morphism of algebras

$$g_0 : (KQ_{\mathfrak{Q}})_0 \to K(Q,A)$$

defined by $g_0(e^i_j) = f_i(e^i_j) = e^i_j$ for any $j = 1,\ldots,k_i$ and $i = 1,\ldots,n$. Also let

$$g_1 : (KQ_{\mathfrak{Q}})_1 \to K(Q,A)$$

be the map defined by

$$g_1(\alpha) = \begin{cases} f_i(\alpha) & \text{if } i = l, \\ e^l_m \alpha e^i_j & \text{if } i \neq l \end{cases}$$

for any arrow $\alpha$ in $Q_{\mathfrak{Q}}$ from $e^i_j$ to $e^l_m$. Clearly, $g_1$ is a morphism of $(KQ_{\mathfrak{Q}})_0$-bimodules (viewing $K(Q,A)$ as a $(KQ_{\mathfrak{Q}})_0$-bimodule via $g_0$). Therefore, by Lemma 2.1, there is a unique morphism of algebras

$$g : KQ_{\mathfrak{Q}} \to K(Q,A).$$

This map may be also described as follows:

- $g(p) = f_i(p)$ for any path $p \in Q_i \subset Q_{\mathfrak{Q}}$;
- $g(\alpha) = e^l_m \alpha e^i_j$ for any arrow $\alpha : e^i_j \to e^l_m$ such that $i \neq l$.

Then it is obvious that $g$ is surjective and

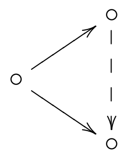
$$\operatorname{Ker} g = (\operatorname{Ker} f_1, \ldots, \operatorname{Ker} f_n) = (\Omega_1, \ldots, \Omega_n).$$

Therefore,

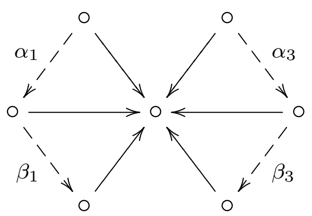
$$g : \frac{KQ_{\mathfrak{Q}}}{(\Omega_1, \ldots, \Omega_n)} \to K(Q, A)$$

is an isomorphism. $\square$

*Example* 3.4. Let $Q$ be the quiver $\circ \longrightarrow \circ$ and $\mathcal{A} = \{A_1, A_2\}$, where $A_1 = K$ and $A_2 = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$. Then $K(Q, \mathcal{A})$ is the path algebra of the quiver



*Example* 3.5. Let $Q$ be the quiver ① ⟶ ② ⟵ ③ and $\mathcal{A} = \{A_1, A_2, A_3\}$, where $A_2 = K$ and $A_1 = A_3$ is the quotient $KQ_1/(\beta\alpha)$ with $Q_1$ the quiver $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$ . Then the generalized path algebra $K(Q, \mathcal{A}) \cong K\Gamma/(\alpha_1\beta_1, \alpha_3\beta_3)$, where $\Gamma$ is the quiver

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