HIGHER-ORDER SYMMETRIC MULTIOBJECTIVE DUALITY INVOLVING GENERALIZED (F, ρ, γ, b)-CONVEXITY

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We define the higher-order (F, ρ, γ, b)-convexity and the generalized higher-order (F, ρ, γ, b)-convexity. For a pair of symmetric primal-dual higher-order multiobjective programming problems involving support functions, we prove higher-order weak, strong and converse duality theorems under appropriate higher-order (F, ρ, γ, b)-convexity conditions.

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1. INTRODUCTION

Beside the well known Wolfe dual [10], Mond and Weir [14] proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudoconvexity / quasi-convexity assumptions.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used [7, 9, 12, 19]. Mangasarian [9] considered a nonlinear programming and discussed second order duality under inclusion condition. Mond [12] was the first who present second order convexity. He also gave in [12] simpler conditions than these of Mangasarian using a generalized form of convexity which was later called second order convexity by Mahajan [8] and bonvexity by Bector and Chandra [3]. Zhang and Mond [20] established some duality theorems for second-order duality in nonlinear programming under second order B-invexity or generalized second-order B-invexity, defined in their paper. In [2, 14] it was shown that second order duality can be useful from computational point of view, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve n-set functions was studied by Preda [17].

The case of higher-order symmetric duality for nondifferentiable multiobjective programming problems was considered by Chen [4]. Here, the duality results are given under higher-order F-convexity or generalized higher-order F-convexity assumptions.

In this paper, we start by defining in Section 2 the higher-order (F, ρ, γ, b)-convexity and generalized higher-order (F, ρ, γ, b)-convexity. In Section 3 we formulate a pair of symmetric higher-order multiobjective programming problems considered by Chen [4], where each of the objective function contains a support function of a compact convex set. Relative to the classes of functions introduced in Section 2, under appropriate higher-order (F, ρ, γ, b)-convexity conditions, where F is not necessary a sublinear function, we prove the higer-order weak, higher-order strong and higher-order converse duality theorems related to a properly efficient solution.

2. NOTATION AND DEFINITIONS

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $\mathbb{R}^n_+$ its nonnegative orthant. Further, $\mathbb{R}^*_+ = \{x \in \mathbb{R} \mid x > 0\}$.

For any vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, we denote: $x^\top y = \sum_{i=1}^{n} x_i y_i$.

Let $C \subset \mathbb{R}^n$ be a compact convex set. The support function of C is defined by

$$s(x \mid C) = \max \left\{ x^\top y \mid y \in C \right\}.$$

Being convex and everywhere finite, it has a subdifferential [18], that is, there exists $z \in \mathbb{R}^n$ such that

$$s(y \mid C) \geq s(x \mid C) + z^\top (y - x) \quad \text{for all } y \in C.$$

The subdifferential of $s(x \mid C)$ is given by

$$\partial s(x \mid C) = \left\{ z \in C \mid z^\top x = s(x \mid C) \right\}.$$

For any set $D \subset \mathbb{R}^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \left\{ y \in \mathbb{R}^n \mid y^\top (z - x) \leq 0, \ \text{for all } z \in D \right\}.$$

For a compact convex set C we obviously have $y \in N_C(x)$ if and only if $s(y \mid C) = x^\top y$, or equivalently, if $x \in \partial s(y \mid C)$.

Let us consider $H:\mathbb{R}^n\to\mathbb{R}^p$, $G:\mathbb{R}^n\to\mathbb{R}^q$, and $X\subset\mathbb{R}^n$. We define the multiobjective programming problem

$$\text{(P)}\qquad \begin{cases}\text{minimize } H(x)\\ \text{subject to } G(x)\geq 0,\ x\in X\end{cases}$$

We denote the set of feasible solutions of (P) by $\mathcal{P}$, that is,

$$\mathcal{P}=\{x\in X\mid G(x)\geq 0\}.$$

Definition 2.1. A vector $\bar{x}\in\mathcal{P}$ is an efficient solution of (P) if there exists no other $x\in\mathcal{P}$ such that $H(\bar{x})-H(x)\in\mathbb{R}^p_+\setminus\{0\}$, that is, $H_i(x)\leq H_i(\bar{x})$ for all $i\in\{1,\dots,p\}$, and for at least one $j\in\{1,\dots,p\}$ we have $H_j(x)<H_j(\bar{x})$; $\bar{x}\in\mathcal{P}$ is said to be a weak efficient solution of (P) if there exists no $x\in\mathcal{P}$ such that for all $i\in\{1,\dots,p\}$, $H_i(x)<H_i(\bar{x})$.

Definition 2.2. An efficient solution $\bar{x}\in\mathcal{P}$ of (P) is properly efficient, if there exists a positive number M such that, whenever $H_i(x)<H_i(\bar{x})$ for $x\in\mathcal{P}$ and $i\in\{1,\dots,p\}$, there exists $j\in\{1,\dots,p\}$ such that $H_j(x)>H_j(\bar{x})$ and $\frac{H_i(\bar{x})-H_i(x)}{H_j(x)-H_j(\bar{x})}\leq M$.

We denote by $\nabla f(\bar{x})$ the gradient vector at $\bar{x}$ of a differentiable function $f:\mathbb{R}^p\to\mathbb{R}$, and by $\nabla^2 f(\bar{x})$ the Hessian matrix of f at $\bar{x}$. For a real-valued twice differentiable function $\psi(x,y)$ defined on an open set in $\mathbb{R}^p\times\mathbb{R}^q$, we denote by $\nabla_x\psi(\bar{x},\bar{y})$ the gradient vector of ψ with respect to x at $(\bar{x},\bar{y})$, and by $\nabla_{xx}\psi(\bar{x},\bar{y})$ the Hessian matrix with respect to x at $(\bar{x},\bar{y})$. Similarly, we may define $\nabla_y\psi(\bar{x},\bar{y})$, $\nabla_{xy}\psi(\bar{x},\bar{y})$ and $\nabla_{yy}\psi(\bar{x},\bar{y})$.

Let us consider a function $F:X\times X\times\mathbb{R}^n\to\mathbb{R}$ (where $X\subseteq\mathbb{R}^n$) with the property that for all $(x,y)\in X\times X$, we have

$$\text{(i)}\qquad F(x,y;\,\cdot\,)\text{ is a convex function,}$$

$$\text{(ii)}\qquad F(x,y;0)\geq 0.$$

If F satisfies (i) and (ii), we obviously have $F(x,y;-a)\geq -F(x,y;a)$ for any $a\in\mathbb{R}^n$.

Let us consider, for example, $F(x,y;a)=\|a\|+\|a\|^2$, where a depends on x and y. This function satisfies (i) and (ii), but it is neither sub-additive, nor positive homogeneous, that is, the relations

$$\text{(i}'\text{)}\qquad F(x,y;a+b)\leq F(x,y;a)+F(x,y;b),$$

$$\text{(ii}'\text{)}\qquad F(x,y;\lambda a)=\lambda F(x,y;a)$$

are not fulfilled for any $a,b\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, $\lambda\geq 0$.

We may conclude that the class of functions that verify (i) and (ii) is more general than the class of sub-linear functions with respect the third argument, i.e. those which satisfy (i′) and (ii′). We notice that till now, most results in optimization theory were stated under generalized convexity assumptions involving functions F which are sub-linear. The results of this paper are obtained by using weaker assumptions with respect to the function F.

Let us consider functions $b : X \times X \to \mathbb{R}_+$, $d : X \times X \to \mathbb{R}_+$, $\gamma : X \times X \to \mathbb{R}_+^*$, and a number $\rho \in \mathbb{R}$. Further, we suppose that $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ satisfies (i) and (ii) and that $\varphi : X \to \mathbb{R}$ and $h : X \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions. We introduce in the subsequent definition the class of higher-order (F, ρ, γ, b)-convexity.

Definition 2.3. • We say that φ is higher-order (F, ρ, γ, b)-convex at $u \in X$ with respect to h, if for all $(x, y) \in X \times \mathbb{R}^n$ we have

$$\begin{aligned} b(x,u)(\varphi(x) - \varphi(u)) \geq F(x, u; \gamma(x,u)[\nabla\varphi(u) + \nabla_y h(u,y)]) + \\ + b(x,u)\left(h(u,y) - y^\top [\nabla_y h(u,y)]\right) + \rho d(x,u). \end{aligned} \tag{2.1}$$

• We say that φ is higher-order (F, ρ, γ, b)-pseudo-convex at $u \in X$ with respect to h, if for all $(x, y) \in X \times \mathbb{R}^n$ we have

$$\begin{aligned} F(x, u; \gamma(x,u)[\nabla\varphi(u) + \nabla_y h(u,y)]) \geq -\rho d(x,u) \\ \Longrightarrow b(x,u)(\varphi(x) - \varphi(u)) \geq b(x,u)\left(h(u,y) - y^\top [\nabla_y h(u,y)]\right). \end{aligned}$$

• We say that φ is higher-order (F, ρ, γ, b)-quasi-convex at $u \in X$ with respect to h, if for all $(x, y) \in X \times \mathbb{R}^n$ we have

$$\begin{aligned} b(x,u)(\varphi(x) - \varphi(u)) \leq b(x,u)\left(h(u,y) - y^\top [\nabla_y h(u,y)]\right) \\ \Longrightarrow F(x, u; \gamma(x,u)[\nabla\varphi(u) + \nabla_y h(u,y)]) \leq -\rho d(x,u). \end{aligned}$$

• If φ is higher-order (F, ρ, γ, b)-convex (pseudo/quasi-convex) at each $u \in X$ with respect to the same function h, then φ is said to be higher-order (F, ρ, γ, b)-convex (pseudo/quasi-convex) on X with respect to h.

• If $-\varphi$ is higher-order (F, ρ, γ, b)-convex (pseudo/quasi-convex) at $u \in X$ with respect to h, then φ is said to be higher-order (F, ρ, γ, b)-concave (pseudo/quasi-concave) at $u \in X$ with respect to h.

Remark 2.1. The elements ρ, γ and b give more felexibility that the defined properties hold for all $(x, y) \in X \times \mathbb{R}^n$. In particular, if we consider the case where $\rho = 0$ and $b \equiv 1$, $\gamma \equiv 1$, then

1. The above definition reduce to Definition 4 of Chen [4].

2. When $h(u,y) = y^\top \nabla_{uu}\varphi(u)\, y/2$ and $F(x,u;a) = \eta(x,u)^\top a$, where $\eta : X \times X \to \mathbb{R}^n$, the higher-order (F, ρ, γ, b)-(pseudo/quasi)-convexity reduces to η-(pseudo/quasi)-bonvexity in [5, 16].

3. When $h(u,y) = y^\top \nabla_{uu}\varphi(u)\, y/2$ and $F(x,u;a) = \eta(x,u)^\top a$, where $\eta : X \times X \to \mathbb{R}^n$, the higher-order (F, ρ, γ, b)-(pseudo/quasi)-convexity reduces to the second-order F (pseudo/quasi) invexity in [6].

4. When $h(u,y)=-y^\top \nabla_u \varphi(u)+\psi(u,y)$ and $F(x,u;a)=\alpha(x,u)a^\top \eta(x,u)$, where $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\eta : X \times X \to \mathbb{R}^n$ are positive functions, and $\psi : X \times \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, the higher-order (F,ρ,γ,b)-(pseudo/quasi)-convex function becomes the higher-order (pseudo/quasi) type I function in [11, 15].

Example. We present here a function which is higher-order (F,ρ,γ,b)-convex. We can proceed similarly for the other classes of functions introduced in Definition 2.3. Let us consider $X=(0,\infty)$ and

$$\varphi : X \to \mathbb{R},\ \varphi(x)=x\log x, \quad h : X \times \mathbb{R} \to \mathbb{R},\ h(u,y)=-y\log u.$$

Obviously, φ is not convex. We have

$$\nabla_u \varphi(u)=1+\log u, \quad \nabla_{uu}\varphi(u)=\frac{1}{u}, \quad \nabla_y h(u,y)=-\log u.$$

Let us consider $F : X \times X \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,y;a)=|a|+|a|^2$, which satisfies (i) and (ii) and is not sublinear.

Before proceeding, it is worth to notice that in this example we have

$$h(u,y)=-y\log u \neq \frac{1}{2}y^\top \nabla_{uu}\varphi(u)\, y=\frac{y^2}{2u},$$

i.e., $h(u,y)$ has not the form used in [5, 16], as we mentioned in Remark 2.1 (item 2). Further, if we consider the case presented in Remark 2.1 (item 4), we see that

$$-y^\top \nabla_u \varphi(u)+\psi(u,y)=-y(1+\log u)+\psi(u,y),$$

and if we take $\psi(u,y)=y$, then $h(u,y)=-y\log u$ has the same form as in [11, 15], but now our mapping F is not sublinear as required there.

Let us define the functions

$$\begin{aligned} b(x,u) &= \begin{cases} \dfrac{xu(1+xu)}{x\log x-u\log u} & \text{if } (x\log x-u\log u)>0, \\ 0 & \text{if } (x\log x-u\log u)\le 0, \end{cases} \\ \gamma(x,u) &= xu, \quad d(x,u)=xu+x^2u^2. \end{aligned}$$

Since

$$\begin{gathered} \nabla\varphi(u)+\nabla_y h(u,y)=1+\log u-\log u=1, \\ h(u,y)-y\nabla_y h(u,y)=-y\log u-y(-\log u)=0, \end{gathered}$$

we have

$$F(x,u;\gamma(x,u)[\nabla\varphi(u)+\nabla_y h(u,y)])=xu+x^2u^2,$$

and relation (2.1) becomes

$$1\ge 1+\rho \text{ if } (x\log x-u\log u)>0, \quad 0\ge 1+\rho \text{ if } (x\log x-u\log u)\le 0.$$

It now follows that for $\rho \leq -1$ the function $\varphi(x) = x \log x$ is higher-order (F, ρ, γ, b)-convex at $u \in X$ with respect to $h(u, y) = -y \log u$.

3. HIGHER-ORDER SYMMETRIC DUALITY

We consider in this section twice differentiable functions

$$f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, \quad h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

and compact convex sets $C_i \subset \mathbb{R}^n$ and $D_i \subset \mathbb{R}^m$, for $i = 1, 2, \ldots, p$.

We define the following pair of higher-order symmetric multiobjective dual problems (cf. Chen [4]).

$$\text{(MP)} \quad \text{minimize} \begin{pmatrix} f_1(x,y) + s(x \,|\, C_1) - y^\top z_1 + h_1(x,y,\pi_1) - \pi_1^\top \left[\nabla_{\pi_1} h_1(x,y,\pi_1)\right] \\ \vdots \\ f_p(x,y) + s(x \,|\, C_p) - y^\top z_p + h_p(x,y,\pi_p) - \pi_p^\top \left[\nabla_{\pi_p} h_p(x,y,\pi_p)\right] \end{pmatrix}$$

subject to

$$\sum_{i=1}^{p} \lambda_i \left[\nabla_y f_i(x,y) - z_i + \nabla_{\pi_i} h_i\left(x,y,\pi_i\right)\right] \leq 0, \tag{3.1}$$

$$y^\top \sum_{i=1}^{p} \lambda_i \left[\nabla_y f_i\left(x,y\right) - z_i + \nabla_{\pi_i} h_i\left(x,y,\pi_i\right)\right] \geq 0, \tag{3.2}$$

$$z_i \in D_i, \; i = 1, \ldots, p, \; \lambda > 0, \; \sum_{i=1}^{p} \lambda_i = 1, \tag{3.3}$$

and

$$\text{(MD)} \quad \text{maximize} \begin{pmatrix} f_1(u,v) - s(v \,|\, D_1) + u^\top w_1 + g_1(u,v,\mu_1) - \mu_1^\top \left[\nabla_{\mu_1} g_1(u,v,\mu_1)\right] \\ \vdots \\ f_p(u,v) - s(v \,|\, D_p) + u^\top w_p + g_p(u,v,\mu_p) - \mu_p^\top \left[\nabla_{\mu_p} g_p(u,v,\mu_p)\right] \end{pmatrix}$$

subject to

$$\sum_{i=1}^{p} \lambda_i \left[\nabla_u f_i\left(u,v\right) + w_i + \nabla_{\mu_i} g_i\left(u,v,\mu_i\right)\right] \geq 0, \tag{3.4}$$

$$u^\top \sum_{i=1}^{p} \lambda_i \left[\nabla_u f_i\left(u,v\right) + w_i + \nabla_{\mu_i} g_i\left(u,v,\mu_i\right)\right] \leq 0, \tag{3.5}$$

$$w_i \in C_i,\ i = 1, \ldots, p,\ \lambda > 0,\ \lambda^\top e = 1. \tag{3.6}$$

Since the objective functions of (MP) and (MD) contain the support functions $s\,(x \mid C_i)$ and $s\,(v \mid D_i)\,, i = 1, \ldots, p$, these problems are nondifferentiable multiobjective programming problems.

Remark 3.1. If $p = 1$ and $h_1\,(x, y, \pi_1) = \pi_1^\top \nabla_{yy} f(x, y)\pi_1/2$, $g_1\,(u, v, \mu_1) = \mu_1^\top \nabla_{xx} f\,(u, v)\,\mu_1/2$, then (MP) and (MD) become the second-order symmetric problems considered by Hou and Yang [6]. In addition, if C_1 is defined by

$$C_i = \left\{ B_i v \mid v^\top B_i v \le 1 \right\},$$

we obtain the pair of Wolfe type second order nondifferentialbe symmetric programs considered by Ahmad and Husain [1]. Also, in this case, if $\pi_1 = \mu_1 = 0$, we obtain a pair of symmetric dual nondifferentiable programs considered in Mond and Schechter [13].

In the sequel we shall establish weak, strong and converse duality theorems under (F, ρ, γ, b)-convexity type assumptions. For this, we consider functions $b_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, $d : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, $\gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+^*$, $\gamma' : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+^*$ and numbers $\rho_i, \rho_i' \in \mathbb{R}$, $i = 1, \ldots, p$. Further, we suppose that the functions $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ enjoy properties (i) and (ii) and satisfy the condition

$$\begin{aligned} &F\,(x, u; \gamma(x, y)\alpha) + \alpha^\top y \ge 0 \text{ for all } \alpha \in \mathbb{R}_+^n \\ &G\,(v, y; \gamma'\,(x, u)\,\beta) + \beta^\top y \ge 0 \text{ for all } \beta \in \mathbb{R}_+^m. \end{aligned} \tag{3.7}$$

We suppose also that following conditions are satisfied:

(j1) the functions $f_i\,(\,\cdot\,, v) + (\,\cdot\,)^\top w_i$ are higher-order (F, ρ_i, γ, b_i)-convex at u with respect to $g_i\,(u, v, \mu_i)\,, i = 1, 2, \ldots, p$;

(j2) the functions $f_i\,(x,\,\cdot\,) - (\,\cdot\,)^\top z_i$ are higher-order $(G, \rho_i', \gamma', b_i)$-concave at y with respect to $-h_i\,(x, y, \pi_i)\,, i = 1, 2, \ldots, p$;

(j3) $\sum_{i=1}^{p} \lambda_i\,(\rho_i + \rho_i') \ge 0$.

Theorem 3.1 (Weak duality). *Let* $(x, y, \lambda, z_1, z_2, \ldots, z_p, \pi_1, \pi_2, \ldots, \pi_p)$ *be a feasible solution of* (MP) *and* $(u, v, \lambda, w_1, w_2, \ldots, w_p, \mu_1, \mu_2, \ldots, \mu_p)$ *a feasible solution of* (MD). *Then the inequalities below cannot hold simultaneously:*

(I) *for all* $i \in \{1, 2, \ldots, p\}\,,$

$$\begin{aligned} &f_i(x, y) + s\,(x \mid C_i) - y^\top z_i + h_i\,(x, y, \pi_i) - \pi_i^\top\,[\nabla_\pi h_i\,(x, y, \pi_i)] \le \\ &\le f_i\,(u, v) - s\,(v \mid D_i) + u^\top w_i + g_i\,(u, v, \mu_i) - \mu_i^\top\,[\nabla_\mu g_i\,(u, v, \mu_i)]\,; \end{aligned} \tag{3.8}$$

(II) *for at least one* $j \in \{1, 2, \dots, p\}$,

$$f_j(x,y) + s(x \mid C_j) - y^\top z_j + h_j(x,y,\pi_j) - \pi_j^\top [\nabla_\pi h_j(x,y,\pi_j)] < \\ < f_j(u,v) - s(v \mid D_j) + u^\top w_j + g_j(u,v,\mu_j) - \mu_j^\top [\nabla_\mu g_j(u,v,\mu_j)]. \tag{3.9}$$

Proof. Since $(x, y, \lambda, z_1, z_2, \dots, z_p, \pi_1, \pi_2, \dots, \pi_p)$ is a feasible solution of (MP) and $(u, v, \lambda, w_1, w_2, \dots, w_p, \mu_1, \mu_2, \dots, \mu_p)$ is a feasible solution of (MD), by (3.7) and (3.4) we get

$$F\left(x, u; \gamma(x,y) \sum_{i=1}^p \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_\mu g_i(u,v,\mu_i)]\right) + \\ + \sum_{i=1}^p \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_\mu g_i(u,v,\mu_i)]^\top u \ge 0.$$

By (3.5) we have

$$F\left(x, u; \gamma(x,y) \sum_{i=1}^p \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_\mu g_i(u,v,\mu_i)]\right) \ge 0. \tag{3.10}$$

It follows from the higher-order (F, ρ_i, b_i)-convexity of $f_i(\cdot, v) + (\cdot)^\top w_i$ at u with respect to $g_i(u, v, \mu_i)$ that

$$b_i(x,y,u,v)\left(\left[f_i(x,v) + x^\top w_i\right] - \left[f_i(u,v) + u^\top w_i\right]\right) \ge \\ \ge F(x, u; \gamma(x,y)[\nabla_u f_i(u,v) + w_i + \nabla_\mu g_i(u,v,\mu_i)]) + \\ + b_i(x,y,u,v)\left[g_i(u,v,\mu_i) - \mu_i^\top \nabla_\mu g_i(u,v,\mu_i)\right] + \rho_i d(x,y,u,v). \tag{3.11}$$

Since F satisfies (i) and (ii), and $\lambda > 0$, $\lambda^\top e = 1$, from (3.4), (3.10) and (3.11) we get

$$\sum_{i=1}^p \lambda_i b_i(x,y,u,v)\left(\left[f_i(x,v) + x^\top w_i\right] - \left[f_i(u,v) + u^\top w_i\right]\right) \ge \\ \ge F\left(x, u; \gamma(x,y) \sum_{i=1}^p \lambda_i [\nabla_u f_i(u,v) + w_i + \nabla_\mu g_i(u,v,\mu_i)]\right) + \\ + \sum_{i=1}^p \lambda_i b_i(x,y,u,v)\left[g_i(u,v,\mu_i) - \mu_i^\top \nabla_\mu g_i(u,v,\mu_i)\right] + \sum_{i=1}^p \lambda_i \rho_i d(x,y,u,v) \ge \\ \ge \sum_{i=1}^p \lambda_i b_i(x,y,u,v)\left[g_i(u,v,\mu_i) - \mu_i^\top \nabla_\mu g_i(u,v,\mu_i)\right] + \sum_{i=1}^p \lambda_i \rho_i d(x,y,u,v)$$

that is,

$$\sum_{i=1}^p \lambda_i b_i(x,y,u,v) f_i(x,v) \ge \sum_{i=1}^p \lambda_i b_i(x,y,u,v)[f_i(u,v) - x^\top w_i + u^\top w_i] + \\ + \sum_{i=1}^p \lambda_i b_i(x,y,u,v)[g_i(u,v,\mu_i) - \mu_i^\top \nabla_\mu g_i(u,v,\mu_i)] + \sum_{i=1}^p \lambda_i \rho_i d(x,y,u,v). \tag{3.12}$$

On the other hand, from (3.1) and (3.7) we get

$$G\left(v,y;-\gamma'(v,y)\sum_{i=1}^{p}\lambda_i\left[\nabla_y f_i(x,y)-z_i+\nabla_\pi h_i(x,y,\pi_i)\right]\right)-$$
$$-y^\top\sum_{i=1}^{p}\lambda_i\left[\nabla_y f_i(x,y)-z_i+\nabla_\pi h_i(x,y,\pi_i)\right]\geq 0,$$

which, by using (3.2), imply

$$G\left(v,y;-\gamma'(v,y)\sum_{i=1}^{p}\lambda_i\left[\nabla_y f_i(x,y)-z_i+\nabla_\pi h_i(x,y,\pi_i)\right]\right)\geq 0. \tag{3.13}$$

Now, using the fact that $f_i(x,\,\cdot\,)-(\,\cdot\,)^\top z_i$ is higher-order (G,ρ_i',b_i)-concave at y with respect to $-h_i(x,y,\pi_i)$, we have

$$\begin{aligned}&-b_i(x,y,u,v)\left(\left[f_i(x,v)-v^\top z_i\right]-\left[f_i(x,y)-y^\top z_i\right]\right)\geq\\ &\geq G\left(v,y;-\gamma'(v,y)\left[\nabla_y f_i(x,y)-z_i+\nabla_\pi h_i(x,y,\pi_i)\right]\right)+\\ &+b_i(x,y,u,v)\left[-h_i(x,y,\pi_i)+\pi_i^\top\nabla_\pi h_i(x,y,\pi_i)\right]+\rho_i' d(x,y,u,v).\end{aligned} \tag{3.14}$$

Since G satisfies (i) and (ii), $\lambda>0$, $\lambda^\top e=1$, from (3.13) and (3.14) we have

$$\begin{aligned}\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)f_i(x,v)&\leq\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)[f_i(x,y)+v^\top z_i-y^\top z_i]+\\ &+\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)[h_i(x,y,\pi_i)-\pi_i^\top\nabla_\pi h_i(x,y,\pi_i)]-\sum_{i=1}^{p}\lambda_i\rho_i' d(x,y,u,v).\end{aligned} \tag{3.15}$$

From (3.12) and (3.15) we obtain

$$\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)\left[f_i(u,v)-x^\top w_i+u^\top w_i+g_i(u,v,\mu_i)-\right.$$
$$\left.-\mu_i^\top\nabla_\mu g_i(u,v,\mu_i)\right]+\sum_{i=1}^{p}\lambda_i(\rho_i+\rho_i')d(x,y,u,v)\leq$$
$$\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)\left[f_i(x,y)+v^\top z_i-y^\top z_i+h_i(x,y,\pi_i)-\pi_i^\top\nabla_\pi h_i(x,y,\pi_i)\right].$$

Now, since $\sum_{i=1}^{p}\lambda_i(\rho_i+\rho_i')\geq 0$ and $x^\top w_i\leq s(x\mid C_i)$, $v^\top z_i\leq s(v\mid D_i)$, the last inequality yields

$$\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)[f_i(u,v)-s(v\mid D_i)+u^\top w_i+g_i(u,v,\mu_i)-\mu_i^\top\nabla_\mu g_i(u,v,\mu_i)]\leq$$
$$\leq\sum_{i=1}^{p}\lambda_i b_i(x,y,u,v)[f_i(x,y)+s(x\mid C_i)-y^\top z_i+h_i(x,y,\pi_i)-\pi_i^\top\nabla_\pi h_i(x,y,\pi_i)],$$

which proves the assertion of the theorem. □

Remark 3.2. Following the same lines as in the previous proof, we easily can prove other variants of Theorem 3.1 under the same assumptions, but replacing in the statement the corresponding conditions by those below:

• the functions $f_i(\,\cdot\,, v) + (\,\cdot\,)^\top w_i$ are higher-order (F, ρ_i, γ, b_i)-pseudoconvex at u with respect to $g_i(u, v, \mu_i)$, $i = 1, 2, \ldots, p$;

• the functions $f_i(x,\,\cdot\,) - (\,\cdot\,)^\top z_i$ are higher-order $(G, \rho'_i, \gamma', b_i)$-pseudoconcave at y with respect to $-h_i(x, y, \pi_i)$, $i = 1, 2, \ldots, p$;

respectively,

• the functions $f_i(\,\cdot\,, v) + (\,\cdot\,)^\top w_i$ are higher-order (F, ρ_i, γ, b_i)-quasiconvex at u with respect to $g_i(u, v, \mu_i)$, $i = 1, 2, \ldots, p$;

• the functions $f_i(x,\,\cdot\,) - (\,\cdot\,)^\top z_i$ are higher-order $(G, \rho'_i, \gamma', b_i)$-quasiconcave at y with respect to $-h_i(x, y, \pi_i)$, $i = 1, 2, \ldots, p$.

Remark 3.3. If in Theorem 3.1 we take $b_i(x, y, u, v) \equiv 1$ and $\rho_i = \rho'_i = 0$ for all $i = 1, 2, \ldots, p$, as well as $\gamma(x, y) \equiv 1$ and $\gamma'(v, y) \equiv 1$, then we obtain Theorem 1 of Chen [4].

Now, under appropriate conditions, we state a strong duality and a converse duality theorem relative to problems, (MP) and (MD) which can be proved following the lines in [4].

THEOREM 3.2 (Strong duality). *Let* $\left(\tilde{x}, \tilde{y}, \tilde{\lambda}, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_p, \tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_p\right)$ *be a feasible solution of* (MP) *and assume that*

(k1) *for all* $i \in \{1, \ldots, p\}$ *we have* $h_i(\tilde{x}, \tilde{y}, 0) = 0$, $g_i(\tilde{x}, \tilde{y}, 0) = 0$, $\nabla_\pi h_i(\tilde{x}, \tilde{y}, 0) = 0$, $\nabla_y h_i(\tilde{x}, \tilde{y}, 0) = 0$, $\nabla_x h_i(\tilde{x}, \tilde{y}, 0) = \nabla_\mu g_i(\tilde{x}, \tilde{y}, 0)$;

(k2) *for all* $i \in \{1, 2, \ldots, p\}$ *the Hessian matrix* $\nabla_{\pi\pi} h_i(\tilde{x}, \tilde{y}, \tilde{\pi}_i)$ *is positive or negative definite;*

(k3) *the vectors* $\nabla_y f_i(\tilde{x}, \tilde{y}) - \tilde{z}_i + \nabla_\pi h_i(\tilde{x}, \tilde{y}, \tilde{\pi}_i)$, $i = 1, 2, \ldots, p$, *are linearly independent;*

(k4) *for any* $\alpha \in R^p_+$, $\alpha \neq 0$, *and* $\pi_i \in R^m$, $\pi_i \neq 0$, $i = 1, 2, \ldots, p$, *we have*

$$\sum_{i=1}^{p} \alpha_i \pi_i^\top \left[\nabla_y f_i(\tilde{x}, \tilde{y}) - \tilde{z}_i + \nabla_\pi h_i(\tilde{x}, \tilde{y}, \pi_i)\right] \neq 0.$$

Then

• $\tilde{\pi}_i = 0$, $i = 1, 2, \ldots, p$;

• *there exist* $\tilde{w}_i \in C_i$ *such that* $\left(\tilde{x}, \tilde{y}, \tilde{\lambda}, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_p, \overbrace{0, 0, \ldots, 0}^{p\ times}\right)$ *is a feasible solution of* (MD).

Furthermore, if the assumptions of Theorem 3.1 *are satisfied and the functions* $b_i(\tilde{x}, \tilde{y}, \tilde{x}, \tilde{y}) > 0$, $i = 1, 2, \ldots, p$, *then* $\Big(\tilde{x}, \tilde{y}, \tilde{\lambda}, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_p,$

$\overbrace{0,0,\ldots,0}^{p\ times}\Big)$ *is a properly efficient solution of* (MD) *and the values of both problems are equal.*

THEOREM 3.3 (Converse duality). *Let* $\left(\tilde{u},\tilde{v},\tilde{\lambda},\tilde{w}_1,\ldots,\tilde{w}_p,\tilde{\mu}_1,\ldots,\tilde{\mu}_p\right)$ *be a properly efficient solution of* (MD) *and we assume that*

(k1′) *for all* $i\in\{1,2,\ldots,p\}$ *we have* $h_i(\tilde{u},\tilde{v},0)=0$, $g_i(\tilde{u},\tilde{v},0)=0$, $\nabla_\mu g_i(\tilde{u},\tilde{v},0)=0$, $\nabla_x g_i(\tilde{u},\tilde{v},0)=0$, $\nabla_y g_i(\tilde{u},\tilde{v},0)=\nabla_\pi h_i(\tilde{u},\tilde{v},0)$;

(k2′) *for all* $i\in\{1,2,\ldots,p\}$ *the Hessian matrix* $\nabla_{\mu\mu}g_i(\tilde{u},\tilde{v},\tilde{\mu}_i)$ *is positive or negative definite;*

(k3′) *the vectors* $\nabla_x f_i(\tilde{u},\tilde{v})-\tilde{w}_i+\nabla_\mu g_i(\tilde{u},\tilde{v},\tilde{\mu}_i)$, $i=1,2,\ldots,p$, *are linearly independent;*

(k4′) *for any* $\alpha\in R^p_+$, $\alpha\neq 0$, *and* $\mu_i\in R^n$, $\mu_i\neq 0$, $i=1,2,\ldots,p$, *we have*

$$\sum_{i=1}^{p}\alpha_i\mu_i^\top\left[\nabla_x f_i(\tilde{u},\tilde{v})-\tilde{w}_i+\nabla_\mu g_i(\tilde{u},\tilde{v},\tilde{\mu}_i)\right]\neq 0.$$

Then

- $\mu_i=0$, $i=1,2,\ldots,p$;
- *there exist* $\tilde{z}_i\in D_i$ *such that* $\Big(\tilde{u},\tilde{v},\tilde{\lambda},\tilde{z}_1\ldots,\tilde{z}_p,\overbrace{0,\ldots,0}^{p\ times}\Big)$ *is a feasible solution of problem* (MP).

Furthermore, if the assumptions of Theorem 3.1 *are satisfied and the functions* $b_i(\tilde{u},\tilde{v},\tilde{u},\tilde{v})>0$, $i=1,2,\ldots,p$, *then* $\Big(\tilde{u},\tilde{v},\tilde{\lambda},\tilde{z}_1\ldots,\tilde{z}_p,\overbrace{0,\ldots,0}^{p\ times}\Big)$ *is a properly efficient solution of* (MP) *and the values of both problems are equal.*

We finally notice that the results obtained in this section about higher-order weak, strong and converse duality are obtained by considering higher-order (F,ρ,γ,b)-(pseudo-, quasi-) convexity assumptions. As a consequence, some known results obtained previously in [3, 4, 6, 11, 12, 15, 16] can be derived from ours as particular cases.

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REFERENCES

[1] I. Ahmad and Z. Husain, *On nondifferentiable second order symmetric duality in mathematical programming.* Indian J. Pure Appl. Math. **35(5)** (2004), 665–676.

[2] C.R. Bector and B.K. Bector, *Generalized B-invex functions and second order duality for a non-linear programming problem.* Congr. Numer. **22** (1986), 37–52.

[3] C.R. Bector and S. Chandra, *Second order symmetric and self dual programs.* Opsearch **23** (1986), 89–95.
[4] X. Chen, *Higher-order symmetric duality in nondifferentiable multiobjective programming problems.* J. Math. Anal. Appl. **290** (2004), 423–435.
[5] G. Devi, *Symmetric duality for nonlinear programming problem involving η-bonvex functions.* European J. Oper. Res. **104** (1998), 615–621.
[6] S.H. Hou and X.M. Yang, *On second-order symmetric duality in nondifferentiable programming.* J. Math. Anal. Appl. **255** (2001), 488–491.
[7] V. Jeyakumar, *p-convexity and second order duality.* Utilitas Math. **29** (1986), 71–85.
[8] D.J. Mahajan, *Contributions to optimality conditions and duality theory in nonlinear programming.* Ph.D. Thesis, 1977.
[9] O.L. Mangasarian, *Second and higher order duality in nonlinear programming.* J. Math. Anal. Appl. **51** (1975), 607–620.
[10] S.K. Mishra, *Second order symmetric duality in mathematical programming with F-convexity.* European J. Oper. Res. **127** (2000), 507–518.
[11] S.K. Mishra and N.G. Rueda, *Higher order generalized invexity and duality in mathematical programming.* J. Math. Anal. Appl. **247** (2000), 173–182.
[12] B. Mond, *Second-order duality for nonlinear programming.* Opsearch **11** (1974), 90–99.
[13] B. Mond and M. Schechter, *Nondifferentiable symmetric duality.* Bull. Austral. Math. Soc. **53** (1996), 177–188.
[14] B. Mond and T. Weir, *Generalized convexity and duality.* In: S. Schaible, W.T. Ziemba (Eds.), *Generalized Convexity in Optimization and Economics*, pp. 263–280. Academic Press, New York, 1981.
[15] B. Mond and J. Zhang, *Higher order invexity and duality in mathematical programming.* In: J.P. Crouzeix *et al.* (Eds.), *Generalized Convexity, Generalized Monotonicity: Recent Results*, pp. 357–372. Kluwer, Dordrecht, 1998.
[16] S. Pandey, *Duality for multiobjective fractional programming involving generalized η-bonvex functions.* Opsearch **28** (1991), 36–43.
[17] V. Preda, *Duality for multiobjective fractional programming problems involving n-set functions.* In: C. Andreian Cazacu, W.E. Lehto and T.M. Rassias (Eds.), *Analysis and Topology*, pp. 569–583. Academic Press, 1998.
[18] R.T. Rockafellar, *Convex Analysis.* Princeton Univ. Press, 1970.
[19] X.Q. Yang, *Second order global optimality conditions for convex composite optimization.* Math. Programming **81** (1998), 327–347.
[20] J. Zhang and B. Mond, *Second order B-invexity and duality in mathematical programming.* Utilitas Math. **50** (1996), 19–31.

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