MATRICEAL BLOCH
AND BERGMAN-SCHATTEN SPACES

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The main goal of this paper is to extend some theorems of the papers [ACP] and [Z] concerning the space of analytic Bloch functions, respectively the Bergman space of functions, to infinite matrices. The extension to the matriceal framework will be based on the fact that there is a natural correspondence between Toeplitz matrices and formal Fourier series associated to 2π-periodic distributions. We mention a characterization of diagonal matrices associated to a Bloch matrix using a quadratic form and the fact that the matriceal Bloch space is the dual Banach space of the matriceal Bergman space.

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1. INTRODUCTION

The Bloch functions and the Bloch space have a long history behind them. They were introduced by the French mathematician André Bloch at the beginning of the last century. Many mathematicians payed attention to these functions: L. Ahlfors, J.M. Anderson, J. Clunie, Ch. Pommerenke, P.L. Duren, B.W. Romberg and A.L. Shields are some of them. There were some good papers about this topic (see for example [DRS], [ACP]) and in the more recent past the monographs [Z] and [DS].

Our intention is to introduce a concept of *Bloch matrix* (respectively of *Bergman-Schatten matrix*) which extends the notion of Bloch function (respectively the function from the Bergman space) and to prove some results generalizing those of the papers [ACP] and [Z].

The idea behind our considerations is to consider an infinite matrix A as the analogue of the formal Fourier series associated to a 2π-periodic distribution, the diagonals A_k, $k \in \mathbb{Z}$, being the analogues of the Fourier coefficients associated to such a distribution. In this manner we get a one-to-one correspondence between the infinite Toeplitz matrices and formal Fourier series associated to periodic distributions, hence an infinite matrix appears in a natural way as a more general concept than that of a periodic distribution on the torus. (See the papers [BPP], [BLP] and [BKP] for more information about these concepts.)

For an infinite matrix $A = (a_{ij})$ and an integer k we denote by A_k the matrix whose entries $a'_{i,j}$ are given by

$$a'_{i,j} = \begin{cases} a_{i,j} & \text{if } j - i = k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then A_k will be called *the* kth-*diagonal matrix associated to* A.

In the sequel we need a special type of matrices, *Toeplitz matrices*, defined below.

Definition 1. Let $A = (a_{ij})_{i,j\geq 1}$ be an infinite matrix. If there is a sequence of complex numbers $(a_k)_{k=-\infty}^{+\infty}$ such that $a_{ij} = a_{j-i}$ for all $i, j \in \mathbb{N}$, then A is called *a Toeplitz matrix.*

For simplicity we will write a Toeplitz matrix as $A = (a_k)_{k=-\infty}^{+\infty}$. The class of all Toeplitz matrices will be denoted by $\mathcal{T}$.

We consider on the interval $[0, 1)$ the Lebesgue measurable infinite matrix-valued functions $A(r)$. These functions may be regarded as infinite matrix-valued functions defined on the unit disc D using the correspondence $A(r) \to f_A(r, t) = \sum_{k=-\infty}^{\infty} A_k(r)\mathrm{e}^{\mathrm{i}kt}$, where $A_k(r)$ is the kth-diagonal of the matrix $A(r)$. The preceding sum is a formal one and t belongs to the torus $\mathbb{T}$.

We may consider $f_A(r, t)$, also denoted by $f_A(z)$, where $z = r\mathrm{e}^{\mathrm{i}t}$, as a matrix valued distribution (resp. formal series). Such a matrix $A(r)$ is called an *analytic matrix* if there exists an upper triangular infinite matrix A such that for all $r \in [0, 1)$ we have $A_k(r) = A_k r^k$ for all $k \in \mathbb{Z}$.

In what follows we identify the analytic matrices $A(r)$ with their corresponding upper triangular matrices A and also call the latter *analytic matrices.*

Let us denote by $A * B$ the *Schur product* of the matrices A and B, that is, the matrix having as entries the products of corresponding entries of these matrices. $C(r)$ means the *Cauchy matrix*, that is the Toeplitz matrix corresponding to the *Cauchy kernel* $\frac{1}{1-r}$.

2. MATRICEAL BLOCH SPACE

We now define the matriceal analog of the classical Bloch space of analytic functions.

Definition 2. The matriceal Bloch space $\mathcal{B}(D, \ell_2)$ is the space of all analytic matrices A with $A(r) \in B(\ell_2)$, $0 \leq r < 1$, such that

$$\|A\|_{\mathcal{B}(D,\ell_2)} \stackrel{\text{def}}{=} \sup_{0\leq r<1} (1 - r^2)\|A'(r)\|_{B(\ell_2)} + \|A_0\|_{B(\ell_2)} < \infty,$$

where $\|A\|_{B(\ell_2)}$ is the usual operator norm of the matrix A on the space ℓ_2, and $A'(r) = -\frac{\mathrm{i}}{r}\frac{\partial f_A(r,t)}{\partial t}$.

A matrix $A \in \mathcal{B}(D, \ell_2)$ is called a *Bloch matrix.*

It is clear that Toeplitz matrices which belong to the Bloch space $\mathcal{B}(D, \ell_2)$ of analytic matrices coincide with Bloch functions. So, $\mathcal{B}(D, \ell_2)$ appears as an extension of the classical space $\mathcal{B}$ of Bloch functions.

Now, we give some properties of Bloch matrices, which extend the corresponding properties of Bloch functions.

It is known that in [ACP] a characterization of Taylor coefficients of Bloch functions in terms of a quadratic form is given. We want to extend this result to infinite matrices.

Let us recall the definition of the space $\mathcal{I}$ from the paper [ACP]: $\mathcal{I} = \{g : D \to \mathbb{C} \mid \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |g'(z)| \mathrm{d}\theta\, \mathrm{d}r + |g(0)| < \infty\}$, equipped with the norm $\|g\|_{\mathcal{I}} = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |g'(z)| \mathrm{d}\theta\, \mathrm{d}r$.

Then the following result holds.

Lemma 3. *Let $A \in \mathcal{B}(D, \ell_2)$; $A = \sum_{n=0}^{\infty} A_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{I}$. Then $h(z) = \sum_{n=0}^{\infty} A_n b_n z^n : \overline{D} \to B(\ell_2)$ is a continuous function in $|z| \le 1$ and we have*

$$\|h(z)\|_{B(\ell_2)} \le 2\|A\|_{\mathcal{B}(D,\ell_2)}\|g\|_{\mathcal{I}} \tag{1}$$

for all $\|z\| \le 1$.

In particular, it follows that there exists

$$\langle A, g\rangle = \lim_{\rho\to 1^-} \sum_{n=0}^{\infty} A_n b_n \rho^n = \lim_{\rho\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} A(\rho \mathrm{e}^{-\mathrm{i}\theta}) g(\rho \mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta,$$

for all $A \in \mathcal{B}(D, \ell_2)$, $g \in \mathcal{I}$.

Proof. Let $\|\zeta\| < 1$. We have $A'(z) = f_A'(z) = \sum_{n=1}^{\infty} nA_n z^{n-1}$ and $\frac{\mathrm{d}}{\mathrm{d}z}[z(g(z) - b_0)] = \sum_{n=1}^{\infty}(n+1)b_n z^n$. Then we easily get that for $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $\zeta \in D$ we have

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi}(1-r^2)A'(\zeta\overline{z})\frac{\mathrm{d}}{\mathrm{d}z}[z(g(z)-b_0)]\,\mathrm{e}^{-\mathrm{i}\theta}\mathrm{d}\theta\,\mathrm{d}r = \sum_{n=1}^{\infty} A_n b_n \zeta^{n-1}.$$

Using Hölder's inequality we get

$$\Big\|\sum_{n=1}^{\infty} A_n b_n \zeta^n\Big\|_{B(\ell_2)} \le$$

$$\le \sup_{|z|<1}(1-|z|^2)\|A'(\zeta\overline{z})\|_{B(\ell_2)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}(|g(z)-b_0| + r|g'(z)|)\mathrm{d}\theta\,\mathrm{d}r.$$

We further have

$$\int_0^1\int_0^{2\pi}|g(re^{i\theta})-b_0|\mathrm{d}\theta\,\mathrm{d}r\le\int_0^1\int_0^{2\pi}\int_0^r|g'(te^{i\theta})|\mathrm{d}t\mathrm{d}\theta\,\mathrm{d}r=$$

$$=\int_0^{2\pi}\int_0^1\left(\int_t^1\mathrm{d}r\right)|g'(te^{i\theta})|\mathrm{d}\theta\mathrm{d}t=\int_0^{2\pi}\int_0^1(1-t)|g'(te^{i\theta})|\mathrm{d}\theta\mathrm{d}t.$$

Since $z\to\|A'(z)\|_{B(\ell_2)}$ is a subharmonic function, we get

$$\left\|\sum_{n=0}^{\infty}A_nb_n\zeta^n\right\|_{B(\ell_2)}\le$$

$$\le\|A_0b_0\|_{B(\ell_2)}+\sup_{z\in D}(1-|z|^2)\|A'(z)\|_{B(\ell_2)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|g'(te^{i\theta})|\mathrm{d}\theta\mathrm{d}t.$$

Hence

$$\|h(\zeta)\|_{B(\ell_2)}\le 2\|A\|_{\mathcal{B}(D,\ell_2)}\|g\|_{\mathcal{I}}$$

for $|\zeta|<1$.

In order to show the continuity of h in $|z|\le 1$, we take ζ_1, $\zeta_2\in D$ and note that

$$\|h(\zeta_1)-h(\zeta_2)\|_{B(\ell_2)}=\left\|\sum_{n=0}^{\infty}A_n(b_n\zeta_1^n-b_n\zeta_2^n)\right\|_{B(\ell_2)}\le 2\|A\|_{\mathcal{B}(D,\ell_2)}\|g(\zeta_1-g(\zeta_2)\|_{\mathcal{I}}.$$

But it is known that the last norm converges to 0 as $|\zeta_1-\zeta_2|\to 0$. (See Theorem 2.2 [ACP].)

Hence h can be extended by continuity to $\overline{D}$ and we get (1). □

THEOREM 4. *Let $A=\sum_k A_k$ be a Bloch matrix. Then the inequality*

$$\left\|\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{A_{\mu+\nu+1}}{\mu+\nu+1}w_\mu w_\nu\right\|_{B(\ell_2)}\le K\sum_{\nu=0}^{\infty}\frac{|w_\nu|^2}{2\nu+1}\tag{2}$$

holds, where w_ν, $\nu=0,1,2,\dots$ are complex numbers and $K\le 2\|A\|_{\mathcal{B}(D,\ell_2)}$.

Conversely, (2) *implies that $A\in\mathcal{B}(D,\ell_2)$ and $\|A\|_{\mathcal{B}(D,\ell_2)}\le 2K$.*

Proof. It is clear that the double series converges if the right hand series converges, too.

Then we have

$$\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{A_{\mu+\nu+1}}{\mu+\nu+1}w_\mu w_\nu=\sum_{n=0}^{\infty}\frac{A_{n+1}}{n+1}\left(\sum_{\nu=0}^{n}w_\nu w_{n-\nu}\right)=\langle A,g\rangle\,,\tag{3}$$

where

$$g(z)=\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\sum_{\nu=0}^{n}w_\nu w_{n-\nu}\right)z^{n+1},\quad z\in D,$$

and by $\langle A, g\rangle$ we mean

$$\langle A, g\rangle = \lim_{\rho\to 1^-}\sum_{n=0}^{\infty} A_n b_n \rho^n = \lim_{\rho\to 1^-}\frac{1}{2\pi}\int_0^{2\pi} A(\rho e^{-i\theta}) g(\rho e^{i\theta}) d\theta,$$

for $\| \ \|_{B(\ell_2)}$-convergent expansions $A(\rho e^{-i\theta}) = \sum_{n=0}^{\infty} A_n b_n \rho^n e^{in\theta}$ and $g(\rho e^{i\theta}) = \sum_{n=0}^{\infty} b_n b_n \rho^n e^{in\theta}$. But

$$g'(z) = \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} w_\nu w_{n-\nu}\right) z^n = \left(\sum_{n=0}^{\infty} w_n z^n\right)^2.$$

Hence, by Parseval formula we get

$$\|g\|_{\mathcal{I}} = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |g'(z)| d\theta\, dr = \int_0^1 \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty} w_n z^n\right|^2 d\theta\, dr =$$

$$= \int_0^1 \sum_{n=0}^{\infty} |w_n|^2 r^{2n} dr = \sum_{n=0}^{\infty}\frac{|w_n|^2}{2n+1}.$$

Now, by Lemma 3 we have

$$\|\langle A, g\rangle\|_{B(\ell_2)} \leq 2\|A\|_{\mathcal{B}(D,\ell_2)}\sum_{n=0}^{\infty}\frac{|w_n|^2}{2n+1},$$

that is, (2) holds for $A \in \mathcal{B}(D, \ell_2)$.

b) Conversely, if (2) holds we take $\mathcal{I} \in D$ and find w_n such that $g(z) = \sum_{n=1}^{\infty} n\zeta^{n-1} z^n = \frac{z}{(1-\zeta z)^2} = \sum_{n=0}^{\infty}\frac{1}{n+1}(\sum_{\nu=0}^{n} w_\nu w_{n-\nu}) z^n$. (See Theorem 3.5 in [ACP].)

Using the computations in [ACP], page 17, we get

$$\sum_{n=0}^{\infty}\frac{|w_n|^2}{2n+1} = \|g\|_{\mathcal{I}} \leq \frac{2}{1-|\zeta|^2}.$$

By (2) and (3) we have

$$\|A'(\zeta)\|_{B(\ell_2)} = \left\|\left\langle A(z), \frac{z}{(1-z\zeta)^2}\right\rangle\right\|_{B(\ell_2)} \leq K\sum_{n=0}^{\infty}\frac{|w_n|^2}{2n+1} \leq \frac{2K}{1-|\zeta|^2}.$$

Therefore, $\|A\|_{\mathcal{B}(D,\ell_2)} \leq 2K$. $\square$

We can identify the space $\mathcal{I}$ with the space of all corresponding Toeplitz matrices and consider the bounded linear operators which invariate the diagonals $\psi : \mathcal{I} \to B(\ell_2)$.

Then we have the following extension of a result from [DRS]. (See also Theorem 2.4 in [ACP].)

Theorem 5. *Any $B \in \mathcal{B}(D, \ell_2)$ corresponds to a bounded linear operator $\psi_B : \mathcal{I} \to B(\ell_2)$ invariating the diagonals and conversely. The correspondence is given by: $B \to \psi_B$, where ψ_B is defined by $\psi_B(g) = \lim_{r \to 1^-} B * G(r)$, G being the Toeplitz matrix associated to g, for $g = f_G$.*

Proof. Let $B \in \mathcal{B}(D, \ell_2)$. Then by Lemma 3 we have

$$\|\psi_B(g)\|_{B(\ell_2)} \leq 2\|A\|_{\mathcal{B}(D,\ell_2)} \, \|g\|_{\mathcal{I}}$$

for all $g \in \mathcal{I}$. Hence ψ_B has the required properties.

Conversely, let $\psi : \mathcal{I} \to B(D, \ell_2)$ be a bounded linear operator invariating the diagonals. Then there exists a unique $A \in \mathcal{B}(D, \ell_2)$ such that

$$\psi(G) = \lim_{r \to 1^-} G * A(r).$$

Let the matrix A given by $A_n = \psi(E_n)$ for every $n = 0, 1, 2, \dots$, where E is the Toeplitz matrix given by the constant sequence $(1, 1, \dots)$. Here of course we used the fact that ψ invariates the diagonals. Now, let us define $A(s) = A * C(s) = \sum_{n=0}^{\infty} A_n s^n$, converging in $B(D, \ell_2)$ for all $0 \leq s < 1$. For $0 < \rho < 1$ and $g(s) = \sum_{n=0}^{\infty} b_n s^n$ we have

$$\psi(G * C(\rho s)) = \sum_{n=0}^{\infty} b_n \rho^n \psi(E_n) = \sum_{n=0}^{\infty} b_n \rho^n A_n.$$

Since $\lim_{\rho \to 1^-} g(\rho \cdot) = g(\cdot)$ in the norm of $\mathcal{I}$, we have

$$\|G * C(\rho) - G\|_{\mathcal{I}} \leq \epsilon / \|\psi\|,$$

if $|\rho - 1| < \delta(\epsilon)$. But $\|\psi(G * C(\rho \cdot)) - \psi(G(\cdot))\|_{B(D,\ell_2)} \leq \|\psi\| \, \|G * C(\rho) - G\|_{\mathcal{I}} < \epsilon$ if $|\rho - 1| < \delta(\epsilon)$, that is, $\lim_{\rho \to 1^-} \sum_{n=0}^{\infty} b_n \rho^n A_n$ exists in $B(\ell_2)$.

Moreover, by the definition of the derivative, for $0 \leq s < 1$ we have

$$A'(s) = \sum_{n=1}^{\infty} n A_n s^{n-1} = \psi(G_s), \tag{4}$$

G_s being the Toeplitz matrix corresponding to the function $g_s = \frac{z}{(1-sz)^2}$, $|z| < 1$.

But by computations on page 17 in [ACP] we have

$$\|g_s\|_{\mathcal{I}} = \left\| \frac{z}{(1 - sz)^2} \right\|_{\mathcal{I}} \leq \frac{2}{1 - s^2}.$$

Hence, by (4) and by the definition of the norm of ψ we have

$$\|A'(s)\|_{B(\ell_2)} \leq \|\psi(G_s)\| \leq \|\psi\| \, \|g_s\|_{\mathcal{I}} \leq \|\psi\| \, \frac{2}{1 - s^2},$$

therefore $\|A\|_{\mathcal{B}(D,\ell_2)} \leq 2\|\psi\|$. $\square$

We may call $\lim_{r\to 1^-} B * G(r)$, the *generalized Schur product of B and G.* It is known that the usual Schur product $B * G$ of a matrix $B \in \mathcal{B}(D, \ell_2)$ and $G \in \mathcal{I}$ does not exist in general. (See Theorem 2.5 in [ACP].)

So, Theorem 5 may be stated as follows. *The Bloch matrices are generalized Schur multipliers from $\mathcal{I}$ into $B(\ell_2)$.*

Now, we can give an interesting example of a Bloch matrix.

THEOREM 6. *Let $A = \sum_{k=0}^{\infty} A_{2^k}$. Then $A \in \mathcal{B}(D, \ell_2)$ if and only if $\sup_k \|A_k\|_{B(\ell_2)} < \infty$.*

Proof. By Theorem 5, there is a constant $C > 0$ such that $C\|A\|_{\mathcal{B}(D,\ell_2)} \geq \sup_k \|A_k\|_{B(\ell_2)}$ for all infinite matrices A.

Now, let us consider a *lacunary* matrix A as in the statement. Then

$$\frac{\|zf_A'(z)\|_{B(\ell_2)}}{1-|z|} = \left(\sum_{n=0}^{\infty} |z|^n\right) \left\|\sum_{k=0}^{\infty} A_k 2^k z^{2^k}\right\|_{B(\ell_2)} \leq$$

$$\leq \sup_k \|A_{2^k}\|_{B(\ell_2)} \cdot \sum_{n=1}^{\infty} \left(\sum_{2^k \leq n} 2^k\right) |z|^n \leq$$

$$\leq 2 \sup_k \|A_{2^k}\|_{B(\ell_2)} \sum_{n=1}^{\infty} n|z|^n = \frac{2|z|}{(1-|z|)^2} \sup_k \|A_{2^k}\|_{B(\ell_2)}.$$

Consequently, $(1-r^2)\|A'(r)\|_{B(\ell_2)} \leq 4 \sup_k \|A_{2^k}\|_{B(\ell_2)}$, that is $\|A\|_{\mathcal{B}(D,\ell_2)} \leq 4 \sup_k \|A_{2^k}\|_{B(\ell_2)}$. $\square$

It was remarked in [ACP] that the classical space $\mathcal{B}$ of Bloch functions is a Banach algebra with respect to convolution or, equivalently, to Hadamard (Schur) composition of functions, that is, for $f = \sum_{k=0}^{\infty} a_k \mathrm{e}^{\mathrm{i}k\theta} \in \mathcal{B}$ and $g = \sum_{k=0}^{\infty} b_k \mathrm{e}^{\mathrm{i}k\theta} \in \mathcal{B}$, $f * g = \sum_{k=0}^{\infty} a_k b_k \mathrm{e}^{\mathrm{i}k\theta} \in \mathcal{B}$. (See 3.5 in [ACP].)

Now, we extend this remark in the framework of matrices with respect to Schur product. Its proof was communicated to us by Victor Lie.

THEOREM 7. *The space $\mathcal{B}(D, \ell_2)$ is a commutative Banach algebra with respect to Schur product of matrices.*

Proof. Let

$$A = \begin{pmatrix} a_1^0 & a_1^1 & \dots & \dots \\ 0 & a_2^0 & a_2^1 & \dots \\ 0 & 0 & a_3^0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$f_j(t) = \sum_{k=0}^{\infty} a_j^k \mathrm{e}^{2\pi \mathrm{i}kt}, \quad \|A\|'_{\mathcal{B}(D,\ell_2)} = \sup_{r<1} (1-r^2)\|A'(r)\|_{B(\ell_2)}.$$

Then $\|A\|'_{\mathcal{B}(D,\ell_2)}$ is given by

$$\|A\|'_{\mathcal{B}(D,\ell_2)} = \sup_{r<1}(1-r)\left\{ \sup_{\|h\|_2\le 1} \left(\sum_{j=1}^{\infty} \left| \int_0^1 f_j'(r\mathrm{e}^{2\pi\mathrm{i}t}\mathrm{e}^{2\pi\mathrm{i}jt} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t|^2 \right)^{1/2} \right\}.$$

(See [BLP].) Hence

$$(\|A*B\|'_{\mathcal{B}(D,\ell_2)})^2 = \sup_{r<1}\left\{ \sup_{\|h\|_2\le 1} (1-r)^2 \sum_{j=1}^{\infty} \left| \int_0^1 (f_j*g_j)'(r\mathrm{e}^{2\pi\mathrm{i}t}\mathrm{e}^{2\pi\mathrm{i}jt} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t \right|^2 \right\},$$

where $(f_j)_j$ corresponds to A as above and $(g_j)_j$ corresponds to B.

Then we have

$$r(f_j*g_j)'(r\mathrm{e}^{2\pi\mathrm{i}t}) = 2\int_0^{\sqrt{r}} \int_0^1 f_j(s\mathrm{e}^{2\pi\mathrm{i}(\theta+t)}) g_j'(s\mathrm{e}^{-2\pi\mathrm{i}\theta}) s\,\mathrm{d}\theta\,\mathrm{d}s$$

for all j.

By the Cauchy-Schwarz inequality we have

$$\sum_{j=1}^{\infty} \left| \int_0^1 (f_j*g_j)'(r\mathrm{e}^{2\pi\mathrm{i}t}\mathrm{e}^{2\pi\mathrm{i}jt} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t \right|^2 =$$

$$= 4\sum_{j=1}^{\infty} r^{-2} \left| \int_0^{\sqrt{r}} \int_0^1 g_j'(s\mathrm{e}^{-2\pi\mathrm{i}\theta}) s\mathrm{e}^{-2\pi\mathrm{i}j\theta} \times \right.$$

$$\left. \times \left(\int_0^1 f_j'(s\mathrm{e}^{2\pi\mathrm{i}(\theta+t)}) \mathrm{e}^{2\pi\mathrm{i}j(t+\theta)} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t \right) \mathrm{d}\theta\,\mathrm{d}s \right|^2 \le$$

$$\le \sum_{j=1}^{\infty} 4r^{-2} \left(\int_0^{\sqrt{r}} \left(\int_0^1 |g_j'(s\mathrm{e}^{-2\pi\mathrm{i}\theta})|^2 \mathrm{d}\theta \right) s\,\mathrm{d}s \right) \times$$

$$\times \left(\int_0^{\sqrt{r}} \int_0^1 s \left| \int_0^1 f_j'(s\mathrm{e}^{2\pi\mathrm{i}(\theta+t)}) \mathrm{e}^{2\pi\mathrm{i}j(t+\theta)} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t \right|^2 \mathrm{d}\theta\,\mathrm{d}s \right) \stackrel{\mathrm{def}}{=} I.$$

But

$$\sup_{j\ge 1}\sup_{s<1}(1-s^2)^2 \left(\int_0^1 |g_j'(s\mathrm{e}^{-2\pi\mathrm{i}t})|^2 \mathrm{d}\theta \right) \le (\|B\|'_{\mathcal{B}(D,\ell_2)})^2$$

and for $\|h\|_2 = 1$ we also have

$$(1-s^2)^2 \sum_{j=1}^{\infty} \left| \int_0^1 f_j'(s\mathrm{e}^{2\pi\mathrm{i}t}) \mathrm{e}^{2\pi\mathrm{i}jt} h(\mathrm{e}^{-2\pi\mathrm{i}t})\mathrm{d}t \right|^2 \le (\|A\|'_{\mathcal{B}(D,\ell_2)})^2.$$

Consequently,

$$I \leq 4r^{-2} \int_0^{\sqrt{r}} \frac{s(\|B\|'_{\mathcal{B}(D,\ell_2)})^2}{(1-s^2)^2} \mathrm{d}s \int_0^{\sqrt{r}} \frac{s(\|A\|'_{\mathcal{B}(D,\ell_2)})^2}{(1-s^2)^2} \mathrm{d}s =$$
$$= (1-r)^{-2} (\|A\|'_{\mathcal{B}(D,\ell_2)})^2 (\|B\|'_{\mathcal{B}(D,\ell_2)})^2,$$

that is,

$$\|A * B\|'_{\mathcal{B}(D,\ell_2)} \leq \|A\|'_{\mathcal{B}(D,\ell_2)} \|B\|'_{\mathcal{B}(D,\ell_2)}. \quad \square$$

3. MATRICEAL BERGMAN-SCHATTEN SPACES

We intend now to give more results about matriceal Bloch space which are related to other matrix spaces, namely, *matriceal Bergman-Schatten spaces.*

In order to do this, we recall some notions from vector-valued integration theory.

We say that a function $f : D \to B(\ell_2)$ is *w^*-measurable* if $A \circ f$ is a Lebesgue measurable function on D for every $A \in C_1$, where C_1 is the *Schatten class of all operators with trace,* and A is considered as a functional on $B(\ell_2)$.

$f : D \to B(\ell_2)$ is said to be *strong measurable* if it is a norm limit of a sequence of simple functions. (See [E] for more details about vector-valued measurability.)

For instance, it follows from Proposition 8.15.3 in [E] that for a w^*-measurable $B(\ell_2)$-valued function f, the function $t \to \|f(t)\|_{B(\ell_2)}$ is Lebesgue measurable on D. We introduce also the matrix spaces

$$L^\infty(D,\ell_2) = \{r \to A(r) \text{ being a } w^*\text{-measurable function on } [0,1) \mid$$
$$\operatorname*{ess\,sup}_{0 \leq r < 1} \|A(r)\|_{B(\ell_2)} := \|A(r)\|_{L^\infty(D,\ell_2)} < \infty\},$$

$\widetilde{L^\infty}(D,\ell_2)$, the subspace of $L^\infty(D,\ell_2)$ consisting of all strong measurable functions on $[0,1)$, and

$$L^\infty_a(D,\ell_2) = \{A \text{ infinite analytic matrix} \mid \|A\|_{L^\infty_a(D,\ell_2)} :=$$
$$:= \sup_{0 \leq r < 1} \|C(r) * A\|_{B(\ell_2)} = \|A(r)\|_{L^\infty(D,\ell_2)} < \infty\}.$$

To obtain more information about the matriceal Bloch space we need the concept of *Bergman projection.*

First, we introduce the *Bergman-Schatten classes.*

Definition 8. Let $1 \leq p < \infty$. Let $L^p(D,\ell_2) := \{r \to A(r)$ a strong measurable C_p-valued function defined on $[0,1)$ such that $\|A(r)\|_{L^p(D,\ell_2)} := \left(2\int_0^1 \|A(r)\|^p_{C_p} r\mathrm{d}r\right)^{1/p} < \infty\}$, where C_p is the *Schatten class of order* p, and let

$\tilde{L}^p_a(D,\ell_2) = \{A(r) := A * C(r);\ A$ upper triangular matrices $\mid \|A\|_{L^p(D,\ell_2)} < \infty\}$. $\tilde{L}^p_a(D,\ell_2)$ is a subspace of $L^p(D,\ell_2)$.

By $L^p_a(D,\ell_2)$ we mean the space of all upper triangular matrices such that $\|A(r)\|_{L^p_a(D,\ell_2)} < \infty$, where $A(r) = C(r) * A$, $r \in [0,1)$.

We identify $\tilde{L}^p_a(D,\ell_2)$ and $L^p_a(D,\ell_2)$ and call $L^p_a(D,\ell_2)$ the *Bergman-Schatten classes.*

LEMMA 9. *The function* $r \to A(r) := C(r) * A$, *where* A *is an upper triangular matrix, is a continuous function on* $[0,1]$ *taking values in* C_p, $1 \le p \le \infty$, *or in* $B(\ell_2)$, *if* $A \in C_p$, *respectively* $A \in B(\ell_2)$.

Proof. If $A \in B(\ell_2)$, for $r_n \to r \in [0,1]$ we have

$$\|(C(r_n) - C(r)) * A\|_{B(\ell_2)} \overset{\text{(by Theorem 8.1 in [B])}}{\le} \frac{|r_n - r|}{1 - |r_n - r|} \|A\|_{B(\ell_2)} \underset{n\to\infty}{\longrightarrow} 0.$$

Using the duality and interpolation between C_p we have

$$\lim_{n\to\infty} \|(C(r_n) - C(r)) * A\|_{C_p} = 0, \quad 1 \le p \le \infty,$$

if $A \in C_p$. $\square$

By Lemma 9 we have

COROLLARY 10. *Let* $1 \le p \le \infty$ *and* A *an upper triangular matrix. If* $A \in C_p$ *(respectively if* $A \in B(\ell_2)$*), then*

$$\sup_{0\le r<1} \|C(r) * A\|_{C_p} = \sup_{0\le r\le 1} \|C(r) * A\|_{C_p}$$

and similarly with $\| \ \|_{B(\ell_2)}$ *instead of* $\| \ \|_{C_p}$.

Proof. We have to show that

$$\|C(1) * A\|_{C_p} \le \sup_{0\le r<1} \|C(r) * A\|_{C_p},$$

for $1 \le p \le \infty$ (respectively, the similar inequality for $\| \ \|_{B(\ell_2)}$).

By Lemma 9 we have

$$\|A\|_{C_p} = \|C(1) * A\|_{C_p} = \lim_{r\to 1} \|C(r) * A\|_{C_p} \le \sup_{r<1} \|C(r) * A\|_{C_p}$$

and similarly for $B(\ell_2)$. $\square$

Now, we have

COROLLARY 11. $L^\infty_a(D,\ell_2)$ *is a Banach subspace in* $B(\ell_2)$, *sometimes denoted by* $H^\infty(D,\ell_2)$.

Proof. We have

$$\|A\|_{L^\infty_a(D,\ell_2)} = \sup_{0\le r<1} \|C(r)*A\|_{B(\ell_2)} \overset{\text{(by Corollary 10)}}{=} \sup_{0\le r<1} \|C(r)*A\|_{B(\ell_2)} =$$

$$= \sup_{r\in[0,1]} \sup_{\substack{\|B\|_{C_1}\le 1,\\ B \text{ a lower triangular matrix}}} \operatorname{tr}(C(r)*A)B \le$$

$$\le \|A\|_{B(\ell_2)} \sup_{\substack{\|B\|_{C_1}\le 1,\ \operatorname{rank}(B)<\infty,\\ B \text{ lower triangular}}} \|C(r)*B\|_{C_1} \overset{\text{(by Lemma 9)}}{\le} \|A\|_{B(\ell_2)}.$$

On the other hand,

$$\|A\|_{L^\infty_a(D,\ell_2)} = \sup_{0\le r\le 1} \|C(r)*A\|_{B(\ell_2)} \ge \|A\|_{B(\ell_2)}.$$

Consequently, $\|A\|_{L^\infty_a(D,\ell_2)} \sim \|A\|_{B(\ell_2)}$. □

Proposition 12. *The Banach space $L^\infty_a(D,\ell_2)$ is a subspace of $\mathcal{B}(D,\ell_2)$, $\|A\|_{\mathcal{B}(D,\ell_2)} \le 6\|A\|_{L^\infty_a(D,\ell_2)}$ and $L^\infty_a(D,\ell_2)) \subsetneq \mathcal{B}(D,\ell_2)$.*

More precisely, the infinite analytic matrix

$$A = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \dots & \frac{1}{k} & \dots \\ 0 & 1 & 1 & \dots & \frac{1}{k-1} & \dots \\ 0 & 0 & 1 & \dots & \frac{1}{k-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

is not in $L^\infty_a(D,\ell_2)$, but $A \in \mathcal{B}(D,\ell_2)$.

Proof. We have $(1-r^2)A'(r) = C_1 * A^1(r)$, where

$$C_1(r)(k,j) = \begin{cases} (1-r^2)(j-k)r^{(j-k)/2-1} & \text{if } j > k+1 \\ (1-r^2)(j-k) & \text{if } j = k+1 \\ 0 & \text{if } j \le k+1, \end{cases}$$

and $A^1(r) = A_0 + A_1 + \sum_{k=2}^\infty A_k r^{k/2}$.

Note that $C_1(r)$ is a Schur multiplier by Theorem 8.1 in [B], with $\|C_1(r)\|_{M(\ell_2)} \le 1$. Thus,

$$\|(1-r^2)A'(r)\|_{B(\ell_2)} = \|C_1(r)*A^1(r)\|_{B(\ell_2)} \le \|A^1(r)\|_{B(\ell_2)} \quad \forall\, 0 \le r < 1,$$

implying that

$$\|A\|_{\mathcal{B}(D,\ell_2)} \le 2 \sup_{0\le r<1} \|A^1(r)\|_{B(\ell_2)} = 6\|A\|_{L^\infty_a(D,\ell_2)}.$$

If $A \in L_a^\infty(D,\ell_2)$, then clearly $A \in B(\ell_2)$. But, taking $h_1 = h_2 = \cdots = h_n = \frac{1}{\sqrt{n}}$ and $h_{n+1} = h_{n+2} = \cdots = 0$, we have $\sum_{i=1}^\infty |h_i|^2 = 1$ and

$$\|A\|^2_{B(\ell_2)} \geq \frac{1}{n}\left(1+1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)^2 + \frac{1}{n}\left(1+1+\frac{1}{2}+\cdots+\frac{1}{n-2}\right)^2 + \\ +\cdots+\frac{1}{n}(1)^2 \geq C\ln(n-1) \to \infty.$$

Now, $A \in \mathcal{B}(D,\ell_2)$. It is easy to see that

$$\sup_{0\leq r<1}(1-r^2)\|C(r)*A'(r)\|_{B(\ell_2)} = \sup_{0\leq r<1} r^2 = 1. \quad \square$$

Proposition 13. *Let $1 \leq p < \infty$. Then $\tilde{L}_a^p(D,\ell_2)$ is a closed subspace in $L^p(D,\ell_2)$. Consequently, $L_a^p(D,\ell_2)$ may be identified via the map $A \to A*C(r)$, $r \in [0,1)$, with a closed subspace of $L^p(D,\ell_2)$.*

Proof. Let $A^n \in L_a^p(D,\ell_2)$. Then there are upper triangular matrices A^n such that $A^n(r) = C(r)*A^n$ for all $n \in \mathbb{N}$ and all $0 \leq r < 1$. If $A^n(r) \to A(r) \in L^p(D,\ell_2)$, then $(A^n(r))_n$ is a Cauchy sequence in $L^p(D,\ell_2)$ and, consequently, $\|C(r)*(A^n-A^m)\|_{C_p} \underset{n,m\to\infty}{\longrightarrow} 0$ a.e. with respect to Lebesgue measure on $[0,1]$. Consequently, by Lemma 9, we have $\lim_{n,m\to\infty}\|C(r)*(A^n-A^m)\|_{C_p} = 0$ for all $r \in [0,1]$, that is, the sequence $(C(r)*A^n)_n$ is a Cauchy sequence in C_p for all $r \in [0,1]$, which in turn implies that $\lim_{n\to\infty}(C(r)*A^n)(i,j) = A(r)(i,j)$ for all $i,j \in \mathbb{N}$ and for all $0 \leq r \leq 1$. Consequently, $A(r)$ is an upper triangular matrix for all $r \leq 1$ and, since $(A^n*C(r))(i,j) = a_{ij}^n r^{j-i}$, we have $\lim_{n\to\infty} a_{ij}^n = a_{ij}$ for all $i,j \in \mathbb{N}$ and $A(r) = C(r)*A$, where $A = (a_{ij})_{i,j}$. Thus, $A \in L_a^p(D,\ell_2)$. $\square$

Now, let $A \in L_a^2(D,\ell_2)$ be arbitrary, let a fixed $0 \leq r < 1$ and let $B \in C_2$. We then have

Lemma 14. *The linear functional $F_{r,B}(A) = \operatorname{tr} A(r)B^*$ is continuous on $L_a^2(D,\ell_2)$.*

Proof. If A is an upper triangular matrix of finite order and $A(r) = C(r)*A$, then we consider the function $f_A(r,\theta)$ on D defined in Section 1. It is clear that it is an holomorphic C_2-valued function on D. Consequently, the function $z \to \|\sum_{k=0}^\infty A_k r^k \mathrm{e}^{\mathrm{i}k\theta}\|_{C_2}$ is subharmonic.

Thus, for $0 < r' \leq 1-r$ we have

$$\|A(r)\|^2_{C_2} \leq \frac{1}{\pi r'^2}\int_0^{r'}\int_0^{2\pi}\left\|\sum_{k=0}^\infty A_k|r+s\mathrm{e}^{\mathrm{i}\theta}|^{k\mathrm{e};\mathrm{i}k\theta}\right\|^2_{C_2} s\mathrm{d}s\mathrm{d}\theta \leq$$

$$\leq \frac{1}{r'^2}\int_0^{r'}\sum_{k=0}^\infty \|A_k\|^2_{C_2}(r+s)^{2k}2s\mathrm{d}s \leq (\text{since } r' \leq 1-r) \leq$$

$$\leq \frac{1}{r'^2}\int_0^1 \|A(s)\|_{C_2}^2 2s\mathrm{d}s = \frac{1}{r'^2}\|A\|^2_{L_a^2(D,\ell_2)}.$$

Taking $r' = 1-r$, we have

$$|F_{r,B}(A)|^2 = |\mathrm{tr}\, A(r)B^*| \leq \|A(r)\|_{C_2}^2 \cdot \|B\|_{C_2}^2 \leq \frac{\|B\|_{C_2}^2}{(1-r)^2}\cdot \|A\|^2_{L_a^2(D,\ell_2)},$$

which implies the continuity of $F_{r,B}$. □

Thus, by the Riesz theorem there is a unique matrix $K_{r,B} \in L_a^2(D,\ell_2)$ such that

$$F_{r,B}(A) = \langle A, K_{r,B}\rangle_{L_a^2(D,\ell_2)} = 2\int_0^1 \mathrm{tr}\, A(s)[K_{r,B}(s)]^* s\mathrm{d}s$$

for all $B \in C_2$, $0 \leq r < 1$ and $A \in L_a^2(D,\ell_2)$.

Let $i,j \in \mathbb{N}$ fixed and B the matrix whose entries $b(k,l)$ are

$$b(k,l) := \delta_{ki}\delta_{lj}.$$

Then the above formula becomes

$$A(r)(i,j) = 2\int_0^1 \mathrm{tr}\, A(s)K_{r,i,j}(s)^* s\mathrm{d}s = 2\int_0^1 \mathrm{tr}\,[A(s)(K_{i,j}(r) * P(s))]s\mathrm{d}s$$

for all $i,j \in \mathbb{N}$, where by $K_{r,i,j}$ we denoted the matrix $K_{r,B}$ for the above matrix B, while $P(s)$ is the Toeplitz matrix associated to the Poisson kernel.

Since $A(r)$ is an analytic matrix, we have

$$(P(r) * A)(i,j) = \int_0^1 \mathrm{tr}(P(s) * A)[P(s) * K_{i,j}(r)](2s)\mathrm{d}s$$

for all $j \geq i$ and all $0 \leq r < 1$.

It is easy to see that

$$K_{i,j}(r) = \begin{cases} \left((j-i-1)r^{j-i}\delta_{i,m}\delta_{j,l}\right)_{l,m=1}^{\infty} & i \leq j \\ (0)_{l,m=1}^{\infty} & i > j. \end{cases}$$

Definition 15. Let $r \to A(r)$ be an element of $L^2(D,\ell_2)$. Since $\tilde{L}_a^2(D,\ell_2)$ is a closed subspace in the Hilbert space $L^2(D,\ell_2)$, there is a unique orthogonal projection $\tilde{P}$ on $\tilde{L}_a^2(D,\ell_2)$, called *Bergman projection.* Denote by P the corresponding operator from $L^2(D,\ell_2)$ onto $L_a^2(D,\ell_2)$.

Proposition 16. *For all functions $A(r) \in L^2(D,\ell_2)$ defined on $[0,1)$ and for all $i,j \in \mathbb{N}$ we have*

$$\{[P(A(\cdot))](r)\}(i,j) = \begin{cases} 2(j-i+1)r^{j-i}\int_0^1 a_{ij}(s)s^{j-i+1}\mathrm{d}s & \textit{if } i \leq j \\ 0 & \textit{if } i > j. \end{cases}$$

Proof. We have

$$[P(A(\cdot))](r)(i,j) = F_{r,i,j}(P(A(\cdot))) = \langle P(A(\cdot)), K_{r,i,j}\rangle =$$

$$\overset{\text{(since } P \text{ is a selfadjoint projection)}}{=} \langle A, P(K_{r,i,j})\rangle =$$

$$\overset{\text{(since } K_{r,i,j} \text{ is an analytic matrix)}}{=} \langle A, K_{r,i,j}\rangle_{L^2_a(D,\ell_2)} =$$

$$= 2\int_0^1 \operatorname{tr} A(s)K^*_{r,i,j}(s)s\mathrm{d}s = 2(j-i+1)r^{j-i}\int_0^1 a_{ij}(s)\cdot s^{j-i+1}\mathrm{d}s,$$

if $j \ge i$ and $\langle A, B\rangle$ means $\operatorname{tr} AB^*$.

If $j < i$, then it is easy to get that $([P(A(\cdot))])\,(i,j) = 0$. $\square$

If $A \in L^\infty(D,\ell_2)$ then $r \to A(r)$ is a w^*-measurable function, consequently each function $a_{ij}(r)$ is a Lebesgue measurable function on $[0,1)$ for all i and j and we may introduce $PA(\cdot)$ as in Proposition 16.

THEOREM 17. *Both* $P : L^\infty(D,\ell_2) \to \mathcal{B}(D,\ell_2)$ *and* $P : \widetilde{L^\infty}(D,\ell_2) \to \mathcal{B}(D,\ell_2)$ *are bounded surjection operators.*

Proof. It is enough to prove the first assertion. Let $A(\cdot) \in L^\infty(D,\ell_2)$ and $B = PA(\cdot)$. We show that $B \in \mathcal{B}(D,\ell_2)$.

We have

$$\|B'(r)\|^2_{B(\ell_2)} =$$

$$= \sup_{\|h\|_2\le 1}\left[\int_0^1\left(\sum_{i=1}^\infty\left|\sum_{j=i+1}^\infty a_{ij}(s)r^{j-i-1}s^{j-i}(j-i+1)(j-i)h_j\right|^2\right)^{1/2}(2s\mathrm{d}s)\right]^2 \le$$

$$\le \left[\int_0^1 \|A(s) * C(r,s)\|_{B(\ell_2)}(2s\mathrm{d}s)\right]^2,$$

where

$$C(r,s)(i,j) = \begin{cases} (j-i+1)(j-i)(rs)^{j-i-1}sr & \text{if } j > i\\ 0 & \text{if } j \le i.\end{cases}$$

Thus,

$$\|B'(r)\|_{B(\ell_2)} \le 2\|A(\cdot)\|_{L^\infty(D,\ell_2)}\cdot\int_0^1\int_{-\pi}^{\pi}\frac{s}{|1-rs\mathrm{e}^{\mathrm{i}\theta}|^3}s\mathrm{d}s\frac{\mathrm{d}\theta}{\pi} \sim$$

$$\sim \text{(by Lemma 4.2.2 in [Z])} \sim C\|A(\cdot)\|_{L^\infty(D,\ell_2)}\cdot\frac{r}{1-r^2}.$$

Consequently, $\|B\|_{\mathcal{B}(D,\ell_2)} \le C\|A(\cdot)\|_{L^\infty(D,\ell_2)}$, that is, $P : L^\infty(D,\ell_2) \to \mathcal{B}(D,\ell_2)$ is a bounded operator.

In order to show that P is onto, we take $B \in \mathcal{B}(D,\ell_2)$ and $B^1(r) = B(r) - B_0 - B_1 r$. Then

$$B^2(r)(i,j) = \frac{3(j-i)}{2(j-i+1)(j-i)}b'_{ij}(r) = (B^2)'(r) * T,$$

where

$$(B^2)'(r)(i,j) = b'_{ij}(r) = [C(r)*(B^1)'(r)](i,j) = \begin{cases} (j-i)r^{j-i-1}b_{ij} & j-i \geq 2 \\ 0 & j-i < 2 \end{cases}$$

and $T = (t_{j-i})_{i,j}$, with $t_k = \frac{3k+2}{2k(k+1)}$. Easy computations show that

$$[PB^2(\cdot)] = B^1.$$

But T is a Schur multiplier. It thus follows that $B^2(r) \in L^\infty(D, \ell_2)$. If we show that $B_0 + B_1 r \in L^\infty(D, \ell_2)$, we are done.

Clearly, it suffices to show that $B_1 r \in L^\infty(D, \ell_2)$ (since $B_0 \in B(\ell_2)$ by the hypothesis, we have $B \in \mathcal{B}(D, \ell_2)$). As

$$\|B_1\|_{\mathcal{B}(D,\ell_2)} = \sup_{0 \leq r < 1} (1-r^2)\|B_1\|_{B(\ell_2)} = \|B_1\|_{B(\ell_2)},$$

it follows that $B_1 r \in B(\ell_2)$. Thus $B_1 r \in L^\infty(D, \ell_2)$. □

Remark 18. Note that $\mathcal{B}(D, \ell_2)$ endowed with $\|A\|_{\mathcal{B}(D,\ell_2)}$ is a Banach space and, by the open mapping theorem, $(\mathcal{B}(D, \ell_2), \| \ \|_{\mathcal{B}(D,\ell_2)})$ is isomorphic to the quotient space $L^\infty(D, \ell_2)/\mathrm{Ker}\, P$, endowed with quotient norm.

We would like to prove a similar result for the Bergman-Schatten class $L^1_a(D, \ell_2)$.

Unfortunately, the Bergman projection is unbounded on this last space, but instead we can consider a version of it.

Let $\alpha > -1$. Put

$$K_{i,j,\alpha}(r) = \begin{cases} \left(\frac{\Gamma(j-i+2+\alpha)}{(j-i)!\Gamma(2+\alpha)} r^{j-i} \delta_{i,l}\delta_{j,m}\right)_{l,m=1}^\infty & i \leq j \\ 0 & i > j \end{cases}$$

and for an analytic matrix $A(s) = P(s) * A$ we have

$$a_{ij} r^{j-i} = (\alpha+1) 2 \int_0^1 \sum_{k=0}^\infty \langle A_k, K_k^{i,j,\alpha}(r)\rangle s^{2k+1}(1-s^2)^\alpha \mathrm{d}s \quad \forall\, i, j,$$

where $A = (a_{ij})_{i,j=1}^\infty$. Then

$$[P_\alpha A(\cdot)](r) = \begin{cases} \frac{(\alpha+1)\Gamma(j-i+2+\alpha)}{(j-i)!\Gamma(\alpha+2)} r^{j-i} (2\int_0^1 a_{ij}(s) s^{j-i+1}(1-s^2)^\alpha \mathrm{d}s & \text{if } j \geq i \\ 0 & \text{if } j < i. \end{cases}$$

THEOREM 19. *If $\alpha = 1$ then P_1 is a continuous operator* (*precisely a continuous projection*) *from $L^1(D, \ell_2)$ on $L^1_a(D, \ell_2)$.*

Proof. By Theorem 8.18.2 in [E], the topological dual of $L^1(D,\ell_2)$ is $L^\infty(D,\ell_2)$ with respect to the duality pair

$$\langle A(\cdot),B(\cdot)\rangle \stackrel{\text{def}}{=} 2\int_0^1 \operatorname{tr}(A(s)[B(s)]^*)2s\mathrm{d}s,$$

where $A(\cdot)\in L^\infty(D,\ell_2)$, $B(\cdot)\in L^1(D,\ell_2)$.

Now, we are looking for the adjoint P_1^* of P_1. We have

$$\langle P_1^*A(\cdot),B(\cdot)\rangle = 2\int_0^1\sum_{i=1}^\infty\sum_{j=1}^\infty (P_1^*A(\cdot))(r)(i,j)\overline{b_{ij}(r)})r\mathrm{d}r =$$

$$=\sum_{i=1}^\infty\sum_{j=1}^\infty\int_0^1 (P_1^*A(\cdot))(r)(i,j)\overline{b_{ij}(r)}(2r)\mathrm{d}r.$$

On the other hand,

$$\langle P_1^*A(\cdot),B(\cdot)\rangle = \langle A(\cdot),P_1B(\cdot)\rangle = \int_0^1 \operatorname{tr}A(r)(P_1B)^*(r)(2r\mathrm{d}r) =$$

$$=\sum_{i=1}^\infty\sum_{j=1}^\infty\int_0^1 A(r)(i,j)\overline{(P_1B)(r)}(i,j)(2r\mathrm{d}r) =$$

$$=\sum_{i=1}^\infty\sum_{j=i}^\infty\frac{\Gamma(j-i+3)}{(j-i)!\Gamma(2)}\left(\int_0^1[A(s)](i,j)s^{j-i}(2s\mathrm{d}s)\right)\times$$

$$\times\left(\int_0^1\overline{b_{ij}(s)}s^{j-i}(1-s^2)(2s\mathrm{d}s)\right).$$

Now, we take $B(s)(i,j)=\chi_{I_k}(s)/(\mu(I_k))$ and $B(s)(l,k)=0$, $(l,k)\neq(i,j)$, $\forall(i,j)\in\mathbb{N}\times\mathbb{N}$, where $I_k\ni r$ is a sequence of intervals such that $\lim\limits_{k\to\infty}\mu(I_k)=0$, $\mathrm{d}\mu=2s\mathrm{d}s$.

By Lebesgue's diferentiation theorem we have

$$(P_1^*A(\cdot))(r)(i,j)=\begin{cases}\frac{\Gamma(j-i+3)}{(j-i)!\Gamma(2)}r^{j-i}(1-r^2)\int_0^1 A(s)(i,j)s^{j-i}(2s\mathrm{d}s) & \text{if } j>i\\ 0 & \text{if } j\le i\end{cases}$$

a.e. for all $r\in[0,1)$.

We show that $P_1^*: L^\infty(D,\ell_2)\to L^\infty(D,\ell_2)$ is a bounded operator. In order to prove that, we remark that

$$\|A(r)\|^2_{L^\infty(D,\ell_2)} = \operatorname*{ess\,sup}_{0\le r<1}\|A(r)\|^2_{B(\ell_2)} = \operatorname*{ess\,sup}_{0\le r<1}\sup_{\sum_{j=1}^\infty|h_j|^2\le1}\sum_{i=1}^\infty\left|\sum_{j=1}^\infty a_{ij}(r)h_j\right|^2.$$

Consequently,

$$\|P_1^* A(\cdot)\|^2_{L^\infty(D,\ell_2)} =$$

$$= \operatorname*{ess\,sup}_{0\le r<1} \sup_{\sum_{j=1}^\infty |h_j|^2\le 1} \sum_{i=1}^\infty \left| \sum_{j=1}^\infty \int_0^1 h_j r^{j-i} \frac{\Gamma(j-i+2)}{(j-i)!\Gamma(2)} (1-r^2) a_{ij}(s) s^{j-i}(2s\mathrm{d}s)\right|^2 =$$

$$= \operatorname*{ess\,sup}_{0\le r<1} \sup_{\|h\|_{l_2}\le 1} (1-r^2)^2 \sum_{i=1}^\infty \int_0^1 \Bigg(\sum_{j=i}^\infty a_{ij}(s)((rs)^{j-i} \frac{\Gamma(j-i+3)}{(j-i)!})h_j\Bigg)(2s\mathrm{d}s)|^2 \le$$

$$\le \operatorname*{ess\,sup}_{0\le r<1}(1-r^2) \sup_{\|h\|_{l_2}\le 1} \Bigg[\int_0^1 \Bigg(\sum_{i=1}^\infty \Bigg| \sum_{j=i}^\infty a_{ij}(s)\times$$

$$\times[(rs)^{j-i}(j-i+2)(j-i+1)]h_j\Bigg|^2\Bigg)^{1/2} (2s\mathrm{d}s)\Bigg]^2.$$

Since the Toeplitz matrix $C(rs) = \big((c_{ij})(rs)^\infty_{i,j=1}\big)$ with

$$c_{ij}(rs) := c_{j-i}(rs) = \begin{cases} (rs)^{j-i}(j-i+2)(j-i+1) & \text{if } j\ge i \\ 0 & \text{if } j<i \end{cases}$$

is a Schur multiplier (we remark that $\sum_{k=0}^\infty u^k(k+2)(k+1)\mathrm{e}^{\mathrm{i}k\theta} = \frac{2}{(1-u\mathrm{e}^{\mathrm{i}\theta})^3}$), by Theorem 8.1 in [B] the multiplier norm of the matrix $C(rs)$ is exactly the $L^1(\mathbb{T})$-norm of $\frac{2}{(1-rs\mathrm{e}^{\mathrm{i}\theta})^3}$, that is, is equal to $2\sum_{n=0}^\infty \frac{\Gamma(n+3/2)^2}{(n!)^2\Gamma(3/2)^2}(rs)^{2n}$.

Thus,

$$\sup_{\sum_{j=1}^\infty |h_j|^2\le 1} \Bigg(\sum_{i=1}^\infty \Bigg| \sum_{j=i}^\infty a_{ij}(s)(rs)^{j-i}(j-i+2)(j-i+1)h_j\Bigg|^2\Bigg)^{1/2} =$$

$$= \|A(s) * C(rs)\|_{B(\ell_2)} \le \|A(s)\|_{B(\ell_2)} \cdot \sum_{n=0}^\infty \frac{\Gamma(n+3/2)^2}{(n!)^2\Gamma(3/2)^2}(rs)^{2n}.$$

Consequently,

$$\|P_1^* A(\cdot)\|^2_{L^\infty(D,\ell_2)} \le$$

$$\le \operatorname*{ess\,sup}_{r<1}(1-r^2)^2 \Bigg[\int_0^1 \|A(s)\|_{B(\ell_2)} \cdot \sum_{n=0}^\infty \frac{\Gamma(n+3/2)^2}{(n!)^2\Gamma(3/2)^2} r^{2n}s^{2n+1}(2s\mathrm{d}s)\Bigg]^2 \le$$

$$\le \operatorname*{ess\,sup}_{r<1}(1-r^2)^2 \|A(\cdot)\|^2_{L^\infty(D,\ell_2)} \Bigg(\sum_{n=0}^\infty \frac{\Gamma(n+3/2)^2 r^{2n}}{(n!)^2(n+1)\Gamma(3/2)^2}\Bigg)^2 \sim$$

$$\overset{\text{(by Stirling's formula)}}{\sim} \operatorname*{ess\,sup}_{r<1}(1-r^2)^2 \frac{1}{(1-r^2)^2} \|A(\cdot)\|^2_{L^\infty(D,\ell_2)} \sim \|A(\cdot)\|^2_{L^\infty(D,\ell_2)},$$

which shows in turn that $P_1^* : L^\infty(D,\ell_2) \to L^\infty(D,\ell_2)$ is bounded. $\square$

THEOREM 20. *The projection P_1 is a bounded operator from $L^\infty(D,\ell_2)$ (respectively from $\widetilde{L^\infty(D,\ell_2)}$) onto $\mathcal{B}(D,\ell_2)$.*

The *proof* is an easy adaptation of the proof of Theorem 12 and we leave it to the reader.

By Theorems 19 and 20 we easily get

COROLLARY 21. *Let $1 < p < \infty$ and $1-\theta = 1/p$. Then $L^p(D,\ell_2) = [L^1(D,\ell_2), \mathcal{B}(D,\ell_2)]_\theta$ with equivalent norms.*

Indeed, we use the known result about the interchangebility of the interpolation functor and a bounded projection. (See [T].)

We have the folowing interesting result.

THEOREM 22. *$L^1_a(D,\ell_2)^*$, the Banach space dual of $L^1_a(D,\ell_2)$, can be identified with $\mathcal{B}(D,\ell_2)$. Namely, let $A \in L^1_a(D,\ell_2)$ and $B \in \mathcal{B}(D,\ell_2)$. Then we have*

$$|\langle A,B\rangle| = \left|\int_0^1 \mathrm{tr}\,[A(r)B^*(r)](2r\mathrm{d}r)\right| \le C\|A\|_{L^1_a(D,\ell_2)}\cdot\|B\|_{\mathcal{B}(D,\ell_2)},$$

where $C>0$ is a constant.

Proof. Since C_1 is a separable Banach space with $C_1^* = B(\ell_2)$, with $\langle A,B\rangle = \mathrm{tr}(AB^*)$, according to Theorem 8.18.2 in [E] we have $L^1(D,\ell_2)^* = L^\infty(D,\ell_2)$, using the duality map $\langle A(r),B(r)\rangle = \int_0^1 \mathrm{tr}\,[A(r)B^*(r)](2r\mathrm{d}r)$.

Then, by Hahn-Banach theorem, $\tilde{L}^1_a(D,\ell_2)^* = L^\infty(D,\ell_2)/(\tilde{L}^1_a(D,\ell_2))^\perp$.

Using the fact that $L^1_a(D,\ell_2)$ is canonically isomorphic to $\tilde{L}^1_a(D,\ell_2)$, we have to show that

$$\mathrm{Ker}\,P = \mathrm{Ker}\,\tilde{P} = (\tilde{L}^1_a(D,\ell_2))^\perp \quad \text{in } L^\infty(D,\ell_2).$$

But $\mathrm{Ker}\,\tilde{P} \subset (\tilde{L}^1_a(D,\ell_2))^\perp$, since for $A(r)\in L^\infty(D,\ell_2)$ such that $\tilde{P}A(\cdot)=0$, at least for finite order matrices $A(\cdot), B(\cdot)$ we have

$$\langle \tilde{P}A(\cdot), B(\cdot)\rangle = \langle A(\cdot), \tilde{P}B(\cdot)\rangle,$$

and if $B \in \tilde{L}^1_a(D,\ell_2))$ then

$$\langle A(\cdot),B(\cdot)\rangle = \langle A(\cdot)-\tilde{P}A(\cdot),B(\cdot)\rangle = \langle A(\cdot),B(\cdot)-\tilde{P}B(\cdot)\rangle = 0,$$

consequently, $A(\cdot)\in(\tilde{L}^1_a(D,\ell_2))^\perp$.

Conversely, let $A(\cdot) \in (L^1_a(D,\ell_2))^\perp$, that is, $\langle A(\cdot),B(\cdot)\rangle = 0\ \forall B \in L^1_a(D,\ell_2))$. Taking $B(r)(i,j) = r^{j-i}$ for $j>i$, with fixed j,i and $B(r)(i,j)=0$ otherwise, we get $\int_0^1 a_{ij}(r)(2r\mathrm{d}r) = 0$ for all $j>i$. Thus $(\tilde{P}A)(r)(i,j)=0$ for all i,j, that is, $A(\cdot)\in\mathrm{Ker}\,\tilde{P}$.

For $B(r) \in L^\infty(D,\ell_2)$ and $A \in L^1_a(D,\ell_2)$ we easily get

$$\left|\int_0^1 \operatorname{tr}[A(r)B^*(r)](2r\mathrm{d}r)\right| \le \int_0^1 |\operatorname{tr}[A(r)B^*(r)]|(2r\mathrm{d}r) \le$$

$$\le \int_0^1 \|A(r)\|_{C_1} \cdot \|B(r)\|_{B(\ell_2)} 2r\mathrm{d}r \le \|A\|_{L^1_a(D,\ell_2)} \cdot \|B(r)\|_{L^\infty(D,\ell_2)},$$

so, using Remark 18, we get the required inequality, since for $A \in L^1_a(D,\ell_2)$, $B \in \mathcal{B}(D,\ell_2)$ we obviously have

$$|\langle A, B\rangle| = |\langle A(r), B(r)\rangle| \le \|A\|_{L^1_a(D,\ell_2)} \|B(\cdot)\|_{L^\infty(D,\ell_2)}$$

$\forall\ B(\cdot) \in L^\infty(D,\ell_2)$ defining B, that is, such that $PB(\cdot) = B$. $\square$

Lemma 23. *Let A be a matrix of finite band-type, that is, $A = \sum_{k=1}^n A_k$ such that $A_k \in C_1$ for $k = 1, 2, \ldots$ and let $B \in \mathcal{B}(D,\ell_2)$. Then*

$$\langle A, B\rangle = \sum_{k=0}^\infty \frac{1}{k+1} \operatorname{tr}(A_k \overline{B}_k).$$

Proof. We recall that $\langle A, B\rangle = \int_0^1 \operatorname{tr}[A(r)B^*(r)]2r\mathrm{d}r$. It is easy to see that $\operatorname{tr} A(r)B^*(r) = \sum_{l=1}^n \left(\sum_{k=0}^\infty a_k^l \overline{b_k^l} r^{2k}\right)$. Consequently,

$$\langle A, B\rangle = \int_0^1 \sum_{l=1}^n \left(\sum_{k=0}^\infty a_k^l \overline{b_k^l} r^{2k}\right) 2r\mathrm{d}r = \int_0^1 \sum_{k=0}^\infty 2r^{2k+1} \left(\sum_{l=1}^n a_k^l \overline{b_k^l}\right) \mathrm{d}r =$$

$$= \sum_{k=0}^\infty \frac{1}{k+1} \left(\sum_{l=1}^n a_k^l \overline{b_k^l}\right) = \sum_{k=0}^\infty \frac{1}{k+1} \operatorname{tr}(A_k \overline{B}_k). \quad \square$$

By $L^1_w(D,\ell_2)$ we denote the space of all upper triangular matrices A such that $\|A\|_{L^1_w} = |\sum_{k=0}^\infty \frac{1}{k+1} \sum_{l=1}^\infty a_k^l| < \infty$, equipped with its natural norm.

Let us denote by $SM(X,Y)$, where X and Y are Banach matrix spaces, the space of all Schur multipliers from X into Y, endowed with the natural norm $\|B\|_{SM(X,Y)} = \sup\{\|A * B\|_Y;\ \|A\|_X \le 1\}$.

Then, using Lemma 23, Theorem 22 yields.

Theorem 24. *The matriceal Bloch space $\mathcal{B}(D,\ell_2)$ can be identified with the space $SM(L^1_a(D,\ell_2), L^1_w(D,\ell_2))$, with the equivalence of the norms.*

Now, Theorem 22 has the analogue below.

Theorem 25. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then $(L^p_a(D,\ell_2))^* \sim L^q_a(D,\ell_2)$ with respect to the duality bilinear mapping*

$$\langle A(\cdot), B(\cdot)\rangle = \int_0^1 \operatorname{tr}[A(r)B^*(r)](2r\mathrm{d}r).$$

We should use the boundedness of the projection $P : L^p(D,\ell_2) \to L^p_a(D,\ell_2)$. The details are left to the reader.

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