PARAMETRIC REPRESENTATION AND LINEAR FUNCTIONALS ASSOCIATED WITH EXTENSION OPERATORS FOR BIHOLOMORPHIC MAPPINGS

IAN GRAHAM, GABRIELA KOHR and JOHN A. PFALTZGRAFF

We consider an operator introduced by Pfaltzgraff and Suffridge which provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$. When $n = 1$, this operator reduces to the well known Roper-Suffridge extension operator. In the first part of this paper we prove that if f has parametric representation on B^n then so does F on B^{n+1}. In particular, if $f \in S^*(B^n)$ then $F \in S^*(B^{n+1})$. We also prove that if f is convex on B^n, then the image of F contains the convex hull of the image of some egg domain contained in B^n. In the second part of the paper we investigate some problems related to extreme points and support points for biholomorphic mappings on B^n generated using the Roper-Suffridge extension operator. Given a parametric representation for an extreme point (respectively a support point) generated in this way, we consider whether the corresponding Loewner flow consists only of extreme points (respectively support points). This generalizes work of Pell.

AMS 2000 Subject Classification: Primary 32H02; Secondary 30C45.

Key words: convexity, starlikeness, Loewner chain, parametric representation, Roper-Suffridge extension operator, extreme point, support point.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. For $n \geq 2$, let $\widetilde{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \widetilde{z}) \in \mathbb{C}^n$. The unit ball in $\mathbb{C}^n$ is denoted by B^n. In the case of one variable, B^1 is denoted by U. The ball of radius $r > 0$ in $\mathbb{C}^n$ with center at 0 will be denoted by B^n_r.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of complex-linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm

$$\|A\| = \sup\{\|A(z)\| : \|z\| = 1\}$$

REV. ROUMAINE MATH. PURES APPL., **52** (2007), *1*, 47–68

and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Also, let $H(B^n, \mathbb{C})$ be the set of holomorphic functions from B^n into $\mathbb{C}$. A mapping $f \in H(B^n)$ is called normalized if $f(0) = 0$ and $Df(0) = I_n$. We say that $f \in H(B^n)$ is locally biholomorphic on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $J_f(z) = \det Df(z)$ for $z \in B^n$. Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n, and let $S(B^n)$ denote the set of normalized biholomorphic mappings on B^n. In the case of one variable, the set $S(B^1)$ is denoted by S, and $\mathcal{L}S(B^1)$ is denoted by $\mathcal{L}S$. A mapping $f \in S(B^n)$ is called starlike (respectively convex) if its image is a starlike domain with respect to the origin (respectively convex domain). The classes of normalized starlike (respectively convex) mappings on B^n will be denoted by $S^*(B^n)$ (respectively $K(B^n)$). In the case of one variable, $S^*(B^1)$ (respectively $K(B^1)$) is denoted by S^* (respectively K).

Let $f, g \in H(B^n)$. We say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e., $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

We recall that a mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = \mathrm{e}^t I_n$ for $t \geq 0$, and $f(z, s) \prec f(z, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in B^n$. We note that the requirement $f(z, s) \prec f(z, t)$ is equivalent to the condition that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B^n, \ t \geq s \geq 0.$$

We also note that the normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = \mathrm{e}^{s-t} I_n$ for $0 \leq s \leq t < \infty$.

Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. In particular, f is starlike if and only if $f(z, t) = \mathrm{e}^t f(z)$ is a Loewner chain (see [21]).

Definition 1.1 (see [9], [11], [12], [25], [26]). We say that a normalized mapping $f \in H(B^n)$ has parametric representation if there exists a mapping $h : B^n \times [0, \infty) \to \mathbb{C}^n$ such that

(i) $h(\cdot, t) \in H(B^n)$, $h(0, t) = 0$, $Dh(0, t) = I_n$, $t \geq 0$, $\mathrm{Re}\langle h(z, t), z\rangle > 0$, for $z \in B^n \setminus \{0\}$, $t \geq 0$;

(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$,

and $f(z) = \lim_{t\to\infty} \mathrm{e}^t v(z, t)$ locally uniformly on B^n, where $v = v(z, t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t) \quad \text{a.e.} \quad t \geq 0, \ v(z, 0) = z,$$

for all $z \in B^n$.

The above condition is equivalent to the fact that there exists a Loewner chain $f(z,t)$ such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f(z) = f(z,0)$, $z \in B^n$.

Let $S^0(B^n)$ be the set of mappings which have parametric representation on B^n.

Let $\mathrm{Aut}(B^n)$ be the set of holomorphic automorphisms of B^n. Also, let $\mathcal{U}$ denote the set of unitary transformations in $\mathbb{C}^n$. Then it is well known that

$$\mathrm{Aut}(B^n) = \{V\varphi_a : \ a \in B^n, \ V \in \mathcal{U}\},$$

where

$$\varphi_a(z) = \varphi(z;a) = T_a\left(\frac{z-a}{1-\langle z,a\rangle}\right), \quad z \in B^n, \tag{1.1}$$

$$T_a = \frac{1}{\|a\|^2}\{(1-s_a)aa^* + s_a\|a\|^2 I_n\} \tag{1.2}$$

and

$$s_a = \sqrt{1-\|a\|^2}.$$

Note that $T_0 = I_n$ and $\varphi_0(z) = z$, $z \in B^n$. The following conditions hold (see [29]):

$$|J_{\varphi_a}(0)| = (1-\|a\|^2)^{\frac{n+1}{2}}; \tag{1.3}$$

$$\varphi_a(0) = -a, \quad \varphi_a(a) = 0, \quad \varphi_a^{-1} = \varphi_{-a}. \tag{1.4}$$

Moreover, if $\phi \in \mathrm{Aut}(B^n)$ then

$$|J_\phi(z)| = \left[\frac{1-\|\phi(z)\|^2}{1-\|z\|^2}\right]^{\frac{n+1}{2}}, \quad z \in B^n. \tag{1.5}$$

A key role in our discussion is played by the following Schwarz-type lemma for the Jacobian determinant of a holomorphic mapping from B^n into B^n. We have (cf. [29]; see also [13])

LEMMA 1.1. *Let $\psi \in H(B^n)$ be such that $\psi(B^n) \subseteq B^n$. Then*

$$|J_\psi(z)| \leq \left[\frac{1-\|\psi(z)\|^2}{1-\|z\|^2}\right]^{\frac{n+1}{2}}, \quad z \in B^n. \tag{1.6}$$

This inequality is sharp and equality at a given point $z \in B^n$ holds if and only if $\psi \in \mathrm{Aut}(B^n)$.

Proof. Fix $a \in B^n$ and let $b = \psi(a)$. First, assume that $a \neq 0$ and $b \neq 0$. Let $f : B^n \to \mathbb{C}^n$ be given by $f(z) = (\varphi_b \circ \psi \circ \varphi_{-a})(z)$, $z \in B^n$. Then f is a holomorphic mapping on B^n, $f(0) = 0$ and $f(B^n) \subseteq B^n$. Consequently, by the Schwarz lemma for holomorphic mappings we deduce that $\|f(z)\| \leq \|z\|$, $z \in B^n$, and $\|Df(0)\| \leq 1$. Further, since

$$|J_f(z)| \leq \|Df(z)\|^n, \quad z \in B^n,$$

we have $|J_f(0)| \leq 1$. A simple computation yields

$$Df(0) = D\varphi_b(b)D\psi(a)D\varphi_{-a}(0),$$

and using the last equality in (1.4) we obtain

$$J_f(0) = \frac{J_{\varphi_{-a}}(0)}{J_{\varphi_{-b}}(0)} J_\psi(a).$$

Next, taking into account (1.3) and the above equation, we deduce that

$$1 \geq |J_f(0)| = |J_\psi(a)| \left[\frac{1-\|a\|^2}{1-\|b\|^2}\right]^{\frac{n+1}{2}},$$

hence (1.6) now follows.

If $a = 0$ and $b \neq 0$ then it suffices to consider the mapping $g = \varphi_b \circ \psi$ and to use a similar argument as above. Similarly, if $a \neq 0$ and $b = 0$, then we may consider the mapping $h = \psi \circ \varphi_{-a}$ and apply the above argument. The case $a = b = 0$ is clear.

If $\psi \in \mathrm{Aut}(B^n)$ then equality holds according to (1.5).

Conversely, if equality holds at a given point $a \in B^n$, then reversing the steps in the above proof, we deduce that $|J_f(0)| = 1$ where $f = \varphi_b \circ \psi \circ \varphi_{-a}$ and $b = \psi(a)$. Hence $f \in \mathrm{Aut}(B^n)$ by [16, Theorem 11.3.1], and thus $\psi \in \mathrm{Aut}(B^n)$ too. This completes the proof. □

For $n \geq 1$, set $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

Definition 1.2 ([22]). The extension operator $\Phi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ is defined by

$$\Phi_n(f)(z) = F(z) = \Big(f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}}\Big), \quad z = (z', z_{n+1}) \in B^{n+1}.$$

We choose the branch of the power function such that

$$[J_f(z')]^{1/(n+1)}\Big|_{z'=0} = 1.$$

Then $F = \Phi_n(f) \in \mathcal{L}S_{n+1}$ whenever $f \in \mathcal{L}S_n$. It is easy to see that if $f \in S(B^n)$ then $F \in S(B^{n+1})$.

If $n = 1$ then Φ_1 reduces to the well-known Roper-Suffridge extension operator. For general $n \geq 2$ we have

Definition 1.3 ([27]). The Roper-Suffridge extension operator $\Psi_n : \mathcal{L}S \to \mathcal{L}S_n$ is defined by

$$\Psi_n(f)(z) = \left(f(z_1), \widetilde{z}\sqrt{f'(z_1)}\right), \quad z = (z_1, \widetilde{z}) \in B^n.$$

We choose the branch of the power function such that

$$\sqrt{f'(z_1)}\big|_{z_1=0} = 1.$$

Roper and Suffridge [27] proved that if f is convex on U then $\Psi_n(f)$ is also convex on B^n. Graham and Kohr [8] proved that if f is starlike on U then so is $\Psi_n(f)$ on B^n, and in [10] (see also [9] and [7]) it is shown that if f has parametric representation on the unit disc, then $\Psi_n(f)$ has the same property on B^n. Moreover, if one begins with a complex valued function $f(z_1)$, then the extension to B^2 under $\Phi_1 = \Psi_2$ is $(f(z_1), z_2\sqrt{f'(z_1)})$. If this mapping is then extended to B^3 then to B^4, etc. up to B^n by successive applications of $\Phi_k, k = 1, \dots, n-1$, one obtains the mapping $\Psi_n(f)(z) = (f(z_1), \widetilde{z}\sqrt{f'(z_1)})$, i.e. we obtain the Roper-Suffridge extension operator Ψ_n.

In this paper we prove that if $f \in S(B^n)$ can be imbedded in a Loewner chain $f(z', t)$, then $F = \Phi_n(f)$ can also be imbedded in a Loewner chain $F(z, t)$. Further, if $f \in S^0(B^n)$ then $F = \Phi_n(f) \in S^0(B^{n+1})$. In particular, we give a simplified proof of the theorem of Graham, Kohr and Kohr [10] that the Roper-Suffridge extension operator preserves the parametric representation property. Moreover, we prove that if $f \in S^*(B^n)$ then $F = \Phi_n(f) \in S^*(B^{n+1})$, and if $f \in K(B^n)$ then the image of F contains the convex hull of the image of some egg domain contained in B^n. It would be interesting to give a complete answer to the conjecture of Pfaltzgraff and Suffridge [22, Conjecture 1] that Φ_n preserves convexity, but so far we have not been able to do so. We also discuss the behaviour of Φ_n with respect to starlikeness and convexity on the unit polydisc P^n in $\mathbb{C}^n$.

In the last section we investigate some problems involving extreme points and support points for families of biholomorphic mappings on B^n generated with the Roper-Suffridge extension operator. We consider the following question: given a parametric representation for an extreme point (respectively a support point) of $\Psi_n(S)$, must the corresponding Loewner flow $\mathrm{e}^{-t}F(\cdot, t)$ consist of extreme points (respectively support points)? The analogous one-variable questions were treated by Pell [19] (see also Kirwan [15]).

2. STARLIKENESS AND CONVEXITY PROPERTIES AND THE EXTENSION OPERATOR Φ_n

We begin this section with the following main result. In the case $n = 1$, we obtain a simplified proof of [10, Theorem 2.1].

THEOREM 2.1. *Assume $f \in S(B^n)$ can be imbedded in a Loewner chain $f(z', t)$. Then $F = \Phi_n(f)$ can also be imbedded in a Loewner chain $F(z, t)$.*

Proof. Since $f \in S(B^n)$, it is easiy to see that $F \in S(B^{n+1})$. Let $v = v(z', s, t)$ be the transition mapping associated to $f(z', t)$. Then

$$f(z', s) = f(v(z', s, t), t), \quad z' \in B^n,\ 0 \le s \le t < \infty. \tag{2.1}$$

Let $f_t(z') = f(z', t)$ for $z' \in B^n$ and $t \ge 0$, and let $v_{s,t}(z') = v(z, s, t')$, $z' \in B^n$, $t \ge s \ge 0$. Also, let $F : B^{n+1} \times [0, \infty) \to \mathbb{C}^{n+1}$ be given by

$$F(z, t) = \Big(f(z', t), z_{n+1} \mathrm{e}^{\frac{t}{n+1}} [J_{f_t}(z')]^{\frac{1}{n+1}}\Big) \tag{2.2}$$

for $z = (z', z_{n+1}) \in B^{n+1}$ and $t \ge 0$. We choose the branch of the power function such that

$$[J_{f_t}(z')]^{\frac{1}{n+1}}\Big|_{z'=0} = \mathrm{e}^{nt/(n+1)}.$$

Let us prove that $F(z, t)$ is a Loewner chain. Indeed, since $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$ and $Df(0, t) = \mathrm{e}^t I_n$, it is not difficult to see that $F(\cdot, t)$ is biholomorphic on B^{n+1}, $F(0, t) = 0$ and $DF(0, t) = \mathrm{e}^t I_{n+1}$.

Let $V_{s,t} : B^{n+1} \to \mathbb{C}^{n+1}$ be given by $V_{s,t}(z) = V(z, s, t)$, where

$$V(z, s, t) = \Big(v(z', s, t), z_{n+1} \mathrm{e}^{\frac{s-t}{n+1}} [J_{v_{s,t}}(z')]^{\frac{1}{n+1}}\Big) \tag{2.3}$$

for $z = (z', z_{n+1}) \in B^{n+1}$ and $t \ge s \ge 0$. We choose the branch of the power function such that $[J_{v_{s,t}}(z')]^{\frac{1}{n+1}}\Big|_{z'=0} = \mathrm{e}^{\frac{n(s-t)}{n+1}}$. Then $V_{s,t}$ is biholomorphic on B^n, $V_{s,t}(0) = 0$, $DV_{s,t}(0) = \mathrm{e}^{s-t} I_{n+1}$, and $\|V_{s,t}(z)\| < 1$, $z \in B^{n+1}$. Indeed, by Lemma 1.1 we obtain that

$$\begin{aligned}\|V_{s,t}(z)\|^2 &= \|v_{s,t}(z')\|^2 + \mathrm{e}^{\frac{2(s-t)}{n+1}} |z_{n+1}|^2 |J_{v_{s,t}}(z')|^{\frac{2}{n+1}} \le \\ &\le \|v_{s,t}(z')\|^2 + \frac{|z_{n+1}|^2}{1 - \|z'\|^2}(1 - \|v_{s,t}(z')\|^2) < \\ &< \|v_{s,t}(z')\|^2 + 1 - \|v_{s,t}(z')\|^2 = 1,\ z = (z', z_{n+1}) \in B^{n+1}.\end{aligned}$$

Hence $\|V_{s,t}(z)\| < 1$ for $z \in B^{n+1}$, as claimed.

Further, taking into account (2.1), we easily deduce that $F(z, s) = F(V(z, s, t), t)$ for $z \in B^{n+1}$, $t \ge s \ge 0$. Indeed,

$$\begin{aligned}F(V(z, s, t), t) &= (f(v(z', s, t), t), z_{n+1} \mathrm{e}^{\frac{s}{n+1}} [J_{v_{s,t}}(z')]^{\frac{1}{n+1}} [J_{f_t}(v_{s,t}(z'))]^{\frac{1}{n+1}} = \\ &= \Big(f(z', s), z_{n+1} \mathrm{e}^{\frac{s}{n+1}} [J_{f_s}(z')]^{\frac{1}{n+1}}\Big) = F(z, s),\end{aligned}$$

for all $z \in B^{n+1}$ and $t \ge s \ge 0$. Here we have used (2.1) and the fact that

$$J_{f_s}(z') = J_{f_t}(v_{s,t}(z')) J_{v_{s,t}}(z'), \quad z' \in B^n,\ t \ge s \ge 0.$$

This completes the proof. $\square$

Corollary 2.1. *Assume $f \in S^0(B^n)$. Then $F = \Phi_n(f) \in S^0(B^{n+1})$.*

Proof. Since $f \in S^0(B^n)$, there is a Loewner chain $f(z',t)$ such that $f(z',0) = f(z')$, $z' \in B^n$, and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family. Then

$$\frac{r}{(1+r)^2} \leq \|e^{-t}f(z',t)\| \leq \frac{r}{(1-r)^2}, \quad \|z'\| = r < 1,\ t \geq 0, \tag{2.4}$$

by [11, Corollary 2.6] (see also [6]). Applying the Cauchy integral formula for vector valued holomorphic functions, it is easy to see that for each $r \in (0,1)$ there is $K = K(r) \geq 0$ such that

$$e^{-t}\|Df(z',t)\| \leq K(r), \quad \|z'\| \leq r,\ t \geq 0.$$

Moreover, since

$$|J_{f_t}(z')| \leq \|Df_t(z')\|^n, \quad z' \in B^n,$$

we deduce that there is some $K^* = K^*(r) \geq 0$ such that

$$|J_{f_t}(z')|^{\frac{1}{n+1}} \leq e^{\frac{nt}{n+1}}K^*(r), \quad \|z'\| \leq r,\ t \geq 0. \tag{2.5}$$

Let $F : B^{n+1} \times [0,\infty) \to \mathbb{C}^{n+1}$ be the Loewner chain given by (2.2). Taking into account (2.4) and (2.5) we now easily deduce that for each $r \in (0,1)$ there is some $L = L(r) \geq 0$ such that $e^{-t}\|F(z,t)\| \leq L(r)$, $\|z\| \leq r$, $t \geq 0$. Consequently, $\{e^{-t}F(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family on B^{n+1}, and thus is normal. Hence $F = F(\cdot,0) \in S^0(B^{n+1})$. This completes the proof. □

From Corollary 2.1 and [11, Corollary 2.6] we obtain the following distortion result of independent interest for mappings in $S^0(B^n)$. In particular, this result also holds for mappings in $S^*(B^n)$, since any starlike mapping has parametric representation on B^n (see e.g. [9]).

Corollary 2.2. *Assume $f \in S^0(B^n)$ and $r \in [0,1)$. Then*

$$\frac{r^2}{(1+r)^4} \leq \|f(z')\|^2 + |z_{n+1}|^2|J_f(z')|^{\frac{2}{n+1}} \leq \frac{r^2}{(1-r)^4},$$

$z = (z', z_{n+1}) \in \partial B_r^{n+1}$.

Proof. Since $f \in S^0(B^n)$, we have $F = \Phi_n(f) \in S^0(B^{n+1})$. Then

$$\frac{r}{(1+r)^2} \leq \|F(z)\| \leq \frac{r}{(1-r)^2}, \quad \|z\| = r,$$

by [11, Corollary 2.6]. The result now follows. □

From Corollary 2.1 we can deduce the compactness of the set $\Phi_n(S^0(B^n))$. We have

Corollary 2.3. *The set $\Phi_n(S^0(B^n))$ is compact.*

Proof. Since $\Phi_n(S^0(B^n))$ is a subset of $S^0(B^{n+1})$, $\Phi_n(S^0(B^n))$ is locally uniformly bounded by [11, Corollary 2.6]. We prove that $\Phi_n(S^0(B^n))$ is closed. To this end, let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $\Phi_n(S^0(B^n))$ which converges locally uniformly on B^{n+1} to a mapping F as $k \to \infty$. Also let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $S^0(B^n)$ be such that $F_k = \Phi_n(f_k)$, $k \in \mathbb{N}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is a locally uniformly bounded sequence on B^n, there is a subsequence $\{f_{k_p}\}_{p\in\mathbb{N}}$ of $\{f_k\}_{k\in\mathbb{N}}$ which converges locally uniformly on B^n to a mapping f. Since $S^0(B^n)$ is a compact set, by [11, Theorem 2.9], we deduce that $f \in S^0(B^n)$. Also it is easy to see that the subsequence $\{\Phi_n(f_{k_p})\}_{p\in\mathbb{N}}$ converges locally uniformly on B^{n+1} to $\Phi_n(f)$, and thus we must have $F = \Phi_n(f)$. Hence $F \in \Phi_n(S^0(B^n))$. This completes the proof. □

Example 2.1. (i) Let $f_j \in S$, $j = 1,\dots,n$. Then $f : B^n \to \mathbb{C}^n$ given by $f(z') = (f_1(z_1),\dots,f_n(z_n))$ belongs to $S^0(B^n)$. Indeed, since $f_j \in S$, there is a Loewner chain $f_j(z_j,t)$ such that $f_j(z_j) = f_j(z_j,0)$, $j = 1,\dots,n$. Moreover, $\{e^{-t}f_j(z_j,t)\}_{t\geq 0}$ is a normal family on U since $e^{-t}f_j(z_j,t) \in S$. Next, let $f(z',t) = (f_1(z_1,t),\dots,f_n(z_n,t))$ for $z = (z_1,\dots,z_n) \in B^n$ and $t \geq 0$. Then it is easy to see that $f(z',t)$ is a Loewner chain and $\{e^{-t}f(z',t)\}_{t\geq 0}$ is a normal family on B^n. The desired conclusion now follows.

Further, by Corollary 2.1, we deduce that $F : B^{n+1} \to \mathbb{C}^{n+1}$ given by

$$F(z) = \Big(f_1(z_1),\dots,f_n(z_n), z_{n+1}\prod_{j=1}^{n}[f_j'(z_j)]^{\frac{1}{n+1}}\Big), \quad z = (z', z_{n+1}) \in B^{n+1},$$

belongs to $S^0(B^{n+1})$.

(ii) Let $f \in \mathcal{L}S_n$ be such that

$$(1 - \|z'\|^2)\|[Df(z')]^{-1}D^2f(z')(z',\cdot)\| \leq 1, \quad z' \in B^n.$$

Then $F = \Phi_n(f)$ belongs to $S^0(B^{n+1})$.

Indeed, by [20, Theorem 2.4], f is biholomorphic on B^n and can be imbedded as the first element of the chain

$$f(z',t) = f(e^{-t}z') + (e^t - e^{-t})Df(e^{-t}z')(z'), \quad z' \in B^n,\ t \geq 0.$$

Moreover, since $\lim\limits_{t\to\infty} e^{-t}f(z',t) = z'$ locally uniformly on B^n, $\{e^{-t}f(z',t)\}_{t\geq 0}$ is a normal family on B^n. Hence $f \in S^0(B^n)$. By Corollary 2.1, we have $F = \Phi_n(f) \in S^0(B^{n+1})$.

Another consequence of Theorem 2.1 is given in the following result, which provides a positive answer to the question of Pfaltzgraff and Suffridge regarding the preservation of starlikeness under the operator Φ_n (see [22]).

Corollary 2.4. *Assume $f \in S^*(B^n)$. Then $F = \Phi_n(f) \in S^*(B^{n+1})$.*

Proof. The fact that f is starlike on B^n is equivalent to the statement that $f(z',t) = e^t f(z')$ is a Loewner chain. With this choice of $f(z',t)$, we deduce that $F(z,t)$ given by (2.2) is a Loewner chain. On the other hand, a simple computation yields that $F(z,t) = \mathrm{e}^t F(z)$ for $z \in B^{n+1}$ and $t \geq 0$. Thus $F = F(\cdot, 0) \in S^*(B^{n+1})$, as claimed. This completes the proof. □

Example 2.2. (i) Let $f_j \in S^*$, $j = 1, \ldots, n$. Then $f : B^n \to \mathbb{C}^n$ given by $f(z') = (f_1(z_1), \ldots, f_n(z_n))$ is starlike on B^n. By Corollary 2.4, we deduce that $F : B^{n+1} \to \mathbb{C}^{n+1}$ given by

$$F(z) = \Big(f_1(z_1), \ldots, f_n(z_n), z_{n+1} \prod_{j=1}^{n} [f_j'(z_j)]^{\frac{1}{n+1}}\Big), \quad z = (z', z_{n+1}) \in B^{n+1},$$

is starlike on B^{n+1}. For example, the mapping

$$F(z) = \left(\frac{z_1}{(1-z_1)^2}, \ldots, \frac{z_n}{(1-z_n)^2}, z_{n+1} \prod_{j=1}^{n} \Big[\frac{1+z_j}{(1-z_j)^3}\Big]^{\frac{1}{n+1}}\right)$$

is starlike on B^{n+1}.

(ii) Let a be a complex number. Then the mapping $F : B^3 \to \mathbb{C}^3$ given by

$$F(z) = \big(z_1 + a z_1 z_2, z_2, z_3(1 + a z_2)^{1/3}\big)$$

is starlike if and only if $|a| \leq 1$.

Indeed, if $|a| \leq 1$ then $f : B^2 \to \mathbb{C}^2$ given by $f(z') = (z_1 + a z_1 z_2, z_2)$ is starlike on B^2 (see [32], [28]). Hence F is also starlike on B^3 by Corollary 2.4.

Conversely, if $F \in S^*(B^3)$ then $F(z', 0)$ is starlike on B^2, and thus we must have $|a| \leq 1$ by [32], [28].

We next discuss the case of convex mappings associated with the operator Φ_n. Pfaltzgraff and Suffridge [22] conjectured that if $f \in K(B^n)$ then $\Phi_n(f) \in K(B^{n+1})$.

For $a \in (0, 1]$, let

$$\Omega_{a,n} = \Big\{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \ |z_{n+1}|^2 < a^{\frac{2n}{n+1}}(1 - \|z'\|^2)\Big\}.$$

Then $\Omega_{a,n} \subseteq B^{n+1}$ and $\Omega_{1,n} = B^{n+1}$. We have the following convexity result.

Theorem 2.2. *Let $f \in K(B^n)$ and $\alpha_1, \alpha_2 > 0$ be such that $\alpha_1 + \alpha_2 \leq 1$. Also let $F = \Phi_n(f)$. Then*

$$(1-\lambda)F(z) + \lambda F(w) \in F(\Omega_{\alpha_1+\alpha_2,n}),\ z \in \Omega_{\alpha_1,n},\ w \in \Omega_{\alpha_2,n},\ \lambda \in [0,1].$$

Proof. Our argument combines an idea of Gong and Liu [4] (see also [5]) with the estimates for the Jacobian determinant of a holomorphic mapping from B^n to itself (Lemma 1.1). Fix $\lambda \in [0, 1]$ and let $z \in \Omega_{\alpha_1,n}$ and $w \in \Omega_{\alpha_2,n}$. Since $f \in K(B^n)$, F is biholomorphic on B^{n+1}. We want to find a point $u = (u', u_{n+1}) \in \Omega_{\alpha_1+\alpha_2,n}$ such that

$$(1-\lambda)F(z) + \lambda F(w) = F(u),$$

i.e.,

$$\Big(f(u'), u_{n+1}[J_f(u')]^{\frac{1}{n+1}}\Big) =$$
$$= \Big((1-\lambda)f(z') + \lambda f(w'), (1-\lambda)z_{n+1}[J_f(z')]^{\frac{1}{n+1}} + \lambda w_{n+1}[J_f(w')]^{\frac{1}{n+1}}\Big).$$

Since $z', w' \in B^n$ and $f \in K(B^n)$, there is a unique point $u' \in B^n$ such that

$$f(u') = (1-\lambda)f(z') + \lambda f(w'),$$

i.e., $u' = f^{-1}((1-\lambda)f(z') + \lambda f(w'))$. If $\lambda = 0$ then $u' = z'$, and if $\lambda = 1$ then $u' = w'$. Hence, we may assume that $\lambda \in (0,1)$. Then $u' = u'(z', w')$ can be viewed as a mapping from $B^n \times B^n$ into B^n. Let

$$u_{n+1} = (1-\lambda)z_{n+1}\left[\frac{J_f(z')}{J_f(u')}\right]^{\frac{1}{n+1}} + \lambda w_{n+1}\left[\frac{J_f(w')}{J_f(u')}\right]^{\frac{1}{n+1}}.$$

We prove that $u = (u', u_{n+1}) \in \Omega_{\alpha_1+\alpha_2,n}$. It is obvious that

$$J_{u'_{z'}}(z', w') = (1-\lambda)^n\frac{J_f(z')}{J_f(u')} \quad \text{and} \quad J_{u'_{w'}}(z', w') = \lambda^n\frac{J_f(w')}{J_f(u')},$$

where $u'_{z'}$ and $u'_{w'}$ denote the Fréchet derivatives of u' with respect to z' and w', respectively. Hence

$$u_{n+1} = (1-\lambda)^{\frac{1}{n+1}}z_{n+1}[J_{u'_{z'}}(z', w')]^{\frac{1}{n+1}} + \lambda^{\frac{1}{n+1}}w_{n+1}[J_{u'_{w'}}(z', w')]^{\frac{1}{n+1}}.$$

Next, using Lemma 1.1 and Hölder's inequality in the previous equation, we obtain

$$|u_{n+1}| \le (1-\lambda)^{\frac{1}{n+1}}|z_{n+1}|\left[\frac{1-\|u'(z', w')\|^2}{1-\|z'\|^2}\right]^{1/2} + \lambda^{\frac{1}{n+1}}|w_{n+1}|\left[\frac{1-\|u'(z', w')\|^2}{1-\|w'\|^2}\right]^{1/2}$$
$$\le \sqrt{1-\|u'\|^2}(1-\lambda+\lambda)^{\frac{1}{n+1}}\left\{\left[\frac{|z_{n+1}|^2}{1-\|z'\|^2}\right]^{\frac{n+1}{2n}} + \left[\frac{|w_{n+1}|^2}{1-\|w'\|^2}\right]^{\frac{n+1}{2n}}\right\}^{\frac{n}{n+1}}$$
$$< \sqrt{1-\|u'\|^2}(\alpha_1+\alpha_2)^{\frac{n}{n+1}}.$$

Therefore, we have proved that $|u_{n+1}|^2 < (\alpha_1+\alpha_2)^{\frac{2n}{n+1}}(1-\|u'\|^2)$, i.e., $u = (u', u_{n+1}) \in \Omega_{\alpha_1+\alpha_2,n}$. This completes the proof. □

COROLLARY 2.5. *Let $f \in K(B^n)$ and $F = \Phi_n(f)$. Then*

$$(1-\lambda)F(z) + \lambda F(w) \in F(B^{n+1}), \quad z, w \in \Omega_{1/2,n},\ \lambda \in [0,1].$$

It is natural to investigate the situation concerning starlikeness and convexity on the unit polydisc $P^n = \big\{z \in \mathbb{C}^n : \max_{1\le j\le n}|z_j| < 1\big\}$.

Remark 2.1. The operator Φ_n does not preserve convexity on P^n. Indeed, let $f : P^n \to \mathbb{C}^n$ be a normalized convex mapping and let $F = \Phi_n(f)$. By [31, Theorem 2] there exist normalized convex functions $f_k(z_k)$ on the unit disc U, $k = 1, \dots, n$, such that

$$f(z') = (f_1(z_1), \dots, f_n(z_n)), \quad z' = (z_1, \dots, z_n) \in P^n.$$

Then $F = \Phi_n(f)$ is given by

$$(2.6)\quad F(z) = \left(f_1(z_1), \dots, f_n(z_n), z_{n+1} \prod_{k=1}^{n} [f_k'(z_k)]^{\frac{1}{n+1}}\right), \; z = (z', z_{n+1}) \in P^{n+1}.$$

Here we choose the branch of the power function such that $[f_k'(z_k)]^{1/(n+1)}|_{z_k=0} = 1$ for $k = 1, \dots, n$. It is clear that if there is some k such that $f_k'(z_k) \not\equiv 1$, then F does not satisfy the necessary and sufficient condition for convexity in the polydisc given by Suffridge.

Remark 2.2. The operator Φ_n does not preserve starlikeness on the unit polydisc P^n either. Indeed, let $f_j(z_j)$ be a function in S^* for $j = 1, \dots, n$. Then it is easy to see that the mapping $f(z') = (f_1(z_1), \dots, f_n(z_n))$ is starlike on P^n while F given by (2.6) is not starlike on P^{n+1}. Indeed, the necessary and sufficient condition for starlikeness of F is (see [31, Theorem 1])

$$(2.7)\qquad \operatorname{Re}\left[\frac{w_j(z)}{z_j}\right] > 0, \; j = 1, \dots, n+1, \; \|z\|_\infty = |z_j| > 0,$$

where

$$w(z) = [DF(z)]^{-1} F(z) =$$

$$= \left(\frac{f_1(z_1)}{f_1'(z_1)}, \dots, \frac{f_n(z_n)}{f_n'(z_n)}, z_{n+1}\left[1 - \frac{1}{n+1}\sum_{k=1}^{n} \frac{f_k(z_k) f_k''(z_k)}{(f_k'(z_k))^2}\right]\right),$$

for all $z = (z', z_{n+1}) \in P^{n+1}$.

It is clear that the first n inequalities in (2.7) are satisfied, but if $j = n+1$ then (2.7) becomes

$$0 < \operatorname{Re}\left[\frac{w_{n+1}(z)}{z_{n+1}}\right] = 1 - \frac{1}{n+1}\sum_{k=1}^{n} \operatorname{Re}\left[\frac{f_k(z_k) f_k''(z_k)}{(f_k'(z_k))^2}\right], \; |z_{n+1}| = \|z\|_\infty > 0,$$

i.e.,

$$1 - \frac{1}{n+1}\sum_{k=1}^{n} \operatorname{Re}\left[\frac{f_k(z_k) f_k''(z_k)}{(f_k'(z_k))^2}\right] > 0, \quad |z_k| < 1, \, k = 1, \dots, n.$$

It suffices to choose $n = 2$ and $f_j(z_j) = z_j/(1 - z_j)^2$ for $j = 1, 2$. Also let $z_1 = z_2 \in U$. Then the above inequality is equivalent to

$$\operatorname{Re}\left[\frac{(1 - z_1)(z_1 + 3)}{(1 + z_1)^2}\right] > 0, \quad |z_1| < 1.$$

However, it is elementary to check that this relation is not satisfied everywhere on the unit disc U, and thus F is not starlike.

3. EXTREME POINTS AND SUPPORT POINTS ASSOCIATED WITH THE ROPER-SUFFRIDGE EXTENSION OPERATOR

In this section we restrict our discussion to the Roper-Suffridge extension operator Ψ_n. A key role is played by the result established in Theorem 2.1. We shall study extreme points and support points for families of univalent mappings on B^n constructed using the Roper-Suffridge operator, and their behaviour under the Loewner variation. To this end, we recall the definitions of extreme points and support points in the general setting of locally convex linear spaces. For applications of linear methods to the study of extremal problems in geometric function theory of one variable, the reader may consult [14], [1], [2], [3], [24], [30], and the recent survey [17]. In the case of several variables, see [18].

Definition 3.1. Let X be a locally convex linear space and let E be a subset of X.

(i) A point $x \in E$ is called an *extreme point* of E provided $x = ty+(1-t)z$, where $t \in (0,1)$, $y, z \in E$, implies $x = y = z$. That is, $x \in E$ is an extreme point of E if x is not a proper convex combination of two points in E.

(ii) A point $w \in E$ is called a *support point* of E if

$$\operatorname{Re} L(w) = \max_{y \in E} \operatorname{Re} L(y)$$

for some continuous linear functional $L : X \to \mathbb{C}$ such that $\operatorname{Re} L$ is nonconstant on E.

Let $\operatorname{ex} E$ and $\operatorname{supp} E$ be the sets of extreme points of E and support points of E. From the general theory of locally convex linear spaces, in particular by the Krein-Milman theorem (see e.g. [14, Chapter 4]), it is known that if E is a nonempty compact subset of X then $\operatorname{ex} E$ and $\operatorname{supp} E$ are nonempty subsets of E.

In the case of the class S, it is known that if $f \in \operatorname{ex} S$ or $f \in \operatorname{supp} S$, then f maps the unit disc U onto the complement of a continuous arc tending to ∞ with increasing modulus (see e.g. [17]). In particular, a bounded univalent function cannot be a support point of S.

We recall in view of the proof of Theorem 2.1 that if $f(z_1, t)$ is a Loewner chain then $F : B^n \times [0, \infty) \to \mathbb{C}^n$ given by

$$F(z,t) = \big(f(z_1,t), \widetilde{z}\mathrm{e}^{t/2}(f'(z_1,t))^{1/2}\big), \quad z = (z_1, \widetilde{z}) \in B^n,\ t \geq 0, \tag{3.1}$$

is also a Loewner chain. We choose the branch of the power function such that $(f'(z_1,t))^{1/2}|_{z_1=0} = \mathrm{e}^{t/2}$ for $t \geq 0$. If $t = 0$ then $F = F(\cdot, 0)$ is the Roper-Suffridge extension operator $\Psi_n(f)$.

We also recall that $\Psi_n(S)$ is a compact set by Corollary 2.3, hence $\mathrm{ex}\,\Psi_n(S) \neq \emptyset$ and $\mathrm{supp}\,\Psi_n(S) \neq \emptyset$.

We begin this section with some auxiliary results.

Lemma 3.1. $\Psi_n(\mathrm{ex}\,S) \subseteq \mathrm{ex}\,\Psi_n(S)$.

Proof. Let $F \in \Psi_n(\mathrm{ex}\,S)$ and $f \in \mathrm{ex}\,S$ be such that $F = \Psi_n(f)$. Suppose $F = sG + (1-s)H$ where $s \in (0,1)$ and $G, H \in \Psi_n(S)$. Then there exist functions $g, h \in S$ such that $G = \Psi_n(g)$, $H = \Psi_n(h)$ and

$$\Psi_n(f)(z) = s\Psi_n(g)(z) + (1-s)\Psi_n(h)(z), \quad z \in B^n.$$

Hence

$$f(z_1) = sg(z_1) + (1-s)h(z_1), \quad z_1 \in U,$$

and since $f \in \mathrm{ex}\,S$, we must have $g \equiv h$. Therefore $G \equiv H$, too. This completes the proof. $\square$

Example 3.1. It is known that the rotations of the Koebe function, given by $f(z_1) = z_1/(1-\lambda z_1)^2$, where $|\lambda| = 1$, are extreme points of S. Then $F = \Psi_n(f) \in \mathrm{ex}\,\Psi_n(S)$ by Lemma 3.1, i.e., the mapping F_λ given by

$$F_\lambda(z) = \left(\frac{z_1}{(1-\lambda z_1)^2}, \widetilde{z}\left(\frac{1+\lambda z_1}{(1-\lambda z_1)^3}\right)^{1/2}\right), \quad z = (z_1, \widetilde{z}) \in B^n, \tag{3.2}$$

is an extreme point of $\Psi_n(S)$ for $|\lambda| = 1$.

Lemma 3.2. *Let $f \in \mathrm{ex}\,S$ and let $f(z_1,t)$ be a Loewner chain such that $f(z_1) = f(z_1, 0)$, $z_1 \in U$. Also let $F(z,t)$ be given by* (3.1)*. Then $\mathrm{e}^{-t}F(\cdot,t) \in \mathrm{ex}\,\Psi_n(S)$ for $t \geq 0$.*

Proof. Since f is an extreme point of S, $\mathrm{e}^{-t}f(\cdot,t)$ also is an extreme point of S by a result of Pell [19]. Then $\Psi_n(\mathrm{e}^{-t}f(\cdot,t)) \in \Psi_n(\mathrm{ex}\,S)$ for $t \geq 0$. Hence $\Psi_n(\mathrm{e}^{-t}f(\cdot,t)) \in \mathrm{ex}\,\Psi_n(S)$ by Lemma 3.1. On the other hand, since

$$\Psi_n(\mathrm{e}^{-t}f(\cdot,t)) = \mathrm{e}^{-t}F(\cdot,t), \quad t \geq 0,$$

the conclusion follows. $\square$

We now prove one of the main results of this paper.

Theorem 3.1. *Let $f \in S$ and $F = \Psi_n(f)$. Also let $F(z,t)$ be the Loewner chain given by* (3.1)*. If $F \in \mathrm{ex}\,\Psi_n(S)$ then $\mathrm{e}^{-t}F(\cdot,t) \in \mathrm{ex}\,\Psi_n(S)$ for $t \geq 0$.*

Proof. Fix $t \geq 0$. Suppose that

$$\mathrm{e}^{-t}F(z,t) = \lambda G(z) + (1-\lambda)H(z), \quad z \in B^n,$$

where $\lambda \in (0,1)$ and $G, H \in \Psi_n(S)$. Let $V = V(z,s,t)$ be the transition mapping associated with $F(z,t)$. Also let $V(z,t) = V(z,0,t)$ for $z \in B^n$. Then

$$F(z) = F(V(z,t),t) = \lambda \mathrm{e}^t G(V(z,t)) + (1-\lambda)\mathrm{e}^t H(V(z,t)),\ z \in B^n.$$

Let $g, h \in S$ be such that $G = \Psi_n(g)$ and $H = \Psi_n(h)$. Also, let $v(z_1,t) = v(z_1,0,t)$ where $v(z_1,s,t)$ is the transition function associated with $f(z_1,t)$. A simple computation yields

$$\mathrm{e}^t G(V(z,t)) = \Psi_n(\mathrm{e}^t g \circ v(\cdot,t))(z), \quad z \in B^n.$$

Indeed, if $v_t = v(\cdot,t)$ then

$$\mathrm{e}^t G(V(z,t)) = \Big(\mathrm{e}^t g(v(z_1,t)), \widetilde{z}\mathrm{e}^{-t/2}\mathrm{e}^t(g'(v(z_1,t))^{1/2}(v'(z_1,t))^{1/2}\Big) =$$

$$= \Big(\mathrm{e}^t g(v_t(z_1)), \widetilde{z}(\mathrm{e}^t(g\circ v_t)'(z_1))^{1/2}\Big) = \Psi_n(\mathrm{e}^t g \circ v_t)(z), \quad z = (z_1,\widetilde{z}) \in B^n,$$

as claimed. Further, since $g \in S$ and v_t is a univalent function on U such that $v_t'(0) = \mathrm{e}^{-t}$, the composition $\mathrm{e}^t g \circ v_t$ is a function in S. Hence $\mathrm{e}^t G \circ V_t \in \Psi_n(S)$. Similarly, $\mathrm{e}^t H \circ V_t \in \Psi_n(S)$. Then we deduce that

$$F(z) = \lambda \Psi_n(\mathrm{e}^t g \circ v_t)(z) + (1-\lambda)\Psi_n(\mathrm{e}^t h \circ v_t)(z), \quad z \in B^n,$$

and since $F \in \operatorname{ex}\Psi_n(S)$, we must have $\Psi_n(\mathrm{e}^t g \circ v_t)(z) = \Psi_n(\mathrm{e}^t h \circ v_t)(z)$ for $z \in B^n$. Finally, applying the identity theorem for holomorphic mappings, we deduce that $\Psi_n(g) \equiv \Psi_n(h)$, i.e. $G \equiv H$. This completes the proof. $\square$

We next consider the analog of a result of Pell [19] concerning support points and Loewner chains associated with the Roper-Suffridge extension operator. See also [15].

Lemma 3.3. *Let $f \in S$ and $F = \Psi_n(f)$. Also let $F(z,t)$ be the Loewner chain given by* (3.1). *If $f \in \operatorname{supp} S$ then $\mathrm{e}^{-t}F(\cdot,t) \in \Psi_n(\operatorname{supp} S)$ for $t \geq 0$.*

Proof. Let $f(z_1,t)$ be a Loewner chain such that $f(z_1) = f(z_1,0)$ for $|z_1| < 1$, and $\mathrm{e}^{-t}F(\cdot,t) = \Psi_n(\mathrm{e}^{-t}f(\cdot,t))$, $t \geq 0$. Since $f \in \operatorname{supp} S$, by [19, Theorem] we have $\mathrm{e}^{-t}f(\cdot,t) \in \operatorname{supp} S$, too, for all $t \geq 0$. Hence $\mathrm{e}^{-t}F(\cdot,t) \in \Psi_n(\operatorname{supp} S)$ for $t \geq 0$, as claimed. $\square$

Using Lemma 3.3 and a result of Pfluger [23], we obtain the following asymptotic result.

Corollary 3.1. *If $f \in \operatorname{supp} S$ and $F(z,t)$ is the Loewner chain given by* (3.1), *then there is some $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\lim_{t\to\infty} \mathrm{e}^{-t}F(z,t) = F_\lambda(z)$ locally uniformly on B^{n+1}, where F_λ is given by* (3.2). *Moreover, $F_\lambda \in \Psi_n(\operatorname{supp} S)$.*

Proof. Let $f(z_1,t)$ be a Loewner chain such that $f(z_1) = f(z_1,0)$ for $|z_1| < 1$, and $\mathrm{e}^{-t}F(\cdot,t) = \Psi_n(\mathrm{e}^{-t}f(\cdot,t))$, $t \geq 0$. Since $f \in \operatorname{supp} S$, by a result of Pfluger [23] (in particular, using the fact that the boundary curve of $f(U)$ has an asymptotic direction) we have

$$\lim_{t\to\infty} \mathrm{e}^{-t}f(z_1,t) = \frac{z_1}{(1-\lambda z_1)^2}$$

and the limit holds locally uniformly on U for some choice of λ, $|\lambda| = 1$. Now, it is easy to deduce that

$$\lim_{t\to\infty} \Psi_n(\mathrm{e}^{-t}f(\cdot,t))(z) = F_\lambda(z)$$

locally uniformly on B^n, as claimed.

On the other hand, since the rotations of the Koebe function are all members of $\operatorname{supp} S$, the mapping F_λ belongs to $\Psi_n(\operatorname{supp} S)$. This completes the proof. $\square$

Example 3.2. The mapping F_λ given by (3.2) belongs to $\operatorname{supp}\Psi_n(S)$.

Proof. It suffices to assume $\lambda = 1$. Let $e_1 = (1,0,\ldots,0) \in \partial B^n$ and $L : H(B^n) \to \mathbb{C}$ be given by

$$L(F) = \langle D^2F(0)(e_1,e_1),e_1\rangle, \quad F \in H(B^n).$$

Then it is clear that L is a continuous linear functional on $H(B^n)$. We show that $\operatorname{Re} L|_{\Psi_n(S)}$ is nonconstant and

$$\operatorname{Re} L(F_1) = \max_{F\in\Psi_n(S)} \operatorname{Re} L(F).$$

Suppose that $\operatorname{Re} L|_{\Psi_n(S)}$ is constant. Since the identity mapping id of B^n belongs to $\Psi_n(S)$, we have $\operatorname{Re} L(\Psi_n(f)) = \operatorname{Re} L(\mathrm{id})$, $\forall f \in S$. This is equivalent to

$$\operatorname{Re}\langle D^2\Psi_n(f)(0)(e_1,e_1),e_1\rangle = 0, \quad \forall f \in S.$$

However, this is impossible since if $k(z_1) = z_1/(1-z_1)^2$ then

$$\operatorname{Re}\langle D^2\Psi_n(k)(0)(e_1,e_1),e_1\rangle = 4.$$

Hence we must have $\operatorname{Re} L|_{\Psi_n(S)}$ nonconstant, as claimed. On the other hand, if $f \in S$ then a straightforward computation yields

$$\operatorname{Re}\langle D^2\Psi_n(f)(0)(e_1,e_1),e_1\rangle = \operatorname{Re} f''(0) \leq 4,$$

thus

$$\operatorname{Re} L(F_1) = \max_{F\in\Psi_n(S)} \operatorname{Re} L(F).$$

Consequently, $F_1 \in \operatorname{supp}\Psi_n(S)$. $\square$

Next, we obtain a representation for a continuous linear functional on $H(B^n)$ in terms of the Taylor coefficients. The corresponding one variable result is due to Toeplitz [33] (see also [3, Theorem 9.3] and [14, Theorem 4.3].

First, note that a continuous linear functional L on $H(B^n)$ is easily expressed as a sum of continuous linear functionals on the component functions, i.e.,

$$L(h) = L(h_1, \dots, h_n) = \sum_{k=1}^{n} L(0, \dots, 0, h_k, 0, \dots, 0) = \sum_{k=1}^{n} L_k(h_k)$$

for all $h = (h_1, \dots, h_n) \in H(B^n)$, where

$$L_k(h_k) = L(0, \dots, 0, h_k, 0, \dots, 0), \quad k = 1, \dots, n.$$

For each monomial $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, let $L_k(z^\alpha) = d_{k,\alpha}$. We expand h_k in a Taylor series for each k as

$$h_k(z) = \sum_{\alpha} c_{k,\alpha} z^\alpha, \quad k = 1, \dots, n.$$

LEMMA 3.4. *L_k is given by*

$$L_k(h_k) = \sum_{\alpha} c_{k,\alpha} d_{k,\alpha}. \tag{3.3}$$

The series converges uniformly and $L(h) = \sum_{k=1}^{n} \sum_{\alpha} c_{k,\alpha} d_{k,\alpha}$.

Proof. For $m \in \mathbb{N}$, let

$$h_{k,m}(z) = \sum_{|\alpha| \le m} c_{k,\alpha} z^\alpha.$$

Then

$$L_k(h_{k,m}) = \sum_{|\alpha| \le m} c_{k,\alpha} d_{k,\alpha}.$$

The continuity of L_k implies that $\lim_{m\to\infty} L_k(h_{k,m}) = L_k(h_k)$, and this implies that the series (3.3) converges. Since any rearrangement of $\sum_{\alpha} c_{k,\alpha} z^\alpha$ converges locally uniformly to h_k, any rearrangement of the series in (3.3) converges to $L_k(h_k)$. This completes the proof. $\square$

THEOREM 3.2. *The identity mapping of B^n is not a support point of $\Psi_n(S)$.*

Proof. It suffices to give the proof in the case $n = 2$. Suppose that L is a continuous linear functional on $H(B^2)$ such that $\operatorname{Re} L|_{\Psi_2(S)}$ is nonconstant and

$$\operatorname{Re} L(\mathrm{id}) = \max_{F \in \Psi_2(S)} \operatorname{Re} L(F).$$

If $f \in S$ has the Taylor series expansion $f(z_1) = z_1 + \sum\limits_{j=2}^{\infty} a_j z_1^j$, then

$$\Psi_2(f)(z) = \Big(z_1 + \sum_{j=2}^{\infty} a_j z_1^j, z_2\sqrt{1 + \sum_{j=2}^{\infty} j a_j z_1^{j-1}}\Big) = \tag{3.4}$$

$$= \Big(z_1 + \sum_{j=2}^{\infty} a_j z_1^j, z_2\Big(1 + \sum_{j=2}^{\infty} b_j z_1^{j-1}\Big)\Big),$$

where the b_j's are determined by the a_j's.

Hence the restriction of a continuous linear functional on $H(B^2)$ to $\Psi_2(S)$ is given by

$$L(\Psi_2(f)) = d_{11} + \sum_{j=2}^{\infty} a_j d_{1j} + d_{21} + \sum_{j=2}^{\infty} b_j d_{2j}, \tag{3.5}$$

where $d_{1j} = L_1(z_1^j)$ and $d_{2j} = L_2(z_2 z_1^{j-1})$, $j \geq 1$.

Consider the function $f(z_1) = z_1 + \gamma z_1^2$, which belongs to S if $|\gamma|$ is sufficiently small. Let

$$\sqrt{1 + 2\gamma z_1} = 1 + \gamma z_1 + \sum_{j=2}^{\infty} \beta_j \gamma^j z_1^j,$$

and note that $\beta_j \neq 0$, $j \geq 2$. Then

$$L(\Psi_2(f)) = d_{11} + \gamma d_{12} + d_{21} + \gamma d_{22} + \sum_{j=3}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j} = \tag{3.6}$$

$$= d_{11} + d_{21} + \gamma(d_{12} + d_{22}) + \sum_{j=3}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j}.$$

Now, if $d_{12} + d_{22} \neq 0$, we obtain that $\mathrm{Re}[\gamma(d_{12} + d_{22})] > 0$ by suitably choosing the argument of γ. Then, choosing $|\gamma|$ sufficiently small and noting that

$$\sum_{j=3}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j} = O(|\gamma|^2),$$

we obtain

$$\Big|\sum_{j=3}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j}\Big| < \mathrm{Re}[\gamma(d_{12} + d_{22})],$$

hence

$$\mathrm{Re}\Big[\gamma(d_{12} + d_{22}) + \sum_{j=3}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j}\Big] > 0.$$

But then $\operatorname{Re} L(\Psi_2(f)) > \operatorname{Re} L(\mathrm{id})$, which is a contradiction. Hence $d_{12}+d_{22}=0$.

A similar argument using (3.6) shows that $d_{2j} = 0$, $j = 3, 4, \ldots$. In fact, suppose that (3.6) has the form

$$L(\Psi_2(f)) = d_{11} + d_{21} + \sum_{j=j_0}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j}, \tag{3.7}$$

where $f(z_1) = z_1 + \gamma z_1^2$, $j_0 \geq 3$, and $d_{2j_0} \neq 0$. Then, by suitably choosing the argument of γ, we obtain $\operatorname{Re}[\beta_{j_0-1}\gamma^{j_0-1}d_{2j_0}] > 0$, and, by choosing $|\gamma|$ sufficiently small, we obtain

$$\Big| \sum_{j=j_0+1}^{\infty} \beta_{j-1}\gamma^{j-1} d_{2j} \Big| < \operatorname{Re}[\beta_{j_0-1}\gamma^{j_0-1}d_{2j_0}].$$

Thus $\operatorname{Re} L(\Psi_2(f)) > \operatorname{Re} L(\mathrm{id})$, which is a contradiction.

To show that $d_{1j} = 0$, $j \geq 3$, consider $f(z_1) = z_1 + \gamma z_1^j$, where $|\gamma|$ is sufficiently small. Since $d_{12} + d_{22} = 0$ and $d_{2j} = 0$, $j \geq 3$, it is easy to see that

$$L(f) = d_{11} + d_{21} + \gamma d_{1j}.$$

The usual argument by contradiction yields that $d_{1j} = 0$.

We now know that $d_{12} + d_{22} = 0$ and that $d_{1j} = d_{2j} = 0$, $j \geq 3$. In the representation (3.4) of $\Psi_2(f)$, we have $a_2 = b_2$. Hence (3.5) shows that $L(\Psi_2(f)) = d_{11} + d_{21}$ for all $f \in S$. □

Theorem 3.3. *Let $f \in S$ and let $f(z_1, t)$ be a Loewner chain such that $f = f(\cdot, 0)$. Let $F(z, t)$ be the Loewner chain given by* (3.1) *and let $F = \Psi_n(f)$. If $F \in \operatorname{supp} \Psi_n(S)$ then there exists $t_0 > 0$ such that $\mathrm{e}^{-t}F(\cdot, t) \in \operatorname{supp} \Psi_n(S)$ for $0 \leq t < t_0$.*

Proof. We mention that some of the ideas used below come from the proof of [15, Theorem 1].

Since $F \in \operatorname{supp} \Psi_n(S)$, there is a continuous linear functional L on $H(B^n)$ such that $\operatorname{Re} L$ is nonconstant on $\Psi_n(S)$ and

$$\operatorname{Re} L(F) = \max_{G \in \Psi_n(S)} \operatorname{Re} L(G). \tag{3.8}$$

Fix $t \geq 0$. Since $L : H(B^n) \to \mathbb{C}$ is linear, we have

$$L(h) = L(h_1, \ldots, h_n) = \sum_{k=1}^{n} L(0, \ldots, h_k, 0, \ldots, 0) = \sum_{k=1}^{n} L_k(h_k),$$

for all $h = (h_1, \ldots, h_n) \in H(B^n)$, where

$$L_k(h_k) = L(0, \ldots, h_k, 0, \ldots, 0), \quad k = 1, \ldots, n.$$

It is easy to see that L_k is a continuous linear functional on $H(B^n,\mathbb{C})$ for $k=1,\dots,n$. Arguing as in the proof of [3, Theorem 9.2], we deduce that there exist an integer $m\geq 2$ and a constant $K>0$ such that

$$|L_k(g)|\leq K\sup\left\{|g(z)|: z\in B^n_{1-1/m}\right\},\quad g\in H(B^n,\mathbb{C}).$$

By the Hahn-Banach theorem, L_k extends to a continuous linear functional on the space of continuous complex-valued functions on the closed ball $\overline{B}^n_{1-1/m}$ (with the supremum norm). Hence, by the Riesz representation theorem, L_k is given by integration with respect to a complex Borel measure supported on $\overline{B}^n_{1-1/m}$. That is, there exists a compact subset E_k of B^n and a complex Borel measure $\mathrm{d}\mu_k$ supported on E_k such that

$$L_k(g)=\int_{E_k} g(z)\mathrm{d}\mu_k(z),\quad \forall g\in H(B^n,\mathbb{C}),\ k=1,\dots,n.$$

Hence

$$L(h)=\sum_{k=1}^n\int_{E_k} h_k(z)\mathrm{d}\mu_k(z),\quad \forall h=(h_1,\dots,h_n)\in H(B^n).$$

On the other hand, since $F(z)=F(V(z,t),t)$, $z\in B^n$, $t\geq 0$, where $V(z,t)=V(z,0,t)$ and $V=V(z,s,t)$ is the transition mapping associated with $F(z,t)$, we deduce that

$$L(F)=\sum_{k=1}^n\int_{E_k} F_k(z)\mathrm{d}\mu_k(z)=\sum_{k=1}^n\int_{E_k} F_k(V(z,t),t)\mathrm{d}\mu_k(z).$$

Setting $\zeta=V_t(z)$ where $V_t(z)=V(z,t)$, we obtain

$$\begin{aligned}L(F)&=\sum_{k=1}^n\int_{V(E_k,t)} F_k(\zeta,t)\mathrm{d}\mu_k(V_t^{-1}(\zeta))=\\&=\sum_{k=1}^n\int_{V(E_k,t)} \mathrm{e}^{-t}F_k(\zeta,t)\mathrm{e}^t\mathrm{d}\mu_k(V_t^{-1}(\zeta)).\end{aligned}$$

Since $\mathrm{d}\nu_k(\zeta,t)=\mathrm{e}^t\mathrm{d}\mu_k(V_t^{-1}(\zeta))$ is a complex Borel measure supported on the compact set $V(E_k,t)\subset B^n$, we may consider the functional $L_t: H(B^n)\to\mathbb{C}$ given by

$$L_t(G)=\sum_{k=1}^n\int_{V_t(E_k)} G_k(\zeta)\mathrm{d}\nu_k(\zeta,t),\quad G=(G_1,\dots,G_n)\in H(B^n).$$

Then L_t is a continuous linear functional on $H(B^n)$ and it is clear that

$$L(F)=L_t(\mathrm{e}^{-t}F(\cdot,t)). \tag{3.9}$$

Moreover, if $G \in \Psi_n(S)$ then by another change of variable we obtain that

$$L_t(G) = L(\mathrm{e}^t G \circ V_t), \quad G \in \Psi_n(S). \tag{3.10}$$

Next, taking into account the fact that $F \in \operatorname{supp} \Psi_n(S)$ and equations (3.9) and (3.10), we deduce that

$$\operatorname{Re} L_t(\mathrm{e}^{-t} F(\cdot, t)) = \operatorname{Re} L(F) \geq \operatorname{Re} L(\mathrm{e}^t G \circ V_t) = \operatorname{Re} L_t(G)$$

for $G \in \Psi_n(S)$, i.e.,

$$\operatorname{Re} L_t(\mathrm{e}^{-t} F(\cdot, t)) = \max_{G \in \Psi_n(S)} \operatorname{Re} L_t(G).$$

Here we have used the fact that $\mathrm{e}^t G \circ V_t \in \Psi_n(S)$ for $G \in \Psi_n(S)$, by a similar argument to that in the proof of Theorem 3.1. Now the functionals $\{L_t\}_{t \geq 0}$ are weakly continuous in their dependence on t. In particular, if id is the identity mapping of B^n, then $L_t(\mathrm{id}) \to L(\mathrm{id})$ as $t \to 0^+$. Since id is not a support point of $\Psi_n(S)$, we have $\operatorname{Re} L(\mathrm{id}) < \operatorname{Re} L(F)$. Hence there exists $t_0 > 0$ such that

$$\operatorname{Re} L_t(\mathrm{id}) < \operatorname{Re} L(F) = \operatorname{Re} L_t(\mathrm{e}^{-t} F(\cdot, t)), \quad 0 \leq t < t_0. \tag{3.11}$$

For such t, $\operatorname{Re} L_t|_{\Psi_n(S)}$ is nonconstant. This completes the proof. □

Remark 3.1. If $\Psi_n(S)$ has the property that no bounded mapping is a support point of $\Psi_n(S)$, then $\mathrm{e}^{-t} F(\cdot, t) \in \operatorname{supp} \Psi_n(S)$ for all $t \geq 0$. For, $\mathrm{e}^t V_t = \Psi_n(\mathrm{e}^t v_t)$ is a bounded mapping in $\Psi_n(S)$, and we would have

$$\operatorname{Re} L_t(\mathrm{id}) = \operatorname{Re} L(\mathrm{e}^t V_t) < \operatorname{Re} L(F) = \operatorname{Re} L_t(\mathrm{e}^{-t} F(\cdot, t)),$$

and thus $\operatorname{Re} L_t|_{\Psi_n(S)}$ would be nonconstant.

Conjecture 3.1. No bounded mapping in $\Psi_n(S)$ is a support point of $\Psi_n(S)$.

Taking into account the above results, we can also state.

Conjecture 3.2. Let $f \in S^0(B^n)$, $n \geq 2$, and let $f(z, t)$ be a Loewner chain such that $f(z) = f(z, 0)$, $z \in B^n$. If $f \in \operatorname{ex} S^0(B^n)$ then $\mathrm{e}^{-t} f(\cdot, t) \in \operatorname{ex} S^0(B^n)$ for $t \geq 0$.

Conjecture 3.3. Let $f \in S^0(B^n)$, $n \geq 2$, and let $f(z, t)$ be a Loewner chain such that $f(z) = f(z, 0)$, $z \in B^n$. If $f \in \operatorname{supp} S^0(B^n)$ then $\mathrm{e}^{-t} f(\cdot, t) \in \operatorname{supp} S^0(B^n)$ for $t \geq 0$.

Acknowledgements. The first author was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant A9221.
Some of the research for this paper was carried out in May 2004 while the second and third authors visited the Department of Mathematics of the University of Toronto. G. Kohr and J. Pfaltzgraff express their gratitude to the members of this department for their hospitality during this visit.

REFERENCES

[1] L. Brickman, *Extreme points of the set of univalent functions*. Bull. Amer. Math. Soc. **76** (1970), 372–374.

[2] L. Brickman and D.R. Wilken, *Support points of the set of univalent functions*. Proc. Amer. Math. Soc. **42** (1974), 523–528.

[3] P.L. Duren, *Univalent Functions*. Springer-Verlag, New York, 1983.

[4] S. Gong and T.S. Liu, *On the Roper-Suffridge extension operator*. J. Anal. Math. **88** (2002), 397–404.

[5] S. Gong and T.S. Liu, *The generalized Roper-Suffridge extension operator*. J. Math. Anal. Appl. **284** (2003), 425–434.

[6] I. Graham, H. Hamada and G. Kohr, *Parametric representation of univalent mappings in several complex variables*. Canadian J. Math. **54** (2002), 324–351.

[7] I. Graham, H. Hamada, G. Kohr and T.J. Suffridge, *Extension operators for locally univalent mappings*. Michigan Math. J. **50** (2002), 37–55.

[8] I. Graham and G. Kohr, *Univalent mappings associated with the Roper-Suffridge extension operator*. J. Analyse Math. **81** (2000), 331–342.

[9] I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions*. Marcel Dekker Inc., New York, 2003.

[10] I. Graham, G. Kohr and M. Kohr, *Loewner chains and the Roper-Suffridge extension operator*. J. Math. Anal. Appl. **247** (2000), 448–465.

[11] I. Graham, G. Kohr and M. Kohr, *Loewner chains and parametric representation in several complex variables*. J. Math. Anal. Appl. **281** (2003), 425–438.

[12] I. Graham, G. Kohr and M. Kohr, *Basic properties of Loewner chains in several complex variables*. In: *Geometric Function Theory in Several Complex Variables*, pp. 165–181. World Sci. Publishing, River Edge, NJ, 2004.

[13] K. Hahn, *Subordination principle and distortion theorems on holomorphic mappings in the space* $\mathbb{C}^n$. Trans. Amer. Math. Soc. **162** (1971), 327–336.

[14] D.J. Hallenbeck and T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*. Pitman, Boston, 1984.

[15] W.E. Kirwan, *Extremal properties of slit conformal mappings*. In: D. Brannan and J. Clunie (Eds.), *Aspects of Contemporary Complex Analysis*, pp. 439–449. Academic Press, London–New York, 1980.

[16] S.G. Krantz, *Function Theory of Several Complex Variables*. Reprint of the 1992 Edition, AMS Chelsea Publishing, Providence, R.I., 2001.

[17] T.H. MacGregor and D.R. Wilken, *Extreme points and support points*. In: R. Kühnau (Ed.), *Handbook of Complex Analysis: Geometric Function Theory*, Vol. I, pp. 371–392. Elsevier Science, 2002.

[18] J.R. Muir and T.J. Suffridge, *Extreme points for convex mappings of* B_n. To appear in J. Anal. Math.

[19] R. Pell, *Support point functions and the Loewner variation*. Pacific J. Math. **86** (1980), 561–564.

[20] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$. Math. Ann. **210** (1974), 55–68.

[21] J.A. Pfaltzgraff and T.J. Suffridge, *Close-to-starlike holomorphic functions of several variables*. Pacific J. Math. **57** (1975), 271–279.

[22] J.A. Pfaltzgraff and T.J. Suffridge, *An extension theorem and linear invariant families generated by starlike maps*. Ann. Univ. Mariae Curie-Sklodowska Sect. A. **53** (1999), 193–207.

[23] A. Pfluger, *Linear extremal probleme bei schlichten funktionen.* Ann. Acad. Sci. Fenn. Ser. A. I. **489** (1971), 32 pp.
[24] C. Pommerenke, *Univalent Functions.* Vandenhoeck & Ruprecht, Göttingen, 1975.
[25] T. Poreda, *On the univalent holomorphic maps of the unit polydisc of* $\mathbb{C}^n$ *which have the parametric representation, I-the geometrical properties.* Ann. Univ. Mariae Curie-Sklodowska Sect. A **41** (1987), 105–113.
[26] T. Poreda, *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation, II – necessary and sufficient conditions.* Ann. Univ. Mariae Curie-Sklodowska, Sect. A. **41** (1987), 114–121.
[27] K. Roper and T.J. Suffridge, *Convex mappings on the unit ball of* $\mathbb{C}^n$. J. Anal. Math. **65** (1995), 333–347.
[28] K. Roper and T.J. Suffridge, *Convexity properties of holomorphic mappings in* $\mathbb{C}^n$. Trans. Amer. Math. Soc. **351** (1999), 1803–1833.
[29] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$. Springer-Verlag, New York, 1980.
[30] G. Schober, *Univalent Functions–Selected Topics.* Lecture Notes in Math. **478**. Springer-Verlag, New York, 1975.
[31] T.J. Suffridge, *The principle of subordination applied to functions of several variables.* Pacific J. Math. **33** (1970), 241–248.
[32] T.J. Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions.* In: Lecture Notes in Math., **599**. pp. 146–159. Springer-Verlag, New York, 1976.
[33] O. Toeplitz, *Die linearen volkommenen Räume der Funktionentheorie.* Comment. Math. Helv. **23** (1949), 222–242.

Received 15 September 2005

University of Toronto
Department of Mathematics
Toronto, Ontario M5S 2E4, Canada
graham@math.toronto.edu

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
1 M. Kogălniceanu Str.
400084 Cluj-Napoca, Romania
gkohr@math.ubbcluj.ro

and

University of North Carolina
Department of Mathematics CB 3250,
Chapel Hill, NC 27599-3250, USA
jap@math.unc.edu