MARKOV MOMENT PROBLEM AND RELATED APPROXIMATION

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We apply approximation results of certain functions of several variables by means of sums of tensor products of positive polynomials on the real line, in each separate variable, which are sums of squares. This leads to characterizations of the existence of the solutions of the multidimensional Markov moment problem in terms of quadratic forms or mappings, similarly to the one-dimensional case. From this point of view, one solves the difficulty created by the existence of positive polynomials that are not sums of squares in several dimensions. On the other hand, applications of general results to concrete spaces are considered. The continuity of the dominating operator seems to be essential in proving the properties of the solution.

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1. INTRODUCTION

Using extension of linear operators in existence questions on the solution of the moment problem is a well known technique [1, 2, 4, 11–14, 16, 20, 21, 23, 26–30. The present work uses such constrained extension results. The upper constraint measures the norm of the solution. The lower constraint may be the positivity of the solution on the positive cone X_+ of the domain-space X. In construction and uniqueness of the solution, L^2-approximation is usually applied. In solving existence problems, L^1-approximation by sums of tensor products of positive polynomials in each separate variable is sufficient. Sometimes uniform approximation on compact subsets is used. All these approximations are convenient when the analytic form of the positive polynomials on the respective unbounded intervals in terms of sums of squares is well known. The conclusion is that a combination of extension results with approximation theorems, leads to characterization of the existence of the solutions in terms of quadratic forms or mappings. Similar approximation results and moment problems in the complex case are solved by some other methods in [16]. On the other hand, Hahn-Banach principle is useful in proving approximation results. For the uniqueness of the solution, see [1, 3, 9, 10, 33]. For the background, see [6, 5, 22, 24–26, 32]. Other aspects of the moment problem are studied in [2, 3, 11, 15–19, 31]. Constructions of the solutions appear in [20, 29]. In [6–8], several related applications of special linear operators are given.

The aim of the present work is to continue, complete and improve previous results on this subject. The paper is organized as follows. In Section 2, we recall two main approximation results on unbounded closed subsets of R^n. Section 3 contains applications of these approximation results to the moment problem. Section 4 is devoted to an application of one of our earlier results concerning distanced convex sets and extension of linear operators. Section 5 contains a result not involving polynomials.

2. POLYNOMIAL APPROXIMATION ON UNBOUNDED SUBSETS

We start by recalling the following results concerning polynomial approximation on the positive semiaxis [21, 23]. We have applied these type of results to the moment problem [23, 27] and [30].

THEOREM 2.1. *Let $\psi : [0,\infty) \to R_+$ be a continuous function, such that $\lim_{t\to\infty} \psi(t) \in R_+$ exists. Then there is a decreasing sequence $(h_l)_l$ in the linear hull of the functions $\varphi_k(t) = \exp(-kt)$, $k \in \mathbf{N}$, $t \geq 0$, such that $h_l(t) > \psi(t)$, $t \geq 0$, $l \in \mathbf{N}$, $\lim_l h_l = \psi$ uniformly on $[0,\infty)$. There exists a sequence of polynomial functions $(\tilde{p}_l)_l$, $\tilde{p}_l \geq h_l > \psi$, $\forall l \in \mathbf{N}$, $\lim \tilde{p}_l = \psi$ uniformly on compact subsets of $[0,\infty)$.*

We recall that a determinate (M-determinate) measure is, by definition, uniquely determinate by its moments [9, 10, 33].

THEOREM 2.2. *Let ν be a determinate positive regular measure on $[0,\infty)$ with finite moments of all natural orders. If ψ, $(\tilde{p}_l)_l$ are as in Theorem 2.1, then there exists a subsequence $(\tilde{p}_{l_m})_m$, such that $\tilde{p}_{l_m} \to \psi$ in $L^1_\nu([0,\infty))$ and uniformly on compact subsets. In particular, it follows that positive polynomials are dense in the positive cone $(L^1_\nu([0,\infty)))_+$ of $L^1_\nu([0,\infty))$.*

The next result is interesting in itself and will be applied in the next theorem and further along this work (see also [30], Lemma 7).

THEOREM 2.3. *Let $A \subset R^n$ be a closed subset and ν a positive regular M - determinate Borel measure on A, with finite moments of all orders. Then for any continuous, nonnegative, vanishing at infinity function $\psi \in (C_0(A))_+$,* *there is a sequence $(p_m)_m$ of polynomials on A, $p_m \geq \psi$, $p_m \to \psi$ in $L^1_\nu(A)$. We have*

$$\lim \int_A p_m \mathrm{d}\nu = \int_A \psi \mathrm{d}\nu,$$

the cone P_+ of positive polynomials is dense in $(L^1_\nu(A))_+$ and P is dense in $L^1_\nu(A)$.

THEOREM 2.4. *Let $\nu = \nu_1 \times \nu_2 \times \ldots \times \nu_n$ be a product of n determinate positive regular Borel measures on R, with finite moments of all natural orders. Then we can approximate any nonnegative continuous compactly supported function in $L^1_\nu(R^n)$ by means of sums of tensor products $p_1 \otimes p_2 \otimes \ldots \otimes p_n$, p_j positive polynomial on the real line, in variable t_j, $j = 1, \ldots, n$.*

Proof. If K is the support of a continuous compactly supported nonnegative function $f \in C_c(R^n)$ then

$$K \subset K_1 \times K_2 \times \ldots \times K_n,\ K_j = pr_j(K),\ j = 1, \ldots, n.$$

Consider a hyper parallelipiped

$$P_n = [a_1, b_1] \times \ldots \times [a_n, b_n],\ a_j = \inf K_j,\ b_j = \sup K_j,\ j = 1, \ldots, n$$

containing the above Cartesian product of compacts and apply approximation of f on P_n by the corresponding Bernstein polynomials in n variables [7, p. 138]. Namely, the explicit form of the Bernstein polynomials is

$$B_m(f)(t_1, \ldots, t_n) = \sum_{k_1 \in \{0,\ldots,m\}} \cdots \sum_{k_n \in \{0,\ldots,m\}} p_{mk_1}(t_1) \ldots p_{mk_n}(t_n) \cdot$$

$$\cdot f\left(a_1 + (b_1 - a_1)\frac{k_1}{m}, \ldots, a_n + (b_n - a_n)\frac{k_n}{m}\right),$$

$$p_{mk_j}(t_j) = \begin{pmatrix} m \\ k_j \end{pmatrix} \left(\frac{t_j - a_j}{b_j - a_j}\right)^{k_j} \left(\frac{b_j - t_j}{b_j - a_j}\right)^{m-k_j},$$

$$t_j \in [a_j, b_j],\ j = 1, \ldots, n,\ B_m(f) \to f,\ m \to \infty.$$

Each term of such a polynomial is a tensor product $p_1 \otimes p_2 \otimes \ldots \otimes p_n$, of positive polynomials in each variable, on the projection $pr_j(P_n)$, $j = 1, \ldots, n$. Extend each p_j such that it vanishes outside $pr_j(P_n)$ applying then Luzin's theorem, $j = 1, \ldots, n$. This procedure does not change the values of p_j on K_j. One obtains approximation by sums of tensor products of positive continuous functions with compact support, in each variable t_j, $j = 1, \ldots, n$. The approximating process holds in L^1 norm, and uniformly on K.

Now application of Theorem 2.3 to $n = 1$, $A = R$, leads to approximation of each such function in each separate variable by a dominating (positive) polynomial, in the space $L^1_{\nu_j}(R)$, $j = 1, \ldots, n$. The conclusion follows. □

3. MULTIDIMENSIONAL MARKOV MOMENT PROBLEM AND POLYNOMIAL APPROXIMATION

Let $\nu = \nu_1 \times \nu_2 \times \ldots \times \nu_n$, where ν_j, $j = 1, \ldots, n$ are positive Borel regular M-determinate measures on R, with finite moments of all natural orders. Let

$$\varphi_j(t_1, \ldots, t_n) = t_1^{j_1} \ldots t_n^{j_n},\ j = (j_1, \ldots, j_n) \in \mathbf{N}^n,\ (t_1, \ldots, t_n) \in R^n.$$

Let $X = L_\nu^1(R^n)$ and Y be an order complete Banach lattice with solid norm [6], $(y_j)_{j\in\mathbf{N}^n}$ a sequence in Y.

THEOREM 3.1. *Let $F_2 : X \to Y$ be a positive linear bounded operator. The following statements are equivalent:*

(a) *there exists a unique linear operator $F : X \to Y$, such that*

$$F(\varphi_j) = y_j,\ \forall j \in N^n,$$

F is between zero and F_2 on the positive cone of X, $||F|| \leq ||F_2||$;

(b) *for any finite subsets $J_k \subset N$, $k = 1, \ldots, n$ and any*

$$\{\lambda_{j_k}\}_{j_k\in J_k} \subset R, \quad k = 1, \ldots, n,$$

we have:

$$0 \leq \sum_{i_1,j_1\in J_1} \left(\cdots \left(\sum_{i_n,j_n\in J_n} \lambda_{i_1}\lambda_{j_1} \ldots \lambda_{i_n}\lambda_{j_n} y_{i_1+j_1,\ldots,i_n+j_n} \right) \cdots \right) \leq$$

$$\sum_{i_1,j_1\in J_1} \left(\cdots \left(\sum_{i_n,j_n\in J_n} \lambda_{i_1}\lambda_{j_1} \ldots \lambda_{i_n}\lambda_{j_n} F_2(\varphi_{i_1+j_1,\ldots,i_n+j_n}) \right) \cdots \right).$$

Proof. We define F_0 on the space of polynomials, such that the moment conditions be accomplished. Condition (b) says that

$$0 \leq F_0(p_1 \otimes \ldots \otimes p_n) \leq F_2(p_1 \otimes \ldots \otimes p_n),\ \forall p_1, \ldots, p_n \in (R[X])_+\ ,$$

since p_j, $j = 1, \ldots, n$ are sums of squares of some other polynomials with real coefficients [1]. Hence, the implication (a) $\Rightarrow$ (b) is obvious. For the converse, let ψ be a nonnegative continuous compactly supported function defined on R^n. By the preceding Theorem 2.4, one approximates ψ on a hyperparallelepiped containing $pr_1(\text{support}\,\psi) \times \ldots \times pr_n(\text{support}\,\psi)$ by means of Luzin's theorem and the corresponding Bernstein polynomials in n variables. Then one approximates ψ by sums of tensor products of positive polynomials on R:

$$\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \ldots \otimes p_{m,n,j} \to \psi,\ m \to \infty$$

in the space $L^1_\nu(R^n)$ (Theorem 2.4). On the other hand, the linear positive operator F_0 has a linear positive extension F defined on the space of all integrable functions with their absolute value dominated on R^n by a polynomial (following [6], p. 160). This space contains the space of continuous compactly supported functions. Hence, $h \circ F$ can be represented by a regular positive Radon measure, for any positive linear functional h on Y. Moreover, using (b) and applying Fatou's lemma, one obtains:

$$0 \leq h(F(\psi)) \leq \liminf_m (h \circ F)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \ldots \otimes p_{m,n,j}\right) \leq$$

(3.1)

$$\lim_m (h \circ F_2)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \ldots \otimes p_{m,n,j}\right) = h(F_2(\psi)), \psi \in (C_c(R^n))_+, h \in Y_+^* .$$

Assume that

$$F_2(\psi) - F(\psi) \notin Y_+ .$$

Using a separation theorem, it should exist a positive linear continuous functional $h \in Y_+^*$ such that

$$h(F_2(\psi)) - F(\psi)) < 0,$$

that is $h(F_2(\psi)) < h(F(\psi))$. This relation contradicts (3.1). The conclusion it that we must have

$$F(\psi) \leq F_2(\psi),\ \psi \in (C_c(R^n))_+ .$$

Then for an arbitrary compactly supported continuous function $g \in C_c(R^n)$ one writes

$$|F(g)| \leq F_2(g^+) + F_2(g^-) = F_2(|g|) \Rightarrow ||F(g)|| \leq ||F_2|| \cdot ||g||_1.$$

Consequently, the operator F is positive and continuous, of norm dominated by $||F_2||$, on a dense subspace of $L^1_\nu(R^n)$. It has a unique linear extension preserving these two properties. This concludes the proof. □

Let $X = L^1_\nu([0,\infty) \times \ldots \times [0,\infty))$, $\nu = \nu_1 \times \ldots \times \nu_n$, ν_l, $l = 1, \ldots, n$ being positive regular M - determinate Borel measures on $[0,\infty)$, with finite moments of all orders. Let

$$\varphi_j(t_1, \ldots, t_n) = t_1^{j_1} \ldots t_n^{j_n},\ j = (j_1, \ldots, j_n) \in \mathbf{N}^n,\ t_l \geq 0,\ l = 1, \ldots, n.$$

Repeating the proof of Theorem 3.1, and using the form of positive polynomials on R_+ [1] in terms of squares: $(p_l(t_l) = p_{l,1}^2(t_l) + t_l p_{l,2}^2(t_l),\ t_l \geq 0,$ $l = 1, \ldots, n)$, one obtains a similar statement for this case. Under the same assumptions on Y, and using the same hypothesis and notations, one obtains the following result.

THEOREM 3.2. *Let $(y_j)_{j\in\mathbf{N}^n}$ be a sequence in Y. The following statements are equivalent:*

(a) *there exists a unique linear operator $F \in L(X,Y)$ such that $F(\varphi_j) = y_j$, $j \in \mathbf{N}^n$, F is between zero and F_2 on the positive cone of X, $||F|| \le ||F_2||$;*

(b) *for any finite subsets $J_k \subset \mathbf{N}$, $k = 1,\dots,n$ and any $\{\lambda_{j_k}\}_{j_k\in J_k} \subset \mathbf{R}$, $k = 1,\dots,n$, we have:*

$$0 \le \sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\dots\lambda_{i_n}\lambda_{j_n}y_{i_1+j_1+l_1,\dots,i_n+j_n+l_n}\right)\cdots\right)\le$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\dots\lambda_{i_n}\lambda_{j_n}F_2(\varphi_{i_1+j_1+l_1,\dots,i_n+j_n+l_n})\right)\cdots\right),$$
$$(l_1,\dots,l_n) \in \{0,1\}^n.$$

4. DISTANCED CONVEX SETS AND THE MOMENT PROBLEM

The next result is an operator valued multidimensional Markov moment problem. Let X be the space of functions which are sums of absolutely convergent power series around zero, in the open polydisc $\prod_{k=1}^{n}\{|z_k| < r_k\}$, with real coefficients, continuous up to the boundary. The order relation is given by the convex cone

$$X_+ = \left\{\varphi(z) = \sum_{j\in\mathbf{N}^n}\alpha_j z_1^{j_1}\dots z_n^{j_n};\ \alpha_j \ge 0,\ j \in \mathbf{N}^n\right\}.$$

The space X becomes a real ordered vector space, that is endowed with the "sup" norm on the closed polydisc. We denote

$$\varphi_j(z_1,\dots,z_n) = z_1^{j_1}\dots z_n^{j_n},\ j = (j_1,\dots,j_n) \in \mathbf{N}^n,\ |j| = \sum_{k=1}^{n} j_k.$$

On the other hand, consider a complex Hilbert space H and A_k, $k = 1,\dots,n$ linear positive commuting selfadjoint operators on H, such that $||A_k|| < r_k$, $k = 1,\dots,n$. We denote:

$$Y_1 = \{U \in A(H); A_kU = UA_k,\ k = 1,\dots,n\},$$

$$Y = \{V \in Y_1; VU = UV,\ \forall U \in Y_1\},$$

where $A(H)$ is the real vector space of all selfadjoint operators acting on H. Obviously, Y is a commutative algebra of selfadjoint operators. Moreover, Y is an order complete vector lattice with respect to the usual order relation (see [6, 13]), and the operatorial norm is solid on Y:

$$|U| \leq |V| \Rightarrow ||U|| \leq ||V||, \quad U, V \in Y.$$

The following general statement has a nice geometric meaning and leads to interesting results concerning the moment problem [12]. If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V.

THEOREM 4.1 (see also [12, 25, 30]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following properties:*

(a) *there exists a neighborhood V of the origin such that $(S+V) \cap A = \Phi$ (A and S are distanced);*

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j(s) = f_j(s),\ s \in S \quad \text{and} \quad F_j(x) \geq \tilde{y},\ x \in A,\ j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V \cap S) \subset [-u_0, u_0], \quad (S+V) \cap A = \Phi,$$

α is an upper bound for $p_V(A)$ and $\alpha_1 > 0$ is large enough such that $\tilde{y} \leq \alpha_1 u_0$, then the following relations hold

$$F_j(x) \leq (1+\alpha+\alpha_1)p_V(x) \cdot u_0,\ x \in X,\ j \in J.$$

As an application of the preceding theorem, we prove the following result.

THEOREM 4.2. *Let $(B_j)_{j\in \mathbf{N}^n}$, $|j| > 1$ be a sequence in Y, such that*

$$|B_j| \leq A_1^{j_1} \ldots A_n^{j_n},\ \forall j = (j_1, \ldots, j_n) \in \mathbf{N}^n,\ |j| \geq 1.$$

Let $\tilde{B} \in Y_+$, $\{\psi_j\}_{j\in\mathbf{N}^n} \subset X$, $\psi_j(0,\ldots,0) = 1$, $||\psi_j||_\infty \leq 1$, $\forall j \in \mathbf{N}^n$. Then there exists a linear bounded operator $F \in B(X,Y)$ such that

$$F(\varphi_j) = B_j,\ |j| \geq 1,\ F(\psi_j) \geq \tilde{B},\ j \in \mathbf{N}^n,$$

$$|F(\varphi)| \leq \left(2 + ||\tilde{B}|| \cdot \prod_{k=1}^{n}((r_k - ||A_k||)/r_k)\right) \cdot ||\varphi||_\infty \cdot u_0,$$

$$u_0 := \frac{r_k}{\prod_{k=1}^{n}(r_k - ||A_k||)} \cdot I.$$

Proof. We apply Theorem 4.1. to

$$S = Span\{\varphi_j; j \in \mathbf{N}^n,\ |j| \geq 1\},\ A = conv(\{\psi_j; j \in \mathbf{N}^n\}),$$

the convex hull of the collection of functions ψ_j, $j \in \mathbf{N}^n$. Conditions imposed on the values at $(0,\dots,0)$ and on the norms of the functions ψ_j lead to

$$||\varphi_j - \psi_m||_\infty \geq |\varphi_j(0,\dots,0) - \psi_m(0,\dots,0) = 1 \Rightarrow$$

$$(S + B(0,1)) \cap A = \Phi \Rightarrow p_V(\cdot) = ||\cdot|| \Rightarrow p_{V|A} \leq 1 = \alpha,\ x \in A.$$

Hence, the unit ball $B(0,1)$ in X stands for V, $||\cdot||$ stands for p_V and $\alpha := 1$ stands for an upper bound of $p_V(A)$. On the other hand, for

$$s \in S,\ f(s) = f\left(\sum_{j\in J_0} \lambda_j \varphi_j\right) = \sum_{j \in J_0} \lambda_j B_j.$$

Cauchy inequalities for s and the hypothesis on the operators $|B_j|$ yield $s \in S \cap B(0,1) \Rightarrow$

$$\Rightarrow |f(s)| = \left|\sum_{j\in J_0} \lambda_j B_j\right| \leq \sum_{j\in J_0} |\lambda_j| \cdot |B_j| \leq ||s|| \sum_{j\in\mathbf{N}^n} \frac{A_1^{j_1}}{r_1^{j_1}} \cdots \frac{A_n^{j_n}}{r_n^{j_n}} \leq$$

$$\sum_{j\in\mathbf{N}^n} \frac{||A_1||^{j_1}}{r_1^{j_1}} \cdots \frac{||A_n||^{j_n}}{r_n^{j_n}} \cdot I = \prod_{k=1}^{n} \frac{r_k}{r_k - ||A_k||} \cdot I := u_0.$$

On the other hand, we have

$$\tilde{B} \leq ||\tilde{B}|| \cdot I = \prod_{k=1}^{n} ((r_k - ||A_k||)/r_k) \cdot ||\tilde{B}|| \cdot u_0 =$$

$$= \alpha_1 \cdot u_0,\ \alpha_1 := \prod_{k=1}^{n} ((r_k - ||A_k||)/r_k) \cdot ||\tilde{B}||.$$

Now all conditions from the hypothesis of Theorem 4.1 are accomplished, so that application of the latter theorem leads to the conclusion of the present theorem. □

5. PROBLEMS NOT INVOLVING POLYNOMIALS

In this section, we solve a Markov moment problem involving $L^1(M)$ spaces. Here M is an arbitrary measurable space, not necessarily related to R^n, so that we solve non-classical problems. We start by recalling the following abstract version of a Markov moment problem [26].

THEOREM 5.1. *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ given families, and*

$$F_1, F_2 \in L(X,Y)$$

two linear operators. The following statements are equivalent:

(a) *there is a linear operator $F \in L(X, Y)$ such that*

$$F_1(x) \leq F(x) \leq F_2(x),\ \forall x \in X_+,\ F(x_j) = y_j,\ \forall j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1,\ \varphi_1, \varphi_2 \in X_+\right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq F_2(\varphi_2) - F_1(\varphi_1).$$

THEOREM 5.2. *Let $X = L^1_\nu(M)$, ν being an arbitrary positive measure, Y an order complete Banach lattice with solid norm, $F_2 \in L_+(X, Y)$ a linear positive bounded operator. Let $\{\varphi_j\}_{j \in J} \subset X$, and $\{y_j\}_{j \in J} \subset Y$. The following statements are equivalent:*

(a) *there exists a linear operator F applying X into Y such that*

$$F(\varphi_j) = y_j,\ j \in J,\ |F(\varphi)| \leq F_2(|\varphi|),\ \varphi \in X,\ ||F|| \leq ||F_2||;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset R$, we have*

$$\sum_{j \in J_0} \lambda_j y_j \leq F_2\left(\left|\sum_{j \in J_0} \lambda_j \varphi_j\right|\right).$$

Proof. The implication (a) $\Rightarrow$ (b) is almost obvious, since from (a) we infer that

$$\sum_{j \in J_0} \lambda_j y_j = F\left(\sum_{j \in J_0} \lambda_j \varphi_j\right) \leq F_2\left(\left|\sum_{j \in J_0} \lambda_j \varphi_j\right|\right).$$

To prove the converse implication, we verify condition (b) of Theorem 5.1.

$$\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1,\ \psi_k \in X_+,\ k = 1, 2 \Rightarrow$$

$$\left|\sum_{j \in J_0} \lambda_j \varphi_j\right| \leq \psi_2 + \psi_1 \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq F_2\left(\left|\sum_{j \in J_0} \lambda_j \varphi_j\right|\right) \leq$$
$$F_2(\psi_2) + F_2(\psi_1) = F_2(\psi_2) - F_1(\psi_1),\ F_1 := -F_2.$$

Application of Theorem 5.1, (b) $\Rightarrow$ (a) leads to the existence of a linear operator F such that

$$F(\varphi_j) = y_j,\ j \in J,\ -F_2(\psi) \leq F(\psi) \leq F_2(\psi),\ \psi \in X_+ \Rightarrow$$
$$|F(\psi)| \leq F_2(\psi),\ \psi \in X_+ \Rightarrow |F(\varphi)| \leq |F(\varphi^+)| + |F(\varphi^-)| \leq F_2(|\varphi|),\ \varphi \in X.$$

Since the norm is solid on Y, we infer that

$$||F(\varphi)|| \leq ||F_2|| \cdot ||\varphi||, \quad \varphi \in X.$$

This concludes the proof. $\square$

Next, we state the "scalar version" of the above theorem, when the moments $y_j \in Y = R$, $j \in J$ are real numbers. Let ν be a σ - finite measure on M, and X, $(\varphi_j)_{j\in J}$ as above. Let $(y_j)_{j\in J}$ be a family of real numbers.

THEOREM 5.3. *The following statements are equivalent:*
(a) *there exists a function $h \in L^\infty_\nu(M)$ such that*

$$-1 \leq h(t) \leq 1 \text{ a.e. } \quad \int_M h\varphi_j \mathrm{d}\nu = y_j,\ j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset R$, we have:*

$$\sum_{j\in J_0} \lambda_j y_j \leq \int_M \left| \sum_{j\in J_0} \lambda_j \varphi_j(t) \right| \mathrm{d}\nu.$$

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