

SPECTRAL PROPERTIES OF k -QUASI-PARANORMAL OPERATORS

SALAH MECHERI and S.M. PATEL

Communicated by Dorin Ieşan

The study of operators satisfying Weyl's theorem, Browder's theorem, the SVEP, and Bishop's property (β) is of significant interest and is currently being done by a number of researchers around the world. In the present article, we wish to show that Weyl's theorem and Browder's theorem holds for $p(k)$ operators. Moreover, we show that a k -quasi-paranormal operator is polaroid and generalized a-Weyl's theorem holds for T^* . It will be proved that the spectral mapping theorem holds for $p(k)$ operators. We also establish some results involving powers of k -quasi-paranormal operators.

AMS 2010 Subject Classification: Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words: k -quasi-paranormal, paranormal operator, Browder's theorem, Weyl's spectrum.

1. INTRODUCTION

Let $B(H)$ be an algebra of all bounded operators acting on an infinite dimensional separable Hilbert space H . An operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in H$. It is well known [9] that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0, \text{ for all } \lambda > 0.$$

In order to extend the class of paranormal operators, we introduce a new class of operators defined as follows:

Definition 1.1. Let k be a positive integer. An operator T is called k -quasi-paranormal or $p(k)$ operator if

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|$$

for all $x \in H$.

It is not difficult to verify that T is k -quasi-paranormal if and only if

$$(1) \quad T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0, \text{ for all } \lambda > 0.$$

It is obvious that every paranormal operator is a $p(k)$ operator. However since the class of $p(k)$ operators includes idempotent operators, the converse is not true. Also the class of k -quasi-paranormal operators is included in the class of $k + 1$ -quasi-paranormal operators. Using (1), it is not difficult to show that a nilpotent operator of index $k + 2$ is a $p(k + 1)$ operator but not a $p(k)$ operator; thus, the sequence of classes of $p(k)$ operators is strictly increasing.

2. NOTATIONS AND TERMINOLOGY

Let $T \in B(H)$. $N(T)$ denotes the null space of T and let $\alpha(T) = \dim N(T)$. $\text{ran } T$ denotes the range of T and $[\text{ran } T]$ denotes the closure of $\text{ran } T$. Let $\beta(T) = \dim H/\text{ran } T$. T is called semi-Fredholm if it has closed range and either $\alpha(T) < \infty$ or $\beta(T) < \infty$. T is called Fredholm if it is semi-Fredholm and both $\alpha(T) < \infty, \beta(T) < \infty$. T is called Weyl if it is Fredholm of index zero, *i.e.*, $i(T) = \alpha(T) - \beta(T) = 0$. The Weyl spectrum of T is defined by

$$\omega(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda \text{ is not Weyl} \}.$$

$\pi_{00}(T)$ denotes the set of all eigenvalues of T such that λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$. We write $\sigma_e(T)$ for the essential spectrum of T . The spectral picture $SP(T)$ of T consists of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$ and indices associated with these holes and pseudoholes. We say T to be isoloid if every isolated point in $\sigma(T)$ is an eigenvalue of T . The essential approximate point spectrum $\sigma_{ea}(T)$ of T is defined by $\sigma_{ea}(T) = \{ \sigma_a(T + K) : K \text{ is a compact operator} \}$, where $\sigma_a(T)$ denotes the approximate point spectrum of T .

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

An operator $T \in B(H)$ is said to have Bishop's property (β) if $(T - z)f_n(z) \rightarrow 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D , then $f_n(z) \rightarrow 0$ uniformly on every compact subset of D . T is said to have the single valued extension property, abbreviated, T has SVEP if $f(z)$ is an analytic vector valued function on some open set $D \subset \mathbb{C}$ such that $(T - z)f(z) = 0$ for all $z \in D$, then $f(z) = 0$ for all $z \in D$. The interested reader is referred to [15, 18–21]. More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [5–7].

In a recent paper [16] the author showed that generalized Weyl's theorem holds for (p, k) -quasi-hyponormal operators. Recently, X. Cao, M. Guo and B. Meng [11] proved Weyl type theorems for p -hyponormal operators.

More generally, M. Berkani investigated B-Fredholm theory as follows (see [2, 5–7]). An operator T is called B -Fredholm if there exists $n \in \mathcal{N}$ such that $\text{ran}(T^n)$ is closed and the induced operator

$$T_{[n]} : \text{ran}(T^n) \ni x \rightarrow Tx \in \text{ran}(T^n)$$

is Fredholm, *i.e.*, $\text{ran}(T_{[n]}) = \text{ran}(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim \text{ran}(T^n) / \text{ran}(T_{[n]}) < \infty$. Similarly, a B -Fredholm operator T is called B -Weyl if $i(T_{[n]}) = 0$. The following results is due to M. Berkani and M. Sarih [7].

PROPOSITION 2.1. *Let $T \in B(\mathcal{H})$.*

(1) *If $\text{ran}(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geq n$. Moreover, $\text{ind } T_{[m]} = \text{ind } T_{[n]} (= \text{ind } T)$.*

(2) *An operator T is B -Fredholm (B -Weyl) if and only if there exist T -invariant subspaces \mathcal{M} and \mathcal{N} such that $T = T|_{\mathcal{M}} \oplus T|_{\mathcal{N}}$ where $T|_{\mathcal{M}}$ is Fredholm (Weyl) and $T|_{\mathcal{N}}$ is nilpotent.*

The B -Weyl spectrum $\sigma_{BW}(T)$ are defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathcal{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T , then so does Weyl's theorem [6]. Recently in [5] M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T .

We define $T \in SF_+^-$ if $R(T)$ is closed, $\dim \ker(T) < \infty$ and $\text{ind } T \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim \ker(T - \lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda \mid T - \lambda \notin SF_+^-\} \subset \sigma_W(T)$. We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

V. Rakočević ([22], Corollary 2.5) proved that if a-Weyl's theorem holds for T , then Weyl's theorem holds for T .

We define $T \in SBF_+^-$ if there exists a positive integer n such that $\text{ran}(T^n)$ is closed, $T_{[n]} : \text{ran}(T^n) \ni x \rightarrow Tx \in \text{ran}(T^n)$ is upper semi-Fredholm (*i.e.*, $\text{ran}(T_{[n]}) = \text{ran}(T^{n+1})$ is closed, $\dim \ker(T_{[n]}) = \dim \ker(T) \cap \text{ran}(T^n) < \infty$) and $0 \geq \text{ind } T_{[n]} (= \text{ind } T)$ ([7]). We define $\sigma_{SBF_+^-}(T) = \{\lambda \mid T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim \ker(T - \lambda)$.

We say that generalized a-Weyl’s theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

M. Berkani and J.J. Koliha [6] proved that if generalized a-Weyl’s theorem holds for T , then a-Weyl’s theorem holds for T .

The study of operators satisfying Weyl’s theorem, Browder’s theorem, the SVEP, and Bishop’s property (β) is of significant interest and is currently being done by number of researchers around the world. In the present article, we wish to show that Weyl’s theorem and Browder’s theorem holds for $p(k)$ operators. Moreover, we show that a k -quasi-paranormal operator is polaroid and generalized a-Weyl’s theorem holds for T^* . It will be proved that the spectral mapping theorem holds for $p(k)$ operators. We establish some results involving powers of k -quasi-paranormal operators.

3. MAIN RESULTS

We begin with the following result, which will be utilized to obtain several important properties of k -quasi-paranormal operators.

THEOREM 3.1 ([15]). *Let $T \in B(H)$ be k -quasi-paranormal operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = [\text{ran } T^k] \oplus \ker T^{*k}.$$

Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Let K be an infinite dimensional separable Hilbert space. Above decomposition of $p(k)$ operators motivates us to ask the following question: Is the operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

acting on $H \oplus K$, a $p(k)$ operator if A is paranormal and $C^k = 0$?. We do not know the answer. However, for $k = 1$ we have

THEOREM 3.2. *Let T be the operator on $H \oplus K$ defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is paranormal, then T is 1-quasi-paranormal.

Proof. A simple calculation shows that

$$\begin{aligned} & T^*(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T \\ &= \begin{pmatrix} A^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)A & A^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)B \\ B^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)A & B^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)B \end{pmatrix}. \end{aligned}$$

Let $\lambda > 0$ be arbitrary and let $u = x \oplus y \in H \oplus K$. Then

$$\begin{aligned} & \langle (T^*(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T)u, u \rangle \\ &= \langle A^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)Ax, x \rangle + \langle A^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)By, x \rangle \\ &+ \langle B^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)Ax, y \rangle + \langle B^*(A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)By, y \rangle \\ &= \langle (A^{*2}A^2 - 2\lambda A^*A + 2\lambda I)(Ax + By), (Ax + By) \rangle \geq 0 \end{aligned}$$

because A is paranormal. This proves the result. \square

COROLLARY 3.1. *Let T be a k -quasi-paranormal operator. If T is quasinilpotent, then it must be a nilpotent operator.*

Proof. Invoking Theorem 3.1, we find $\sigma(T_1) = 0$. Since T_1 is paranormal, we conclude that $T_1 = 0$. Since $T_3^k = 0$, a computation shows that

$$T^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0. \quad \square$$

COROLLARY 3.2. *If T is a $p(k)$ operator with $\sigma(T) \subseteq \{0, 1\}$, then $T^{k+1} = T^{k+2}$.*

Proof. The spectral condition on T gives $\sigma(T_1) \subseteq \{0, 1\}$. Since T_1 is paranormal, we conclude that T_1 is a projection. Now it is not difficult to derive the result. \square

LEMMA 3.1. *Let $T \in B(H)$ be an algebraically k -quasi-paranormal operator, and $\sigma(T) = \{\mu_0\}$, then $T - \mu_0$ is nilpotent.*

Proof. Assume $p(T)$ is k -quasi-paranormal for some nonconstant polynomial $p(z)$. Since $\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\}$, the operator $p(T) - p(\mu_0)$ is nilpotent by Corollary 3.1. Let

$$p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1} \cdots (z - \mu_t)^{k_t},$$

where $\mu_j \neq \mu_s$ for $j \neq s$. Then

$$0 = \{p(T) - p(\mu_0)\}^m = a^m(T - \mu_0)^{mk_0}(T - \mu_1)^{mk_1} \cdots (T - \mu_t)^{mk_t}$$

and hence, $(T - \mu_0)^{mk_0} = 0$. \square

In the following theorem we will prove that an algebraically k -quasi-paranormal operator is polaroid.

THEOREM 3.3. *Let T be an algebraically k -quasi-paranormal operator. Then T is polaroid.*

Proof. If T is an algebraically k -quasi-paranormal operator. Then $p(T)$ is a k -quasi-paranormal operator for some nonconstant polynomial p . Let $\mu \in \text{iso}(\sigma(T))$ and let E_μ be the Riesz idempotent associated to μ defined by

$$E_\mu := \frac{1}{2\pi i} \int_{\partial D} (\lambda I - T)^{-1} d\lambda,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T . Then T can be represented as follows

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\mu\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\mu\}$. Since T_1 is algebraically k -quasi-paranormal operator and $\sigma(T_1) = \{\mu\}$, it follows from Lemma 3.1 that $T_1 - \mu I$ is nilpotent. Therefore $T_1 - \mu I$ has finite ascent and descent. On the other hand, since $T_2 - \mu I$ is invertible, it has finite ascent and descent. Therefore $T - \mu I$ has finite ascent and descent. Therefore μ is a pole of the resolvent of T . Now if $\mu \in \text{iso}(\sigma(T))$, then $\mu \in \pi(T)$. Thus, $\text{iso}(\sigma(T)) \in \pi(T)$, where $\pi(T)$ denote the set of poles of the resolvent of T . Hence, T is polaroid. \square

COROLLARY 3.3. *A k -quasi-paranormal operator is isoloid.*

By applying Theorem 3.1 it is easy to prove the following theorem.

THEOREM 3.4. *Let $T \in B(H)$ be k -quasi-paranormal. Then T has SVEP.*

As a simple consequence of the preceding result, we obtain

COROLLARY 3.4. *Let T be a k -quasi-paranormal operator. Then the following assertions hold:*

(i) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, for every analytic function f on some open neighborhood of $\sigma(T)$.

(ii) T obeys a -Browder's theorem, that is $\sigma_{ea}(T) = \sigma_{ab}(T)$ (where $\sigma_{ab}(T) = \bigcap \{\sigma_a(T + K) : TK = KT \text{ and } K \text{ is a compact operator}\}$).

(iii) a -Browder's theorem holds for $f(T)$ for every analytic function f on some open neighborhood of $\sigma(T)$.

Proof. Note that above theorem implies that T has SVEP. By [1], (i) follows. Assertion (ii) is a consequence of ([7], Corollary 2.3). Since $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, the rest of the argument follows as in ([7], Corollary 2.3). \square

THEOREM 3.5. *An operator quasi-similar to a k -quasi-paranormal operator has SVEP.*

Proof. Let T be k -quasi-paranormal. Suppose S is an operator quasi-similar to T . Then there exist an injective operator A with dense range such that $AS = TA$. Let U be an open set and $f : U \mapsto H$ be an analytic function for which $(S - zI)f(z) = 0$ on U . Then $0 = A(S - zI)f(z) = (T - zI)Af(z)$ for all z in U . Since T has SVEP, we find $Af(z) = 0$. Since A is injective, it is immediate that $f(z) = 0$ for all z in U . This finishes the proof. \square

Several authors have established Weyl's theorem for different classes of non-normal operators. In [10], one can find an extension of Weyl's theorem to paranormal operators. Here we wish to show that this theorem also holds for $p(k)$ operators. We first quote following result from ([14], Theorem 2.4) as a lemma.

LEMMA 3.2 ([14], Theorem 2.4). *Suppose operators A and B are such that*

(i) *Either $SP(A)$ or $SP(B)$ has no pseudoholes .*

(ii) *A is an isoloid operator for which Weyl's theorem holds. If Weyl's theorem holds for $A \oplus B$, then Weyl's theorem holds for*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

for any operator $C \in B(H)$.

THEOREM 3.6. *Weyl's theorem holds for a k -quasi-paranormal operator T .*

Proof. Let T be k -quasi-paranormal with decomposition given in Theorem 3.1. Since T_1 and T_3 are isoloid, Weyl's theorem holds for $T_1 \oplus T_3$. Since $SP(T_3)$ has no pseudohole, our result follows from Lemma 3.1. \square

An operator $T \in B(H)$ satisfies generalized a-Browder's theorem if $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T)$. If a Banach space operator T has SVEP (everywhere), the single-valued extension property, then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a-Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \overline{\sigma_a(T^*)}$. Hence, if T has SVEP and is polaroid, then T^* satisfies generalized a-Weyl's (so also, a-Weyl's) theorem by ([3], Theorem 3.10].

THEOREM 3.7. *Let $T \in B(H)$.*

i) *If T^* is a k -quasi-paranormal operator, then generalized a -Weyl's theorem holds for T .*

ii) *If T is a k -quasi-paranormal operator, then generalized a -Weyl's theorem holds for T^* .*

Proof. (i) It is well known that T is polaroid if and only if T^* is polaroid ([3], Theorem 2.11). Now since a k -quasi-paranormal operator is polaroid by Theorem 3.3 and has SVEP by Theorem 3.4, ([3], Theorem 3.10) gives us the result of the theorem. For (ii) we can also apply ([3], Theorem 3.10). \square

Since the polaroid condition entails $E(T) = \pi(T)$ and the SVEP for T entails that generalized Browder's theorem holds for T ([1], Theorem 3.2), i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum. Therefore, $E(T) = \pi(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Thus, we have the following corollary.

COROLLARY 3.5. *If T is k -quasi-paranormal, then also T satisfies generalized Weyl's theorem.*

It is well known that a power of a paranormal operator is paranormal [4]. Next we show that the corresponding result is true for $p(k)$ operators.

THEOREM 3.8. *If T is k -quasi-paranormal, then T^n is k -quasi-paranormal for each positive integer n .*

Proof. We have T is k -quasi-paranormal if

$$\frac{\|T^{k+1}x\|}{\|T^kx\|} \leq \frac{\|T^{k+2}x\|}{\|T^{k+1}x\|} \quad (*)$$

for all $x \in H$. First we note that if $T^{k+1}x = 0$, then

$$\|T^{n(k+2)}x\| \|T^{nk}x\| \geq \|T^{n(k+1)}x\|^2. \quad (**)$$

Now assume that $T^{k+1}x \neq 0$. It results from (*) that

$$\frac{\|T^{k+i+1}x\|}{\|T^{k+i}x\|} \leq \frac{\|T^{k+i+2}x\|}{\|T^{k+i+1}x\|}$$

for all $x \in H$ and for all non-negative integers i . Therefore

$$\begin{aligned} \frac{\|(T^n)^{k+1}x\|}{\|(T^n)^kx\|} &= \frac{\|T^{nk+n}x\|}{\|T^{nk}x\|} = \frac{\|T^{nk+1}x\|}{\|T^{nk}x\|} \cdot \frac{\|T^{nk+2}x\|}{\|T^{nk+1}x\|} \cdots \frac{\|T^{nk+n}x\|}{\|T^{nk+n-1}x\|} \\ &\leq \frac{\|T^{nk+2}x\|}{\|T^{nk+1}x\|} \cdot \frac{\|T^{nk+3}x\|}{\|T^{nk+2}x\|} \cdots \frac{\|T^{nk+n+1}x\|}{\|T^{nk+n}x\|} \leq \cdots \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|T^{nk+n+1}x\|}{\|T^{nk+n}x\|} \cdot \frac{\|T^{nk+n+2}x\|}{\|T^{nk+n+1}x\|} \cdots \frac{\|T^{nk+n+n}x\|}{\|T^{nk+n+(n-1)}x\|} \\ &= \frac{\|T^{nk+2n}x\|}{\|T^{nk+n}x\|} = \frac{\|(T^n)^{k+2}x\|}{\|(T^n)^{k+1}x\|}. \end{aligned}$$

Thus, (**) holds for all x . Hence, T^n is k -quasi-paranormal. \square

We know that if T is normaloid then so T^m for $m \geq 1$. However if T^n is normaloid for $n > 1$, then T^m need not be normaloid for $m > n$. (Counterexample:

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

For a k -quasi-paranormal operator T , we have

THEOREM 3.9. *If T is k -quasi-paranormal and if T^p is normaloid for $p \geq k$ then T^n is normaloid for $n \geq p$.*

Proof. If $\sigma(T) = \{0\}$, then $T^p = 0$ and so the result is obvious. Now assume $\sigma(T) \neq \{0\}$. Then a power T is non-zero. We use the induction to show that T^{p+m} is normaloid for each positive integer m . First we prove the result for $m = 1$. Since T^p is normaloid and

$$\frac{\|T^{2p}\|}{\|T^{2p-1}\|} \geq \frac{\|T^{p+1}\|}{\|T^p\|},$$

$$\frac{\|T^p\|^2}{\|T^{2p-1}\|} \geq \frac{\|T^{p+1}\|}{\|T^p\|}$$

or

$$\|T^p\|^3 \geq \|T^{2p-1}\| \|T^{p+1}\|.$$

This in turn gives

$$r(T)^{3p} \geq \|T^{2p-1}\| \|T^{p+1}\| \geq \|T^{p+1}\| r(T)^{2p-1}$$

or $r(T^{p+1}) \geq \|T^{p+1}\|$. Hence, the result is true for $m = 1$. Now assume the result is true for $m = n$. Now

$$\frac{\|T^{2(p+n)}\|}{\|T^{2(p+n)-1}\|} \geq \frac{\|T^{p+n+1}\|}{\|T^{p+n}\|}$$

$$\Rightarrow \|T^{p+n}\|^3 \geq \|T^{p+n+1}\| \|T^{2p+2n-1}\|$$

$$\Rightarrow r(T)^{3p+3n} \geq \|T^{p+n+1}\| r(T)^{2p+2n-1}$$

$$\Rightarrow r(T^{p+n+1}) \geq \|T^{p+n+1}\|.$$

This completes the induction argument. \square

It is clear from the above results that for a k -quasi-paranormal operator T with non-zero spectrum, the sequence $\left\{\frac{\|T^{k+n+1}\|}{\|T^{k+n}\|}\right\}$ is bounded (bounded by $\|T\|$) and monotonically increasing and therefore it must be convergent. However, the question regarding the exact value of the limit remains unanswered. The following theorem suggest a strong possibility for the limit to be $r(T)$ (the spectral radius of T).

THEOREM 3.10. *If T is k -quasi-paranormal, then $r(T) \geq \frac{\|T^n\|}{\|T^{n-1}\|}$ for every positive integer $n \geq k + 1$.*

Proof. Since

$$\frac{\|T^{k+n}\|}{\|T^{k+n-1}\|} \geq \frac{\|T^{k+n-1}\|}{\|T^{k+n-2}\|} \geq \dots \geq \frac{\|T^{k+1}\|}{\|T^k\|},$$

$$\frac{\|T^{k+n}\|}{\|T^k\|} \geq \left(\frac{\|T^{k+1}\|}{\|T^k\|}\right)^n.$$

Thus,

$$\|T^n\| \geq \left(\frac{\|T^{k+1}\|}{\|T^k\|}\right)^n$$

or

$$\|T^n\|^{\frac{1}{n}} \geq \frac{\|T^{k+1}\|}{\|T^k\|}.$$

Letting $n \rightarrow \infty$, we get

$$r(T) \geq \frac{\|T^{k+1}\|}{\|T^k\|}.$$

Similarly

$$r(T) \geq \frac{\|T^{k+2}\|}{\|T^{k+1}\|}.$$

In general,

$$r(T) \geq \frac{\|T^n\|}{\|T^{n-1}\|}$$

for every positive integer $n \geq k + 1$. \square

In [20], Yamazaki proved that if T and T^* are class A operators, then T is normal. However, corresponding result is not true for k -quasi-paranormals. In case, the adjoint of a k -quasi-paranormal operator T is hyponormal, then T turns out to be normal. To see this note that if T^* is hyponormal, then $\ker T^* \subseteq \ker T$ implying T is paranormal. This suggests the following conjecture.

CONJENCTURE. *If the adjoint of a k -quasi-paranormal operator T is paranormal, then the operator T is normal.*

Acknowledgments. The author would like to thank the referees for their good reading of the paper and for their useful comments. This paper is supported by Taibah University research project 2014.

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Received 24 June 2014

*Taibah University,
College of Science,
Department of Mathematics,
P.O.Box 20003,
Al Madinah Al Munawarah,
Saudi Arabia
mecherisalah@hotmail.com*

*Sardar Patel University,
Department of Mathematics,
Vallabh Vidyanagar 388 120,
Gujarat, India
smpatel-32@yahoo.com*