APPROXIMATION BY LIPSCHITZ FUNCTIONS IN ABSTRACT SOBOLEV SPACES ON METRIC SPACES

MARCELINA MOCANU

We prove that the density of locally Lipschitz functions in a global Sobolev space based on a Banach function space implies the density of Lipschitz functions, with compact support in a given open set, in the corresponding Sobolev space with zero boundary values. In the case, when the Banach function space is a Lebesgue space, we recover some density results of Björn, Björn and Shanmugalingam (2008). Our results require neither a doubling measure nor the validity of a Poincaré inequality in the underlying metric measure space.

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1. INTRODUCTION

In the following, (X, d, μ) is a metric measure space, *i.e.* a metric space (X, d) endowed with a Borel regular measure μ, that is finite and positive on balls. We use the definition given by Bennet and Sharpley [3] for Banach function spaces over the σ-finite measure space (X, μ). The theory of Banach function spaces is an axiomatic unifying framework for the study of Orlicz spaces and Lorentz spaces, that generalize Lebesgue spaces.

Given a Banach function space $\mathbf{B}$, the abstract Sobolev space of Newtonian type $N^{1,\mathbf{B}}(X)$ consists of all functions $u : X \to \mathbb{R}$ such that $u \in \mathbf{B}$ and u has a $\mathbf{B}$-weak upper gradient in $\mathbf{B}$. The Banach space $N^{1,\mathbf{B}}(X)$ is an extension of Newtonian spaces $N^{1,p}(X)$ introduced by Shanmugalingam [21] (where $\mathbf{B} = L^p(X)$, $1 \leq p < \infty$), of Orlicz-Sobolev spaces introduced by Aïssaoui [1] and by Tuominen [23] (where $\mathbf{B}$ is an Orlicz space over X) and of a class of Newtonian Sobolev-Lorentz spaces introduced by Costea and Miranda [7] (where $\mathbf{B}$ is the Lorentz space $L^{p,q}(X)$ with $1 \leq q \leq p$).

In the Euclidean case, when $X = \Omega$ is an open subset of $\mathbb{R}^n$ with the Lebesgue measure μ, the Newtonian space $N^{1,p}(X)$ is the classical Sobolev space $W^{1,p}(\Omega)$, where $1 \leq p < \infty$. The Sobolev space with zero boundary values $W_0^{1,p}(\Omega)$ is an important tool in PDE's and calculus of variations, that allows us to compare boundary values of functions in $W^{1,p}(\Omega)$. There are two equivalent definitions of $W_0^{1,p}(\Omega)$. On one hand, $W_0^{1,p}(\Omega)$ is the completion of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$ ([5], IX.4). On the other hand, u belongs to $W_0^{1,p}(\Omega)$ if u can be extended to a function from the global Sobolev space $W^{1,p}(\mathbb{R}^n)$, such that the trace of the extension vanishes on $X \setminus \Omega$.

The second definition of Sobolev spaces with zero boundary values has been extended to the metric setting, in the case of Newtonian functions based on Lebesgue spaces [22], on Orlicz spaces [2] and on Banach function spaces [16]. A real-valued function u on $E \subset X$ belongs to the Newtonian space $N_0^{1,\mathbf{B}}(E)$ of functions with zero boundary values on E if u has a representative whose extension by zero to X belongs to $N^{1,\mathbf{B}}(X)$. It is known that $N_0^{1,\mathbf{B}}(E)$ is a closed subspace of the Banach space $N^{1,\mathbf{B}}(X)$. In analysis on metric measure spaces the role of smooth functions is played by Lipschitz continuous functions. The purpose of this paper is to provide some sufficient conditions for the density of Lipschitz functions with compact support in Ω in the space $N_0^{1,\mathbf{B}}(\Omega)$, where $\Omega \subset X$ is open.

The density of Lipschitz functions in Newtonian spaces $N^{1,p}(X)$ has been proved in doubling metric measure spaces (X, d, μ) supporting a weak $(1, p)$-Poincaré inequality [21]. Corresponding density results have been proved for Orlicz-Sobolev spaces [1, 23] and Sobolev-Lorentz spaces [7]. Without assuming that μ is doubling or that X supports a Poincaré inequality, Björn, Björn and Shanmugalingam [4] proved that in a proper metric space X the density of locally Lipschitz functions implies the density in $N_0^{1.p}(\Omega)$ of the set of compactly supported Lipschitz functions. So, in this case, we recover the first route to the definition of the Sobolev spaces with zero boundary values. The density results in [4] have been extended in [18] by replacing the Lebesgue space $L^p(X)$ by an Orlicz space $L^\Psi(X)$, where Ψ is a doubling N-function. In this paper, we generalize the results from [18] to the case of Newtonian spaces based on Banach function spaces that satisfy some natural assumptions.

2. PRELIMINARIES

Denote the open balls, respectively the closed balls in the metric space (X, d) by $B(x, r) = \{y \in X : d(y, x) < r\}$ and $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$. A metric space is called *proper* if every closed ball of the space is compact.

Let (X, μ) be a σ-finite measure space and $\boldsymbol{M}^+(X)$ be the set of μ-measurable non-negative functions on X.

Definition 1 ([3]). A function $\rho : \boldsymbol{M}^+(X) \to [0, \infty]$ is called a *Banach function norm* if, for all functions f, g, f_n $(n \geq 1)$ in $\boldsymbol{M}^+(X)$, for all constants $a \geq 0$ and for all measurable sets $E \subset X$, the following properties hold:

(P1) $\rho(f)=0$ iff $f=0$ μ-a.e.; $\rho(af)=a\rho(f)$; $\rho(f+g)\leq\rho(f)+\rho(g)$.
(P2) If $0\leq g\leq f$ μ-a.e., then $\rho(g)\leq\rho(f)$.
(P3) If $0\leq f_n\uparrow f$ μ-a.e., then $\rho(f_n)\uparrow\rho(f)$.
(P4) If $\mu(E)<\infty$, then $\rho(\chi_E)<\infty$.
(P5) If $\mu(E)<\infty$, then $\int\limits_E f\ \mathrm{d}\mu\leq C_E\rho(f)$, for some constant $C_E\in(0,+\infty)$ depending only on E and ρ.

The collection $\mathbf{B}$ of the μ-measurable functions $f:X\to[-\infty,+\infty]$ for which $\rho(|f|)<\infty$ is called a *Banach function space* on X. For $f\in\mathbf{B}$ define

$$\|f\|_{\mathbf{B}}=\rho(|f|).$$

We identify two functions that coincide μ-a.e. and denote by $\approx$ the relation of equality μ-a.e. If $f,g:X\to\overline{\mathbb{R}}$ such that $\rho(|f|)<\infty$ and $f=g$ μ-a.e, then g is μ-measurable and $\rho(|g|)=\rho(|f|)<\infty$. Moreover, by Definition 1 (P5) and the σ-finiteness of μ, it follows that f and g are finite μ-a.e., hence, $f-g=0$ $\mu-$a.e. and therefore, $\|f-g\|_{\mathbf{B}}=0$.

Definition 2 ([3], Definition I.3.1). A function $f\in\mathbf{B}$ is said to have absolutely continuous norm in $\mathbf{B}$ if and only if $\|f\chi_{E_k}\|_{\mathbf{B}}\to 0$ for every sequence $(E_k)_{k\geq 1}$ of measurable sets satisfying $\mu\left(\limsup_{k\to\infty} E_k\right)=0$. The space $\mathbf{B}$ is said to have absolutely continuous norm if every $f\in\mathbf{B}$ has absolutely continuous norm.

An Orlicz space $L^{\Psi}(X)$ has absolutely continuous norm if the Young function Ψ is doubling. The (p,q)-norm of a Lorentz space $L^{p,q}(X)$ with $1<p<\infty$ and $1\leq q<\infty$ is absolutely continuous [7]. In $L^{\infty}(X)$ the only function having an absolutely continuous norm is the null function.

In a space with absolutely continuous norm, a suitable form of Lebesgue dominated convergence theorem holds.

Lemma 1 ([3], Proposition I.3.6). *A function f in a Banach function space $\mathbf{B}$ has absolutely continuous norm if and only if the following condition holds: whenever f_n, $n\geq 1$ and g are μ-measurable functions satisfying $|f_n|\leq|f|$ for all n and $f_n\to g$ μ-a.e., then $\|f_n-g\|_{\mathbf{B}}\to 0$ as $n\to\infty$.*

A Banach function space $\mathbf{B}$ is said to *have property (C)* if $\lim\limits_{n\to\infty}\mu(E_n)=0$ for every sequence $E_n\subset X$, $n\geq 1$ of measurable sets satisfying the condition $\lim\limits_{n\to\infty}\|\chi_{E_n}\|_{\mathbf{B}}=0$.

If $\mathbf{B}=L^{\Psi}(X)$ is an Orlicz space with the Luxemburg norm, then $\|\chi_E\|_{\mathbf{B}}=1/\Psi^{-1}(1/\mu(E))$ and if $\mathbf{B}=L^{p,q}(X)$ is a Lorentz space with the p,q-norm,

$1 < p < \infty$ and $1 \leq q < \infty$, then $\|\chi_E\|_{\mathbf{B}} = c(p,q)\,\mu(E)^{1/p}$, where $c(p,q) = (p/q)^{1/q}$. If $\mathbf{B} = L^{\infty}(X)$, then $\|\chi_E\|_{\mathbf{B}} = sgn(\mu(E))$. All these three types of Banach function spaces have property (C). Every rearrangement invariant Banach function space over a resonant measure space has property (C), as follows from ([3], Corollary II. 5.3).

Let $\mathbf{B}$ be a Banach function space with a norm $\|\cdot\|_{\mathbf{B}}$. The **B**-*modulus of a family* Γ of curves in X is defined by $M_{\mathbf{B}}(\Gamma) = \inf \|\rho\|_{\mathbf{B}}$,where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ satisfying $\int_{\gamma} \rho \mathrm{d}s \geq 1$ for all rectifiable curves γ in X [17].

A Borel measurable function $g : X \to [0, +\infty]$ is said to be an *upper gradient* of a function $u : X \to \mathbb{R}$ if for every rectifiable curve $\gamma : [a,b] \to X$ the following inequality holds

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \mathrm{d}s. \tag{2.1}$$

A **B**-*weak upper gradient* of a function $u : X \to \mathbb{R}$ is a Borel measurable function $g : X \to [0, \infty]$ such that (2.1) holds for all rectifiable curves $\gamma : [a,b] \to X$ except for a curve family with zero **B**-modulus. We can weaken the assumption that g is Borel measurable in the definition of a **B**-weak upper gradient, saying that $g : X \to [0, \infty]$ is a *generalized* **B**-*weak upper gradient* of a function $u : X \to \mathbb{R}$ if there exists a curve family $\Gamma_0 \subset \Gamma_{rec}$ with $Mod_{\mathbf{B}}(\Gamma_0) = 0$ such that for every $\gamma \in \Gamma_{rec} \backslash \Gamma_0$ the function $g \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is Lebesgue measurable and (2.1) holds.

For every function $u : X \to \mathbb{R}$ we will denote by $G_{u,\mathbf{B}}$ the family of all **B**-weak upper gradients $g \in \mathbf{B}$ of u in X. Let $\widetilde{N}^{1,\mathbf{B}}(X)$ be the set formed from the real-valued functions $u \in \mathbf{B}$ for which $G_{u,B}$ is non-empty. The functional $\|u\|_{1,\mathbf{B}} := \|u\|_{\mathbf{B}} + \inf\{\|g\|_{\mathbf{B}} : g \in G_{u,\mathbf{B}}\}$ is a seminorm on $\widetilde{N}^{1,\mathbf{B}}(X)$. The Newtonian-type space $N^{1,\mathbf{B}}(X)$ is defined as the quotient normed space of $\widetilde{N}^{1,\mathbf{B}}(X)$ with respect to the equivalence relation defined by: $u \sim v$ if $\|u - v\|_{1,\mathbf{B}} = 0$. The norm on $N^{1,\mathbf{B}}(X)$ corresponding to the seminorm $\|\cdot\|_{1,\mathbf{B}}$ is denoted by $\|\cdot\|_{N^{1,\mathbf{B}}(X)}$ [17].

In the definition of $\widetilde{N}^{1,\mathbf{B}}(X)$, we can use alternatively upper gradients (respectively, generalized **B**-weak upper gradients) instead of **B**-weak upper gradients. For every **B**-weak upper gradient $g \in \mathbf{B}$ of a function $u : X \to \mathbb{R}$ there is a decreasing sequence $(g_i)_{i \geq 1}$ of upper gradients of u such that $\lim_{i \to \infty} \|g_i - g\|_{\mathbf{B}} = 0$ ([17], Proposition 2). The case $\mathbf{B} = L^p(X)$ was proved by Koskela and MacManus [12]. For every generalized **B**-weak upper gradient h

of u that is finite μ-a.e. there exists a $\mathbf{B}$-weak upper gradient g of u such that $g = h$ μ-almost everywhere in X. Generalized $\mathbf{B}$-weak upper gradients are stable under modifications μ-a.e. ([14], Lemma 6). For all $u \in N^{1,\mathbf{B}}(X)$ we have

$$\begin{aligned}\|u\|_{N^{1,\mathbf{B}}(X)} - \|u\|_{\mathbf{B}} \\ &= \inf\{\|h\|_{\mathbf{B}} : h \in \mathbf{B} \text{ is a generalized } \mathbf{B}\text{-weak upper gradient of } u\} \\ &= \inf\{\|g\|_{\mathbf{B}} : g \in \mathbf{B} \text{ is an upper gradient of } u\}\end{aligned}$$

A Sobolev capacity with respect to the space $N^{1,\mathbf{B}}(X)$ is defined by $Cap_{\mathbf{B}}(E) = \inf\{\|u\|_{N^{1,\mathbf{B}}(X)} : u \in N^{1,\mathbf{B}}(X) : u \geq 1 \text{ on } E\}$. Note that $Cap_{\mathbf{B}}(E) = \inf\{\|u\|_{N^{1,\mathbf{B}}(X)} : u \in N^{1,\mathbf{B}}(X) : 0 \leq u \leq 1,\ u = 1 \text{ on } E\}$. It was shown that $\mathbf{B}$-capacity is an outer measure, that represents the correct gauge for distinguishing between two functions in $N^{1,\mathbf{B}}(X)$ [17].

We recall the formal definition of a Newtonian space with zero boundary values, based on a Banach function space [16]. Denote by $\widetilde{N}_0^{1,B}(E)$ be the collection of functions $u : E \to \mathbb{R}$ for which there exists $\overline{u} \in \widetilde{N}^{1,\mathbf{B}}(X)$ such that $\overline{u} = u$ μ-a.e. on E and $Cap_{\mathbf{B}}(\{x \in X \backslash E : \overline{u}(x) \neq 0\}) = 0$. If $u, v \in \widetilde{N}_0^{1,\mathbf{B}}(E)$ define $u \simeq v$ if $u = v$ μ-a.e. on E. Then $\simeq$ is an equivalence relation. We consider the quotient space $N_0^{1,\mathbf{B}}(E) = \widetilde{N}_0^{1,B}(E)/\simeq$. A norm on $N_0^{1,\mathbf{B}}(E)$ is unambiguously defined by $\|u\|_{N_0^{1,\mathbf{B}}(E)} := \|\overline{u}\|_{N^{1,\mathbf{B}}(X)}$.

Given $u \in \widetilde{N}_0^{1,\mathbf{B}}(E)$, we define $\widetilde{\overline{u}}(x) = \overline{u}(x)$ if $x \in E$ and $\widetilde{\overline{u}}(x) = 0$ if $x \in X \setminus E$. Since $\widetilde{\overline{u}} = \overline{u}$ outside a set of $\mathbf{B}$-capacity zero, it follows that $\widetilde{\overline{u}} \in \widetilde{N}^{1,\mathbf{B}}(X)$ and $\widetilde{\overline{u}}$ defines the same equivalence class in $N^{1,\mathbf{B}}(X)$ as $\overline{u}$. In the following, we will identify each $u \in N_0^{1,\mathbf{B}}(E)$ to the corresponding function $\widetilde{\overline{u}} \in N^{1,\mathbf{B}}(X)$, that will be also denoted by u.

3. PRELIMINARY DENSITY RESULTS

In the proofs of density results it is very important to build new gradients from old ones. We will need to cut and paste (generalized) weak upper gradients and to have a counterpart for the product rule.

Since every generalized B-weak upper gradient $g \in B$ of a function $u : X \to R$ coincides μ-a.e. with a B-weak upper gradient of u, several results regarding weak upper gradients can be extended to generalized weak upper gradients.

A function $u : X \to \mathbb{R}$ is said to be absolutely continuous (AC) on a compact rectifiable curve γ parameterized by arc-length if $u \circ \gamma : [0, l(\gamma)] \to \mathbb{R}$ is absolutely continuous. The function u is said to be AC on $\mathbf{B}$-almost every curve if there exists a family $\Gamma_0 \subset \Gamma_{rec}$ with $M_{\mathbf{B}}(\Gamma_0) = 0$, such that u is absolutely continuous on each curve $\gamma \in \Gamma_{rec} \setminus \Gamma_0$. We will denote by $ACC_{\mathbf{B}}(X)$ the family of all functions $u : X \to \mathbb{R}$ that are AC on $\mathbf{B}$-almost every curve. Every function $u : X \to \mathbb{R}$ that has a $\mathbf{B}$-weak upper gradient $g \in \mathbf{B}$ in X belongs to $ACC_{\mathbf{B}}(X)$, in particular $N^{1,\mathbf{B}}(X) \subset ACC_{\mathbf{B}}(X)$ ([17], Proposition 3).

Lemma 2 ([15]). *Assume that $u_k : X \to \mathbb{R}$, $k \in \{1, 2, 3\}$, where $u_1 \in ACC_{\mathbf{B}}(X)$ and u_k has a generalized $\mathbf{B}$-weak upper gradient $g_k \in \mathbf{B}$ in X for $k \in \{2, 3\}$. If $F \subset X$ is a μ-measurable set such that $u_1|_F = u_2|_F$ and $u_1|_{X\setminus F} = u_3|_{X\setminus F}$, then the function $g_1 := g_2\chi_F + g_3\chi_{X\setminus F}$ is a generalized $\mathbf{B}$-weak upper gradient of u_1 in X.*

Corollary 1. *Assume that $u : X \to \mathbb{R}$, $c_0 \in \mathbb{R}$ and $F \subset \{x \in X : u(x) = c_0\}$ is a μ-measurable set. If u has a generalized $\mathbf{B}$-weak upper gradient $g \in \mathbf{B}$ in X, then $g\chi_{X\setminus F}$ is also a generalized $\mathbf{B}$-weak upper gradient of u in X.*

Lemma 3 ([14]). *Assume that $u_k : X \to \mathbb{R}$ is a μ-measurable function which has a $\mathbf{B}$-weak upper gradient $g_k \in \mathbf{B}$ in X, for $k \in \{1, 2\}$. Then the function $g := |u_1| g_2 + |u_2| g_1$ is a generalized $\mathbf{B}$-weak upper gradient of $u := u_1u_2$ in X. Moreover, if u_1 and u_2 are bounded, then $g \in \mathbf{B}$.*

Some natural assumptions on $\mathbf{B}$ imply the density in $N^{1,\mathbf{B}}(X)$ of the set of bounded functions from $N^{1,\mathbf{B}}(X)$.

Lemma 4 ([14]). *Assume that the Banach function space $\mathbf{B}$ has absolutely continuous norm and has property (C). Let $u \in N^{1,\mathbf{B}}(X)$ be nonnegative. For each integer $k \geq 0$ we define $u_k := \min\{u, k\}$. Then $u_k \in \mathbf{B}$ for each $k \geq 0$ and the sequence $(u_k)_{k\geq 0}$ converges to u in the norm of $N^{1,\mathbf{B}}(X)$.*

We will need to approximate from below Newtonian functions that are non-negative, bounded and with bounded support.

Lemma 5. *Assume that $\mathbf{B}$ has absolutely continuous norm. Let $u \in N^{1,\mathbf{B}}(X)$ be non-negative, bounded and with bounded support. Then $u_\varepsilon := \max\{u - \varepsilon, 0\}$ tends to u in $N^{1,\mathbf{B}}(X)$ as ε decreases to zero.*

Proof. Denote $S := supp\, u$. For every $\varepsilon > 0$ we have $|u_\varepsilon - u| \leq u$ in X and $u \in \mathbf{B}$, therefore $u_\varepsilon \in \mathbf{B}$.

It suffices to prove that $\lim_{n\to\infty} \max\{u - \varepsilon_n, 0\} = u$ in $N^{1,\mathbf{B}}(X)$ whenever $(\varepsilon_n)_{n\geq 1}$ is a sequence of positive numbers decreasing to zero. Denote $u_n := \max\{u - \varepsilon_n, 0\}$, $n \geq 1$. Since $(\varepsilon_n)_{n\geq 1}$ is decreasing, the sequence (u_n) is non-decreasing.

Since $|u_n - u| \leq u$ and $u_n - u \to 0$ on X as $n \to \infty$, we get $\lim_{n\to\infty} \|u_n - u\|_{\mathbf{B}} = 0$, by Lemma 1.

Let $g_u \in \mathbf{B}$ be an upper gradient of u.

For each $\varepsilon > 0$, denote $F_\varepsilon = \{x \in X : 0 < u(x) < \varepsilon\}$. Then F_ε is μ-measurable and $u_\varepsilon(x) - u(x) \in \{-\varepsilon, 0\}$, whenever $x \in X \setminus F_\varepsilon$, while $u_\varepsilon(x) - u(x) = -u(x)$ for all $x \in F_\varepsilon$. Applying two times Corollary 1, it follows that $g_u \chi_{F_\varepsilon}$ is a generalized $\mathbf{B}$-weak upper gradient of $u_\varepsilon - u$. In particular, denoting $F_n := F_{\varepsilon_n}$ we see that

$$\|u_n - u\|_{N^{1,\mathbf{B}}(X)} \leq \|u_n - u\|_{\mathbf{B}} + \|g_u \chi_{F_n}\|_{\mathbf{B}}, \tag{3.1}$$

for $n \geq 1$.

Since $(\varepsilon_n)_{n\geq 1}$ is decreasing and tends to zero, the sequence of sets $(F_n)_{n\geq 1}$ is decreasing and $\lim\limits_{n\to\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n = \emptyset$. Since $g_u \in \mathbf{B}$ has absolutely continuous norm, we have $\lim\limits_{n\to\infty} \|g_u \chi_{F_n}\|_{\mathbf{B}} = 0$.

By inequality (3.1), we obtain $\lim\limits_{n\to\infty} u_n = u$ in $N^{1,\mathbf{B}}(X)$, q.e.d. □

4. DENSITY OF COMPACTLY SUPPORTED LIPSCHITZ FUNCTIONS IN NEWTONIAN SPACES

Let $\Omega \subset X$ be an open set. We will denote by $Lip_C(\Omega)$ the family of all Lipschitz functions $u : \Omega \to \mathbb{R}$ with the property that the support of u is a compact subset of Ω. It is easy to see that every L-Lipschitz function $u : \Omega \to \mathbb{R}$ admits as an upper gradient the constant function $L > 0$. Moreover, it is known that $lipu$ is an upper gradient of a Lipschitz function $u : \Omega \to \mathbb{R}$, where $lipu(x) = \liminf\limits_{r\to 0} \frac{1}{r} L(x, u, r)$ with $L(x, u, r) = \sup\{|u(y) - u(x)| : d(x, y) \leq r\}$ ([8], Lemma 6.7).

For every $u : E \to \mathbb{R}$ we denote by $\widetilde{u}$ the extension by zero of u to X, defined by $\widetilde{u}(x) = u(x)$ if $x \in E$ and $\widetilde{u}(x) = 0$ if $x \in X \setminus E$.

Lemma 6. *Let $u : \Omega \to \mathbb{R}$ be L-Lipschitz, where $L > 0$ and denote $S :=$ supp u.*

a) *$L\chi_S$ is an upper gradient of u in Ω;*

b) *If $u \in Lip_C(\Omega)$ and $g : \Omega \to \mathbb{R}$ is an upper gradient of u in Ω, then the extension by zero of g to X is an upper gradient in X for the extension by zero of u to X;*

c) *If $u \in Lip_C(\Omega)$, then $L\chi_S$ is an upper gradient in X for the extension by zero of u to X.*

Proof. Clearly, $lipu \leq L$ in Ω. Since S is a closed set, $L\chi_S$ is a Borel function.

a) If $S = \Omega$, there is nothing to prove. Suppose that $S \neq \Omega$. Since u vanishes identically on the open set $\Omega \setminus S$, we have $lipu\,(x) = 0$ for all $x \in \Omega \setminus S$. Then $lipu \leq L\chi_S$ in Ω, hence, $L\chi_S$ is an upper gradient of u in Ω.

b) Let $\widetilde{u} : X \to \mathbb{R}$ and $\widetilde{g} : X \to \mathbb{R}$ be the extensions by zero of u and g, respectively. Let $\gamma : [a, b] \to X$ be a rectifiable curve. We check that $D := |\widetilde{u}(\gamma(a)) - \widetilde{u}(\gamma(b))| \leq \int_\gamma \widetilde{g} ds =: I$. Since $\widetilde{u}$ is continuous on X, the set $\{t \in [a, b] : (\widetilde{u} \circ \gamma)(t) = 0\}$ is compact, therefore it contains its lower bound a_0 and its upper bound b_0. If $a < a_0$, then $\gamma([a, a_0)) \subset \{x \in X : u(x) \neq 0\}$, hence $\gamma([a, a_0]) \subset S \subset \Omega$, by the continuity of γ. Similarly, if $b_0 < b$, then $\gamma([b_0, b]) \subset S \subset \Omega$. Note that $u(\gamma(a_0)) = u(\gamma(b_0)) = 0$. Assuming that $a < a_0$ and $b_0 < b$ we get $D = |u(\gamma(a)) - u(\gamma(b))| \leq |u(\gamma(a)) - u(\gamma(a_0))| + |u(\gamma(b_0)) - u(\gamma(b))|$ $\leq \int_{\gamma|_{[a,a_0]}} g ds + \int_{\gamma|_{[b_0,b]}} g ds \leq I$. If $a = a_0$ and $b_0 = b$, then $D = 0 \leq I$. If $a = a_0$ and $b_0 < b$, then $D = |u(\gamma(b_0)) - u(\gamma(b))| \leq \int_{\gamma|_{[b_0,b]}} g ds \leq I$. If $a < a_0$ and $b_0 = b$, then $D = |u(\gamma(a)) - u(\gamma(a_0))| \leq \int_{\gamma|_{[a,a_0]}} g ds \leq I$.

c) According to a), $L\chi_S$ is an upper gradient of u in Ω, hence, by b), $L\chi_S$ is an upper gradient of $\widetilde{u}$ in X. □

Let $\mathbf{B}$ a Banach function space $\mathbf{B}$ over X. Using properties (P1), (P2) and (P4) from Definition 1 it follows that the extension by zero of any function $u \in Lip_C(\Omega)$ belongs to $N^{1,\mathbf{B}}(X)$. Thus, $Lip_C(\Omega) \subset N_0^{1,\mathbf{B}}(\Omega)$.

It is natural to ask for assumptions on X and $\mathbf{B}$ under which $Lip_C(\Omega)$ is dense in $N_0^{1,\mathbf{B}}(\Omega)$.

First, we look at the density in the global Newtonian space $N^{1,\mathbf{B}}(X)$ of functions on X having bounded support.

Lemma 7. *Let $\mathcal{S}$ be the family of functions in $N^{1,\mathbf{B}}(X)$ that have bounded support. If $\mathbf{B}$ has absolutely continuous norm, then $\mathcal{S}$ is dense in $N^{1,\mathbf{B}}(X)$.*

Proof. If X is bounded, then $\mathcal{S} = N^{1,\mathbf{B}}(X)$. Assuming that X is unbounded, fix $x_0 \in X$ and write X as $X = \cup_{k=1}^{\infty} X_k$, where $X_k := \overline{B}(x_0, k)$ for each integer $k \geq 1$.

Let $v \in N^{1,\mathbf{B}}(X)$. We will approximate v in $N^{1,\mathbf{B}}(X)$ by a sequence of functions in $\mathcal{S}$, using multiplication by Lipschitz cut-off functions.

For each integer $k \geq 1$, consider the function $\eta_k : X \to \mathbb{R}$, $\eta_k(x) = \max\{1 - dist(x, X_k), 0\}$. Note that η_k is 1-Lipschitz, $0 \leq \eta_k \leq 1$ on X and

$supp\,\eta_k \subset X_{k+1}$ is bounded.

The function $v\eta_k$ belongs to $\mathbf{B}$, since $v \in \mathbf{B}$ and η_k is bounded. We estimate the norm in $N^{1,\mathbf{B}}(X)$ of $v - v\eta_k = v(1-\eta_k)$, taking into account that this function vanishes on X_k. Let $g_v \in \mathbf{B}$ be an upper gradient of v in X. Using Lemma 3, taking into account that $0 \le \eta_k \le 1$ and $L = 1$ is an upper gradient of η_k, then applying Corollary 1, we see that $(|v| + g_v)\chi_{X\setminus X_k}$ is a generalized $\mathbf{B}$-weak upper gradient of $v - v\eta_k$ in X. Then

$$\begin{aligned}\|v - v\eta_k\|_{N^{1,\mathbf{B}}(X)} &\le \left\|v\chi_{X\setminus X_k}\right\|_{\mathbf{B}} + \left\|(|v| + g_v)\chi_{X\setminus X_k}\right\|_{\mathbf{B}}\\ &\le 2\left\|v\chi_{X\setminus X_k}\right\|_{\mathbf{B}} + \left\|g_v\chi_{X\setminus X_k}\right\|_{\mathbf{B}}.\end{aligned}$$

Since $\limsup\limits_{k\to\infty}(X \setminus X_k) = \emptyset$ and $v, g_v \in \mathbf{B}$ have absolutely continuous norm, we get $\lim\limits_{k\to\infty}\left\|v\chi_{X\setminus X_k}\right\|_{\mathbf{B}} = \lim\limits_{k\to\infty}\left\|g_v\chi_{X\setminus X_k}\right\|_{\mathbf{B}} = 0$. Then $\lim\limits_{k\to\infty}\|v - v\eta_k\|_{N^{1,\mathbf{B}}(X)} = 0$. □

The following result provides sufficient conditions for the density of Lipschitz compactly supported functions on X in the global Newtonian space $N^{1,\mathbf{B}}(X)$.

Proposition 1. *Let X be proper and assume that $\mathbf{B}$ has absolutely continuous norm. If locally Lipschitz functions are dense in $N^{1,\mathbf{B}}(X)$, then $Lip_C(X)$ is a dense subset of $N^{1,\mathbf{B}}(X)$.*

Proof. If X is bounded, then it is compact and $Lip_C(X) = Lip_{loc}(X)$. Assuming that X is unbounded, let X_k and η_k be as in the proof of Lemma 7, for $k \ge 1$. Since X is proper, each X_k is compact.

Let $u \in N^{1,\mathbf{B}}(X)$ and $\varepsilon > 0$. By our assumption, there exists $v \in Lip_{loc}(X) \cap N^{1,\mathbf{B}}(X)$ such that $\|u - v\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$.

The function $v\eta_k \in N^{1,\mathbf{B}}(X)$ is Lipschitz and compactly supported. By the proof of Lemma 7, $\lim\limits_{k\to\infty}\|v - v\eta_k\|_{N^{1,\mathbf{B}}(X)} = 0$. Pick an integer $k_0 \ge 1$ such that $\|v - v\eta_{k_0}\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$.

We obtain $\|u - v\eta_{k_0}\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$ and the claim follows. □

Next, we move to the study of dense subclasses of $N_0^{1,\mathbf{B}}(E)$, where $E \subset X$.

Lemma 8. *Let $\mathcal{S}$ be the family of functions in $N^{1,\mathbf{B}}(X)$ that have bounded support and let $\mathcal{B}$ be the family of bounded functions in $N^{1,\mathbf{B}}(X)$. Assume that $\mathbf{B}$ has absolutely continuous norm. Let $E \subset X$. Then $\mathcal{S} \cap N_0^{1,\mathbf{B}}(E)$ is dense in $N_0^{1,\mathbf{B}}(E)$. Moreover, if $\mathbf{B}$ has property (C), then $\mathcal{B} \cap N_0^{1,\mathbf{B}}(E)$ is dense in $N_0^{1,\mathbf{B}}(E)$.*

Proof. Let $v \in N_0^{1,\mathbf{B}}(E)$.

If X is bounded, then $v \in \mathcal{S} \cap N_0^{1,\mathbf{B}}(E)$. If X is unbounded, taking the sequence $(\eta_k)_{k\geq 1}$ as in the proof of Lemma 7, we see that $v\eta_k \in \mathcal{S} \cap N_0^{1,\mathbf{B}}(E)$ for $k \geq 1$ and $\lim_{k\to\infty} v\eta_k = v$ in $N^{1,\mathbf{B}}(X)$.

Write $v = v^+ - v^-$, where $v^+ := \max\{v, 0\}$ and $v^- := \max\{-v, 0\}$. Then v^+, $v^- \in N_0^{1,\mathbf{B}}(E)$. For each integer $k \geq 1$ define $v_k^\pm := \min\{v^\pm, k\}$. Then $v_k^\pm \in \mathcal{B} \cap N_0^{1,\mathbf{B}}(E)$, hence $v_k^+ - v_k^- \in \mathcal{B} \cap N_0^{1,\mathbf{B}}(E)$. By Lemma 4, we have $\lim_{k\to\infty} v_k^\pm = v^\pm$ in $N^{1,\mathbf{B}}(X)$, hence $\lim_{k\to\infty} \left(v_k^+ - v_k^-\right) = v$ in $N^{1,\mathbf{B}}(X)$. □

In the case when $E = \Omega$ is an open subset of X, we will obtain a stronger version of the density in $N_0^{1,\mathbf{B}}(E)$ of functions with bounded support, by requiring that all their supports are contained in E. If $u \in N_0^{1,\mathbf{B}}(\Omega)$, then *supp* $(u\eta_k) \subset \overline{\Omega}$ is bounded, but it is possible to have *supp* $(u\eta_k) \cap \partial\Omega \neq \emptyset$, *e.g.* in the case when u is continuous on X.

Example 1. Assume that $u \in N_0^{1,\mathbf{B}}(\Omega) \cap C(X)$ is non-negative, not identically zero. Since $u = 0$ on $X \setminus \Omega$ and $u \in C(X)$, we have $u = 0$ on $\partial\Omega$. Let $\varepsilon > 0$. For each $x \in \partial\Omega$ there exists an open set $V_{\varepsilon,x}$ containing x, such that $0 \leq u(y) < \varepsilon$ for all $y \in V_{\varepsilon,x}$. Then $D_\varepsilon := \bigcup_{x\in\partial\Omega} V_{\varepsilon,x}$ is an open superset of the boundary $\partial\Omega$. Let $0 < \varepsilon_0 < \sup_{z\in\Omega} u(x)$. Then $D_\varepsilon \cap \Omega$ is a proper subset of Ω whenever $0 < \varepsilon \leq \varepsilon_0$. Consider $u_\varepsilon := \max\{u - \varepsilon, 0\}$ as in Lemma 5. Note that *supp* $u_\varepsilon \subset$ *supp* u. Moreover, $u_\varepsilon(x) = 0$ if $x \in (X \setminus \Omega) \cup D_\varepsilon$. For $0 < \varepsilon \leq \varepsilon_0$ we have $\operatorname{supp} u_\varepsilon \subset \Omega$. If $\mathbf{B}$ has absolutely continuous norm, then $u_\varepsilon \to u$ in $N^{1,\mathbf{B}}(X)$ as ε decreases to zero. So, every function in $N_0^{1,\mathbf{B}}(\Omega) \cap C(X)$ is the limit in $N^{1,\mathbf{B}}(X)$ of a sequence of functions having the supports contained in Ω.

In order to get an approximating sequence $(v_n)_{n\geq 1}$ for u, with supports $(\mathit{supp}\, v_n)_{n\geq 1}$ staying away from the boundary $\partial\Omega$, similar to $(u_{\varepsilon_n})_{n\geq 1}$ with $\varepsilon_n \downarrow 0$, we will assume that u is $\mathbf{B}$-quasicontinuous.

A function $u : X \to \mathbb{R}$ is called $\mathbf{B}$-*quasicontinuous* if for every $\varepsilon > 0$ there is a set $E \subset X$ with $Cap_{\mathbf{B}}(E) < \varepsilon$ such that the restriction of u to $X \setminus E$ is continuous.

Proposition 2. *Let* $\mathbf{B}$ *a Banach function space that has absolutely continuous norm and has property (C). Assume that all functions in* $N^{1,\mathbf{B}}(X)$ *are* $\mathbf{B}$-*quasicontinuous. Let* $\Omega \subset X$ *be open. Then every function* $u \in N_0^{1,\mathbf{B}}(\Omega)$ *is the limit in* $N^{1,\mathbf{B}}(X)$ *of a sequence of functions that have bounded support contained in* Ω.

Proof. We follow the lines of the proof of Lemma 5.9 from [4]. Using Lemma 8, we may assume that u is a bounded function and that u has a bounded support. We also may assume that u is non-negative. Let $u_\varepsilon := \max\{u-\varepsilon, 0\}$ for $\varepsilon > 0$. Note that $supp\, u_\varepsilon \subset \{x \in X : u(x) \geq \varepsilon\}$. Since u is bounded, u_ε is also bounded for every $\varepsilon > 0$.

Taking advantage of the fact that u is $\mathbf{B}$-quasicontinuous on X, we find a sequence of open sets U_j, $j \geq 1$, such that each restriction $u|_{X \setminus U_j}$ is continuous and $\lim_{j\to\infty} Cap_{\mathbf{B}}(U_j) = 0$. By the definition of the Sobolev $\mathbf{B}$-capacity and the remark following it, for each $j \geq 1$ there exists $w_j \in N^{1,\mathbf{B}}(X)$ such that $0 \leq w_j \leq 1$, $w_j = 1$ on U_j and $\|w_j\|_{N^{1,\mathbf{B}}(X)} < Cap_{\mathbf{B}}(U_j) + \frac{1}{j}$. By [17, Proposition 2] there exists an upper gradient g_j of w_j such that $\|g_j\|_{\mathbf{B}} \leq \|w_j\|_{N^{1,\mathbf{B}}(X)} + 1/j$. Then $\lim_{j\to\infty} \|w_j\|_{\mathbf{B}} = \lim_{j\to\infty} \|g_j\|_{\mathbf{B}} = 0$. By Theorem I.1.4 from [3], every sequence that converges in B to some function f contains a subsequence that is pointwise μ-almost everywhere convergent to f. Passing to a subsequence, we may assume that $w_j \to 0$ μ-a.e. in X.

Fix $\varepsilon > 0$. By the continuity of $u|_{X\setminus U_j}$, the set $\{x \in X \setminus U_j : u(x) < \varepsilon\}$ is relatively open in $X \setminus U_j$. Then the set $W_{\varepsilon,j} := U_j \cup \{x \in X : u(x) < \varepsilon\}$ is open in X. Since $u = 0$ in $X \setminus \Omega$ and $u \geq \varepsilon$ in $X \setminus W_{\varepsilon,j}$, the closed set $X \setminus W_{\varepsilon,j}$ is contained in $\Omega \cap supp\, u$. Since $supp\, u$ is bounded, $X \setminus W_{\varepsilon,j}$ is also bounded.

Define the functions $u_{\varepsilon,j} := (1-w_j)u_\varepsilon$ for $j \geq 1$. Then $supp\ u_{\varepsilon,j} \subset X \setminus W_{\varepsilon,j}$, therefore $supp\ u_{\varepsilon,j} \subset \Omega$ and $supp\ u_{\varepsilon,j}$ bounded. We will prove that $u_{\varepsilon,j} \to u_\varepsilon$ in $N^{1,\mathbf{B}}(X)$ as $j \to \infty$.

We have $\|u_{\varepsilon,j} - u_\varepsilon\|_{\mathbf{B}} = \|w_j u_\varepsilon\|_{\mathbf{B}}$. Since $|w_j u_\varepsilon| \leq u_\varepsilon$ for all $j \geq 1$, $w_j u_\varepsilon \to 0$ μ-a.e. and $u_\varepsilon \in \mathbf{B}$ has absolutely continuous norm, we have $\lim_{j\to\infty} \|u_{\varepsilon,j} - u_\varepsilon\|_{\mathbf{B}} = 0$, by Lemma 1.

Let $g \in \mathbf{B}$ be an upper gradient of u_ε. By Lemma 3, taking into account that $0 \leq u_\varepsilon \leq \|u_\varepsilon\|_\infty$, it follows that $\rho_{\varepsilon,j} := \|u_\varepsilon\|_\infty g_j + w_j g$ is a generalized $\mathbf{B}$-weak upper gradient of $u_{\varepsilon,j} - u_\varepsilon$. But $\lim_{j\to\infty} \|g_j\|_{\mathbf{B}} = 0$, hence $\|u_\varepsilon\|_\infty g_j \to 0$ in $\mathbf{B}$ as $j \to \infty$. Since $|w_j g| \leq g$ for all $j \geq 1$, $w_j g \to 0$ μ-a.e. and $g \in \mathbf{B}$ has absolutely continuous norm, we have $\lim_{j\to\infty} \|w_j g\|_{\mathbf{B}} = 0$, by Lemma 1. Then $\lim_{j\to\infty} \|\rho_{\varepsilon,j}\|_{\mathbf{B}} = 0$.

But $\|u_{\varepsilon,j} - u_\varepsilon\|_{N^{1,\mathbf{B}}(X)} \leq \|u_{\varepsilon,j} - u_\varepsilon\|_{\mathbf{B}} + \|\rho_{\varepsilon,j}\|_{\mathbf{B}}$ for all j. Then $\lim_{j\to\infty} \|u_{\varepsilon,j} - u_\varepsilon\|_{N^{1,\mathbf{B}}(X)} = 0$.

By Lemma 5, we have $\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u\|_{N^{1,\mathbf{B}}(X)} = 0$. Using a diagonal argument we can find, for every sequence $(u_{\varepsilon_k})_{k\geq 1}$ with $(\varepsilon_k)_{k\geq 1}$ decreasing to zero, an associated sequence of functions $u_k := u_{\varepsilon_k, j_k}$, $k \geq 1$ such that $\lim_{k\to\infty} \|u_{\varepsilon_k, j_k} - u\|_{N^{1,\mathbf{B}}(X)} = 0$. $\square$

Now, we can prove our main result, which is a consequence and an extension of Proposition 1.

THEOREM 1. *Let X be proper and let $\mathbf{B}$ be a Banach function space over X, that has absolutely continuous norm and has property (C). Assume that locally Lipschitz functions are dense in $N^{1,\mathbf{B}}(X)$ and that all functions in $N^{1,\mathbf{B}}(X)$ are $\mathbf{B}$-quasicontinuous. Then, for every open set $\Omega \subset X$, the closure of $Lip_C(\Omega)$ in $N^{1,\mathbf{B}}(X)$ is $N_0^{1,\mathbf{B}}(\Omega)$.*

Proof. If $\Omega = X$, then $N_0^{1,\mathbf{B}}(\Omega) = N^{1,\mathbf{B}}(X)$ and the proof will be completed by Proposition 1. Assume that $\Omega \neq X$. We will identify every function defined on Ω with its extension by zero. Let $u \in N_0^{1,\mathbf{B}}(\Omega)$. Fix $\varepsilon > 0$. We prove that there exists $v \in Lip_C(\Omega)$ such that $\|u - v\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$.

Using Proposition 2, we can choose $w \in N^{1,\mathbf{B}}(X)$ with a bounded support $supp\, w \subset \Omega$, such that $\|u - w\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$. Note that $supp\, w$ is compact, since it is closed and bounded in the proper metric space X.

Let $\delta := \frac{1}{2}\min\{dist\,(supp\, w, X \setminus \Omega), 1\}$. There exists a Lipschitz cut-off function $\eta \in Lip_C(\Omega)$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $supp\, w$, having an upper gradient $g_\eta \leq 1/\delta$.

By Proposition 1, there exists $f(\varepsilon) \in Lip_C(X)$ such that $\|w - f(\varepsilon)\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$. Note that $f(\varepsilon)\eta \in Lip_C(\Omega)$. Since $|f(\varepsilon) - f(\varepsilon)\eta| \leq |f(\varepsilon)|\chi_{X\setminus supp\, w} \leq |w - f(\varepsilon)|$ on X, we have $\|f(\varepsilon) - f(\varepsilon)\eta\|_{\mathbf{B}} \leq \left\||f(\varepsilon)|\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}} \leq \|w - f(\varepsilon)\|_{\mathbf{B}} < \varepsilon$, by (P2).

Assume that $g_\varepsilon \in \mathbf{B}$ is an upper gradient of $f(\varepsilon)$. By Lemma 3, taking into account that $0 \leq \eta \leq 1$, and by Corollary 1, it follows that the function $\rho_\varepsilon := (|f(\varepsilon)|\, g_\eta + g_\varepsilon)\,\chi_{X\setminus supp\, w}$ is a generalized $\mathbf{B}$-weak upper gradient of $f(\varepsilon) - f(\varepsilon)\eta$. We have $\|\rho_\varepsilon\|_{\mathbf{B}} \leq \frac{1}{\delta}\left\||f(\varepsilon)|\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}} + \left\|g_\varepsilon\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}}$. Then

$$\text{(4.1)}\quad \|f(\varepsilon) - f(\varepsilon)\eta\|_{N^{1,\mathbf{B}}(X)} \leq \|f(\varepsilon) - f(\varepsilon)\eta\|_{\mathbf{B}} + \|\rho_\varepsilon\|_{\mathbf{B}} < (\delta + 1)\,\varepsilon/\delta + \left\|g_\varepsilon\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}}.$$

We show that we can choose g_ε such that $\left\|g_\varepsilon\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}} < 2\varepsilon$. Since $\|w - f(\varepsilon)\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$, we can choose by ([17], Proposition 2) an upper gradient $h_\varepsilon \in \mathbf{B}$ of $w - f(\varepsilon)$ such that $\|h_\varepsilon\|_{\mathbf{B}} < 2\varepsilon$. Let $g \in \mathbf{B}$ be any upper gradient of $f(\varepsilon)$. Since $f(\varepsilon) = f(\varepsilon) - w$ on $X \setminus supp\, w$, it follows by Lemma 2 that $g_\varepsilon := h_\varepsilon\chi_{X\setminus supp\, w} + g\chi_{supp\, w}$ is a generalized $\mathbf{B}$-weak upper gradient of $f(\varepsilon)$. Then $g_\varepsilon\chi_{X\setminus supp\, w} = h_\varepsilon\chi_{X\setminus supp\, w}$, hence $\left\|g_\varepsilon\chi_{X\setminus supp\, w}\right\|_{\mathbf{B}} \leq \|h_\varepsilon\|_{\mathbf{B}} < 2\varepsilon$, q.e.d.

With the above choice of g_ε, (4.1) implies $\|f(\varepsilon) - f(\varepsilon)\eta\|_{N^{1,\mathbf{B}}(X)} < \frac{3\delta+1}{\delta}\varepsilon$, hence $\|w - f(\varepsilon)\eta\|_{N^{1,\mathbf{B}}(X)} < \frac{4\delta+1}{\delta}\varepsilon$. For $\psi := f\left(\frac{\varepsilon\delta}{8\delta+2}\right)$ it follows that

$\|w-\psi\eta\|_{N^{1,\mathbf{B}}(X)}<\frac{\varepsilon}{2}$. Then the function $v:=\psi\eta$ satisfies all requirements. □

If continuous functions are dense in $N^{1,\mathbf{B}}(X)$, then the following properties are equivalent ([17], Theorem 3):

(1) Every function in $N^{1,\mathbf{B}}(X)$ is $\mathbf{B}$-quasicontinuous;

(2) $Cap_{\mathbf{B}}(F)=\inf\{Cap_{\mathbf{B}}(G): G \text{ open}, F\subset G\subset X\}$ for every $F\subset X$.

The $\mathbf{B}$-capacity satisfies the outer regularity condition (2) provided that X is proper, $\mathbf{B}$ has absolutely continuous norm, $\mathbf{B}$ has the Vitali-Carathéodory property and continuous functions are dense in $N^{1,\mathbf{B}}(X)$ [19]. We say that $\mathbf{B}$ has the *Vitali-Carathéodory property* if for every $f\in\mathbf{B}$ there is a semicontinuous function $g\in\mathbf{B}$ such that $f\le g$. If $\Psi:[0,\infty)\to[0,\infty)$ is a Young function, strictly increasing and doubling, then $L^{\Psi}(X)$ has the Vitali-Carathéodory property.

In view of the above discussion, Theorem 1 implies the following

COROLLARY 2. *Let X be proper and let $\mathbf{B}$ be a Banach function space over X, that has absolutely continuous norm, has property (C) and has Vitali-Carathéodory property. Assume that locally Lipschitz functions are dense in $N^{1,\mathbf{B}}(X)$. Then, for every open set $\Omega\subset X$, the closure of $Lip_C(\Omega)$ in $N^{1,\mathbf{B}}(X)$ is $N_0^{1,\mathbf{B}}(\Omega)$.*

In the case when $\mathbf{B}=L^{\Psi}(X)$ is an Orlicz space, where $\Psi:[0,\infty)\to[0,\infty)$ is a doubling N-function, Corollary 2 gives the main result from [18], that extends Theorem 5.8 from [4].

5. APPLICATIONS OF THE MAIN DENSITY RESULT

We recall an analogue of Mazur's lemma for function-weak upper gradient pairs in $\mathbf{B}\times\mathbf{B}$.

LEMMA 9 ([17], Theorem 1). *Let $g_j\in\mathbf{B}$ be a $\mathbf{B}$-weak upper gradient of $u_j\in\mathbf{B}$ in X, for all $j\ge1$. Assume that $u_j\to u$ and $g_j\to g$ weakly in $\mathbf{B}$, for some $u,g\in\mathbf{B}$. Then there are some sequences $(U_j)_{j\ge1}$ and $(G_j)_{j\ge1}$ of convex combinations*

$$U_j=\sum_{k=j}^{n_j}\lambda_{kj}u_k,\ G_j=\sum_{k=j}^{n_j}\lambda_{kj}g_k,$$

where $\lambda_{kj}\ge0$, $\sum_{k=j}^{n_j}\lambda_{kj}=1$, such that $U_j\to u$ and $G_j\to g$ in $\mathbf{B}$. In addition, a representative of g is a $\mathbf{B}$-weak upper gradient of u in X.

PROPOSITION 3. *Assume that $Lip_C(\Omega)$ is dense in $N_0^{1,\mathbf{B}}(\Omega)$, where $\Omega\subset X$ is an open set. Let $u\in N_0^{1,\mathbf{B}}(\Omega)$ and $g\in\mathbf{B}$ be a (generalized) $\mathbf{B}$-weak upper gradient of u in Ω. Then $\widetilde{g}$ is a (generalized) $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X.*

Proof. Note that $\widetilde{g} \in \mathbf{B}$.

First assume that g is a $\mathbf{B}$-weak upper gradient of u in Ω. Since $Lip_C(\Omega)$ is dense in $N_0^{1,\mathbf{B}}(\Omega)$, there exists a sequence $(u_k)_{k\geq 1}$ in $Lip_C(\Omega)$ such that $\lim_{k\to\infty} \|\widetilde{u_k} - \widetilde{u}\|_{N^{1,\mathbf{B}}(X)} = 0$. Then we can find, for each $k \geq 1$, an upper gradient h_k of $\widetilde{u_k} - \widetilde{u}$ in X, such that $\|h_k\|_{\mathbf{B}} < \|\widetilde{u_k} - \widetilde{u}\|_{\mathbf{B}} + 1/k$. Note that $g_k := g + h_k|_\Omega$ is a $\mathbf{B}$-weak upper gradient of u_k in Ω. By Lemma 6 b), $\widetilde{g_k}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u_k}$ in X. Since $\widetilde{g_k} \leq \widetilde{g} + h_k$ we see that $\rho_k = \widetilde{g} + h_k$ is also a $\mathbf{B}$-weak upper gradient of $\widetilde{u_k}$ in X.

Since $\lim_{k\to\infty} \|\widetilde{u_k} - \widetilde{u}\|_{\mathbf{B}} = 0$ and $\lim_{k\to\infty} \|\rho_k - \widetilde{g}\|_{\mathbf{B}} = 0$, it follows by Lemma 9 that a representative of $\widetilde{g}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X. Then $\widetilde{g}$ is a generalized $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X, by ([14], Lemma 6 a), but $\widetilde{g}$ is Borel measurable, hence $\widetilde{g}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X.

Now, assume that $g \in \mathbf{B}$ is a generalized $\mathbf{B}$-weak upper gradient of u in Ω. By ([14], Lemma 6 b), there exists a $\mathbf{B}$-weak upper gradient h of u in Ω, such that $h = g$ μ-a.e. in Ω. Then $h \in \mathbf{B}$. By the above argument, a representative of $\widetilde{h}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X. Since $\widetilde{g} = \widetilde{h}$ μ-a.e. in X, it follows by ([14], Lemma 6 a) that $\widetilde{g}$ is a generalized $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X. □

COROLLARY 3. *Let X be proper and let $\mathbf{B}$ be a Banach function space over X, that has absolutely continuous norm and has property (C). Assume that locally Lipschitz functions are dense in $N^{1,\mathbf{B}}(X)$ and that all functions in $N^{1,\mathbf{B}}(X)$ are $\mathbf{B}$-quasicontinuous. If $\Omega \subset X$ is an open set, $u \in N_0^{1,\mathbf{B}}(\Omega)$ and $g \in \mathbf{B}$ is a (generalized) $\mathbf{B}$-weak upper gradient of u in Ω, then $\widetilde{g}$ is a (generalized) $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X.*

We will say that a μ-measurable function $u : \Omega \to \overline{\mathbb{R}}$ on the open set $\Omega \subset X$ belongs to $\mathbf{B}$ if its extension by zero $\widetilde{u} : X \to \mathbb{R}$ belongs to $\mathbf{B}$.

PROPOSITION 4. *Assume that $Lip_C(\Omega)$ is dense in $N_0^{1,\mathbf{B}}(\Omega)$, where $\Omega \subset X$ is an open set. For $k \geq 1$, let $u_k \in N_0^{1,\mathbf{B}}(\Omega)$ and g_k be a $\mathbf{B}$-weak upper gradient of u_k in Ω. Assume that there exist $u : \Omega \to \mathbb{R}$ and $g : \Omega \to [0,\infty]$ such that $\widetilde{u_k} \to \widetilde{u}$ and $\widetilde{g_k} \to \widetilde{g}$ weakly in $\mathbf{B}$. Then $u \in N_0^{1,\mathbf{B}}(\Omega)$ and g is a generalized $\mathbf{B}$-weak upper gradient of u in Ω.*

Proof. By Proposition 3, $\widetilde{g_k}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u_k}$ in X, for $k \geq 1$. Since $\widetilde{u_k} \to \widetilde{u}$ and $\widetilde{g_k} \to \widetilde{g}$ weakly in $\mathbf{B}$, it follows by Lemma 9 that a representative of $\widetilde{g}$ is a $\mathbf{B}$-weak upper gradient of $\widetilde{u}$ in X. Since $\widetilde{u} \in \mathbf{B}$ has the generalized $\mathbf{B}$-weak upper gradient $\widetilde{g} \in \mathbf{B}$, we have $u \in N_0^{1,\mathbf{B}}(\Omega)$. Obviously, g is a generalized $\mathbf{B}$-weak upper gradient of u in Ω. □

Given an open set $\Omega \subset X$, we say that $u \in N_{loc}^{1,\mathbf{B}}(\Omega)$ if $u \in N_{loc}^{1,\mathbf{B}}(\Omega')$ for every open set Ω' that is compactly contained in Ω.

Using Theorem 1 we extend Theorem 5.10 from [4], replacing Newtonian spaces based on Lebesgue spaces with Newtonian spaces based on more general Banach function spaces.

THEOREM 2. *Let X be proper and let $\mathbf{B}$ be a Banach function space over X, that has absolutely continuous norm and has property (C). Assume that locally Lipschitz functions are dense in $N^{1,\mathbf{B}}(X)$ and that all functions in $N^{1,\mathbf{B}}(X)$ are $\mathbf{B}$-quasicontinuous. If $\Omega \subset X$ is open, $u \in N^{1,\mathbf{B}}_{loc}(\Omega)$ and $\varepsilon > 0$, then there exists a locally Lipschitz function $v : \Omega \to \mathbb{R}$ such that $u - v \in N^{1,\mathbf{B}}(\Omega)$ and $\|u - v\|_{N^{1,\mathbf{B}}(\Omega)} < \varepsilon$.*

Proof. We follow Björn, Björn and Shanmugalingam [4]. Let $\varepsilon > 0$. Any open set in a proper metric space possess an exhaustion by compact sets. Then there are open sets $\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega_k \Subset \Omega_{k+1} \Subset \dots \Subset \Omega$ such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. For each $k \geq 1$ we choose $\eta_k \in Lip_C(\Omega_{k+1})$ so that $\eta_k = 1$ on Ω_k and $0 \leq \eta_k \leq 1$ everywhere. Define inductively a sequence of functions $(u_k)_{k \geq 1}$ such that $u - \sum_{i=1}^{k} u_i = u(1-\eta_1)\dots(1-\eta_k)$ for $k \geq 1$, namely $u_1 := u\eta_1$ and $u_k := u(1-\eta_1)\dots(1-\eta_{k-1})\eta_k$ for $k \geq 2$. We have $u_k \in N_0^{1,\mathbf{B}}\left(\Omega_{k+1} \setminus \overline{\Omega_{k-1}}\right)$ for $k \geq 1$, where $\Omega_0 := \emptyset$. Indeed, u_k vanishes in $(X \setminus \Omega_k) \cup \overline{\Omega_{k-1}}$ and $\widetilde{u_k} \in \mathbf{B}$. Moreover, we can find a generalized $\mathbf{B}$-weak upper gradient of $\widetilde{u_k}$ in X, as follows. Let g be an upper gradient of u in Ω_{k+2}, such that the extension by zero of g to X belongs to $\mathbf{B}$. Since $(1-\eta_1)\dots(1-\eta_{k-1})\eta_k$ is a L-Lipschitz function, for some $L > 0$, taking values in $[0,1]$ and vanishing in $\Omega_{k+1} \setminus \overline{\Omega_{k-1}}$, it follows by Lemma 3 and Corollary 1 that $(g + L|u|)\chi_{\Omega_{k+1} \setminus \overline{\Omega_{k-1}}}$ is a generalized $\mathbf{B}$-weak upper gradient of u_k in Ω_{k+2}. The extension by zero of $(g + L|u|)\chi_{\Omega_{k+1} \setminus \overline{\Omega_{k-1}}}$ to X is a generalized $\mathbf{B}$-weak upper gradient of $\widetilde{u_k}$ in X and belongs to $\mathbf{B}$.

For every $x \in \Omega$ there exists $k \geq 1$ such that $x \in \Omega_k$, hence $u(x) = \sum_{i=1}^{k} u_i(x)$. Then $u(x) = \sum_{k=1}^{\infty} u_k(x)$ for every $x \in X$.

By Theorem 1, there exists $v_k \in Lip_C\left(\Omega_{k+1} \setminus \overline{\Omega_{k-1}}\right)$ such that $\|u_k - v_k\|_{N^{1,\mathbf{B}}(X)} < \varepsilon 2^{-k} 3^{-1}$. For each $x \in \Omega$ the sum $\sum_{k=1}^{\infty} v_k(x)$ has at most three nonzero terms. Let $v(x) = \sum_{k=1}^{\infty} v_k(x)$, $x \in X$. Then v is locally Lipschitz in Ω.

Since $\sum_{k=1}^{\infty}\|u_k-v_k\|_{\mathbf{B}}<\infty$, it follows by ([3], Theorem I.1.6) that the series $\sum_{k=1}^{\infty}(u_k-v_k)$ converges in $\mathbf{B}$ and its sum $w\in\mathbf{B}$ satisfies $\|w\|_{\mathbf{B}}\le\sum_{k=1}^{\infty}\|u_k-v_k\|_{\mathbf{B}}$ $<\varepsilon/3$. But every sequence that converges in $\mathbf{B}$ has a subsequence that converges μ-a.e. on X. Then $w=u-v$ μ-a.e. in Ω and choosing a representative of w we may assume that $w=u-v$ in Ω. Since $\|u_k-v_k\|_{N^{1,\mathbf{B}}(X)}<\varepsilon 2^{-k}3^{-1}$, we can choose an upper gradient g_k of u_k-v_k in X such that $\|g_k\|_{\mathbf{B}}<\varepsilon 2^{1-k}3^{-1}$. Then $\sum_{k=1}^{\infty}g_k$ converges in $\mathbf{B}$ to some $g\in\mathbf{B}$, with $\|g\|_{\mathbf{B}}<2\varepsilon/3$, by ([3], Theorem I.1.6). Since $\sum_{k=1}^{n}(u_k-v_k)\to w$ in $\mathbf{B}$ and $\sum_{k=1}^{n}g_k\to g$ in $\mathbf{B}$, as $n\to\infty$, it follows by Lemma 9 that g is a $\mathbf{B}$-weak upper gradient of w in X. Then $w\in N^{1,\mathbf{B}}(X)$ and $\|w\|_{N^{1,\mathbf{B}}(X)}\le\|w\|_{\mathbf{B}}+\|g\|_{\mathbf{B}}<\varepsilon$, hence $u-v\in N^{1,\mathbf{B}}(\Omega)$ and $\|u-v\|_{N^{1,\mathbf{B}}(\Omega)}<\varepsilon$. □

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"Vasile Alecsandri" University of Bacău,
Department of Mathematics and Informatics,
Calea Mărăşeşti,
Bacău 600115, Romania
mmocanu@ub.ro