ITERATED INTEGRATED TAIL WITH PERIODIC HAZARD RATE

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Let F be an absolutely continuous probability measure on $[0,\infty)$ having finite moments and the density f. Let $\underline{F}$ be its tail, defined by $\underline{F}(x) = F((x,\infty))$. Consider the sequence of probability distributions $(F_n)_n$ constructed by the recurrence $F_0 = F$, $F_{n+1} = (F_n)_I$, where F_I is the probability distribution on $[0,\infty)$ having the tail $(\underline{F_I})(x) = \int_x^\infty \underline{F}(y)\mathrm{d}y \,/\, \int_0^\infty \underline{F}(y)\mathrm{d}y$. The main result in [9] was Theorem 3.5: suppose that the hazard rate of F, defined by $\lambda(x) = f(x)/\underline{F}(x)$ has a limit $\lambda_0 := \lambda(\infty) \in (0,\infty)$ as $x \to \infty$. Then F_n converges weakly to $\mathrm{Exp}(\lambda_0)$. In this paper we conjecture that the same result holds if the Cesaro limit

$$\lim_{x\to\infty} \frac{\int_0^\infty \lambda(y)\mathrm{d}y}{x} = \lambda_0$$

does exist. We prove the conjecture in the very particular case when λ is periodic.

AMS 2010 Subject Classification: 60A10, 60K60.

Key words: integrated tail, Markovian operator, Dobrushin coefficient.

1. DEFINITIONS AND STATEMENT OF THE PROBLEM

Let $(\Omega,\mathcal{K},P)$ be a probability space and $\mathcal{L} = \cap L_+^p(\Omega,\mathcal{K},P)$. So, $X \in \mathcal{L}$ iff $X \geq 0$ (a.s.) and $\mathrm{E}X^p < \infty$ for every $1 \leq p < \infty$. Let $\mathcal{M}$ be the set of the distributions of the random variables $X \in \mathcal{L}$. Thus $F \in \mathcal{M}$ iff $F([0,\infty)) = 1$ and $\int x^p \mathrm{d}F(x) < \infty$ $\forall 1 \leq p < \infty$. We shall denote by $F(x)$ the distribution function of F, by $\underline{F}(x)$ its right tail and by $\mu_n(F)$ its moments. Precisely, $F(x)$ will stand for $F([0,x])$, $\underline{F}(x)$ for $F((x,\infty))$ and $\mu_n(F)$ for $\int x^n \mathrm{d}F(x)$. If F is absolutely continuous, its density will be denoted by f_F and its hazard rate by $\lambda_F := \frac{f_F}{\underline{F}}$. The Lebesgue measure on the real line will be denoted by μ. The exponential distribution of parameter λ will be denoted by $\mathrm{Exp}(\lambda)$.

In renewal and ruin theories the following distribution is of interest: it is called *the integrated tail* (see for instance [2], [4] or [6]). Its tail is defined by

$$\underline{F_I}(x) = \frac{\int_x^\infty \underline{F}(y)\mathrm{d}y}{\int_0^\infty \underline{F}(y)\mathrm{d}y}. \tag{1.1}$$

We intend to study the mapping $U : \mathcal{M} \to \mathcal{M}$ defined by $U(F) = F_I$ and the sequence defined by

$$F_0 = F, \quad F_{n+1} = (F_n)_I. \tag{1.2}$$

Ignatov [5] conjectured in 2005 that this sequence always converges to some exponential distribution. Now we know that this is not true (counter-example from [7]). However, Theorem 3.5 from [9] states that if λ has a finite and non-zero limit at infinity, λ_0, then the sequence $(F_n)_n$ weakly converges to the exponential distribution $\mathrm{Exp}(\lambda_0)$. Similar results were obtained in [1]. Thus the tail of the limit is $\exp(-\lambda_0 x)$. Moreover, it was proved in [1] and [8] that in this case the sequence $(\frac{\mu_{n+1}}{(n+1)\mu_n})_n$ converges to $1/\lambda_0$.

However, the particular examples we have studied make us believe that Ignatov's conjecture holds if the Cesaro limit $\lim\limits_{x\to\infty} \frac{\int_0^x \lambda(y)\mathrm{d}y}{x} = \lambda_0$ does exist. A particular case when this limit does exist is when λ is periodic.

In this paper we shall suppose that this is indeed the case: $\lambda(x+T) = \lambda(x), \forall x \geq 0$ for some period T.

The main result is:

Theorem 1.1. *If λ is periodic and has the period $T > 0$, then $F_n \Rightarrow \mathrm{Exp}\Big(\frac{\int_0^T \lambda(x)\mathrm{d}x}{T}\Big)$. More than that, the hazard rates λ_{F_n} converge uniformly to $\frac{\int_0^T \lambda(x)\mathrm{d}x}{T}$.*

As $\mu_1(F_n) = \frac{\mu_{n+1}}{(n+1)\mu_n}$ (see [8]) we have the following by product:

Corollary 1.2. *If F has a periodic hazard rate, then the sequence $\Big(\frac{\mu_{n+1}}{(n+1)\mu_n}\Big)_n$ is convergent and its limit is $\frac{T}{\int_0^T \lambda(x)\mathrm{d}x}$.*

2. PLAN OF THE PROOF

Proposition 2.1. *Let $\lambda : [0,\infty) \to [0,\infty)$ be a measurable function having the period $T > 0$.*

Let $\Lambda(x) = \int_0^x \lambda(y)\mathrm{d}y$ and $\underline{F}(x) = e^{-\Lambda(x)}$. Then

$$\underline{F}(x) = q^{\left[\frac{x}{T}\right]} h\left(T\left\{\tfrac{x}{T}\right\}\right) \text{ with } q = e^{-\Lambda(T)} \text{ and } h(t) = e^{-\Lambda(t)} \tag{2.1}$$

and

$$\underline{F}_I(x) = q^{\left[\frac{x}{T}\right]}\left(1 - p\frac{H(T\left\{\frac{x}{T}\right\})}{H(T)}\right) \text{ with } p = 1-q \text{ and } H(t) = \int_0^t h(x)\mathrm{d}x. \tag{2.2}$$

Moreover, the hazard rate of F_I *is*

$$\lambda_I(x) = \frac{ph(T\left\{\frac{x}{T}\right\})}{H(T) - pH(\left\{\frac{x}{T}\right\})} \tag{2.3}$$

($[y]$ *and* $\{y\}$ *are the integer part of* y *and the decimal part of* y*).*

Proof. Let $x \geq 0$. Let $k \geq 0$ and $t \in [0, T)$ such that $x = kT + t$. Then

$$\Lambda(x) = \int_0^x \lambda(y)\mathrm{d}y = \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} \lambda(y)\mathrm{d}y + \int_{kT}^{kT+t} \lambda(y)\mathrm{d}y$$
$$= k\int_0^T \lambda(y)\mathrm{d}y + \int_0^t \lambda(y)\mathrm{d}y = kI + \int_0^t \lambda(y)\mathrm{d}y = kI + \Lambda(t).$$

It means that $\underline{F}(x) = e^{-kI-\Lambda(t)} = q^k h(t)$. Thus (2.1) is proved.

The integral is

$$\int_x^\infty \underline{F}(x)\mathrm{d}x = \int_{kT}^\infty \underline{F}(x)\mathrm{d}x - \int_{kT}^x \underline{F}(x)\mathrm{d}x$$
$$= \sum_{j=k}^{\infty} \int_{jT}^{(j+1)T} \underline{F}(x)\mathrm{d}x - \int_{kT}^{kT+t} \underline{F}(x)\mathrm{d}x.$$

Let $H(t) = \int_0^t h(x)\mathrm{d}x$. By (2.1) we have $\int_x^\infty \underline{F}(x)\mathrm{d}x = \sum\limits_{j=k}^{\infty} q^j H(T) - q^k H(t) = \frac{q^k}{1-q}(H(T) - (1-q)H(t))$. For $x = 0$ we get $\int_x^\infty \underline{F}(x)\mathrm{d}x = \frac{H(T)}{1-q}$.

It means that $\underline{F}_I(x) = q^k \frac{H(T)-(1-q)H(t)}{H(T)}$, which proves (2.2). As about (2.3), we use the relation $\lambda_I(x) = \frac{\underline{F}(x)}{\int_x^\infty \underline{F}(y)\mathrm{d}y}$. □

Remark. Notice that $h(0) = 1$, $h(T-0) = q$, $\lambda_I(0) = \lambda_I(T-0) = \frac{p}{H(T))}$. Therefore, λ_I is periodic, too, and continuous.

Corollary 2.2. *Let* $(F_n)_n$ *be the sequence given by the recurrence* (1.2). *Then*

$$\underline{F}(x) = q^{\left[\frac{x}{T}\right]} h_n(T\left\{\frac{x}{T}\right\}) \text{ with } q = e^{-\Lambda(T)} \text{ and } h(t) = e^{-\Lambda(t)}. \tag{2.4}$$

The functions $h_n : [0, T) \to \mathcal{R}_+$ *have the properties*

$$h_n(0) = 1, \quad h_n(T) = q, \quad h_{n+1}(x) = 1 - p\frac{\int_0^x h_n(y)\mathrm{d}y}{\int_0^T h_n(y)\mathrm{d}y}. \tag{2.5}$$

Now let us look at the operator $B : Y \to Y$ defined by

$$(Bh)(x) = 1 - p\frac{\int_0^x h(y)\mathrm{d}y}{\int_0^T h(y)\mathrm{d}y}. \tag{2.6}$$

Here $Y = \{h : [0,T]) \to [q,1] \mid h$ is differentiable, decreasing, $h(0) = 1$, $h(T) = 1 - p := q\}$. Taking this into account, we can write

COROLLARY 2.3. *Let* $(F_n)_n$ *be the sequence given by the recurrence* (1.2). *Then*

$$\underline{F}(x) = q^{\left[\frac{x}{T}\right]}(B^n h_n)(T\left\{\tfrac{x}{T}\right\}) \text{ with } q = h(T). \tag{2.7}$$

If we could prove that the sequence $(B^n h)_n$ is convergent to some limit $h*$, then $\underline{F}_n$ would converge to $F^*(x) = \underline{F_n}(x) = q^{\left[\frac{x}{T}\right]}h^*(T\left\{\frac{x}{T}\right\})$, hence the sequence $(F_n)_n$ would have a weak limit. Actually we shall prove more, namely

THEOREM 2.4. $(B^n h)(x)$ *converges* ***uniformly*** *to* $q^{\frac{x}{T}}$ *as* $n \to \infty$ *for any* $h \in Y$ *and* $x \in [0,T]$.

Suppose that Theorem 2.4 holds. Then $\lim\limits_{n\to\infty} \underline{F}_n(x) = q^{\left[\frac{x}{T}\right]}q^{\left\{\frac{x}{T}\right\}} = q^{\frac{x}{T}} = e^{-x\frac{\int_0^T \lambda(y)\mathrm{d}y}{T}}$. Or, otherwise written, $F_n \Rightarrow \exp(\lambda^*)$ with $\lambda^* = \frac{\int_0^T \lambda(y)\mathrm{d}y}{T}$. That will end the proof of the Theorem 1. $\square$

Example. The geometric distribution. Let

$$F = \text{Geometric}(p) := \sum_{k=1}^{\infty} pq^{k-1}\delta_k,$$

with $p, q > 0$, $p + q = 1$.

It is true that F is not absolutely continuous, hence λ_F has no meaning. However, after the first iteration $F_1 = F_I$ has the density $\underline{F}(x)/\mu_1(F) = pq^{[x]}$ and the hazard rate $\lambda_1(x) = \frac{pq^{[x]}}{\int_x^\infty pq^{[y]}\mathrm{d}y} = \frac{p}{1-p\{x\}}$. In this case the period $T = 1$ and $\lambda^* = \frac{\int_0^T \lambda(y)\mathrm{d}y}{T} = \int_0^1 \frac{p}{1-px}\mathrm{d}x = -\ln q$. Thus $F_n \Rightarrow \exp(-\ln q)$. The hazard rates λ_{F_n} converge to $-\ln q$ *uniformly*. It follows that the moments of F_n converge to the moments of the exponential distribution. Therefore, $\lim\limits_{n\to\infty} \frac{\mu_{n+1}}{(n+1)\mu_n} = -\frac{1}{\log(1-p)}$.

3. PROOF OF THEOREM 2.4

The proof relies on a result about Markovian kernels, which may be of some interest itself. Firstly we give some *definitions* (see for instance [3] or [7]) for the readers which are not familiar with these objects.

Let $(E, \mathcal{E})$ be a measurable space. A kernel from E to E is a family of finite measures on $E, Q := (Q_x)_{x\in E}$, having the property that the mapping $x \mapsto Q_x(A)$ is measurable for any $A \in \mathcal{E}$. If $Q_x(E) \le 1\ \forall x \in E$, then Q is called a *submarkovian kernel.* If $Q_x(E) = 1\ \forall x \in E$, Q is called a *Markovian kernel.*

To any kernel one can attach two operators: one which is acting on bounded measurable functions and the other one acting on finite signed measures. The first one is the operator V_Q defined by $V_Q(f)(x) = V_Q(f)(x) = \int f(y)Q_x(\mathrm{d}y)$. The standard notation for it is Qf : $Qf(x)$ actually means $V_Q(f)(x)$. Notice that Q is Markovian iff $Q1 = 1$.

The other one is the operator $U_Q(\nu)(A) = \int Q_x(A)\mathrm{d}\nu(x)$. For it, the standard notation is νQ: thus $\nu Q(A)$ stands for $U_Q(\nu)(A)$. As a particular case, $Q_x = \delta_x Q$ where δ_x is Dirac's point measure. The compositions $Q(Qf))$, $Q(Q(Qf))), \ldots$ are denoted by $Q^n f$ and the compositions $(\mu Q)Q, ((\mu Q)Q)Q$, $\ldots$ are denoted by μQ^n. Thus Q^n_x stands for $\delta_x Q^n$. Notice again that Q is Markovian iff $\delta_x Q$ is a probability for every $x \in E$.

The notation is motivated by a particular case: suppose that all Q_x have densities $K(x, \cdot)$ with respect to some σ-finite measure μ. Then $V_Q(f)(x) = \int K(x,y)f(y)\mathrm{d}\mu(y)$. It is as if we multiply the "matrix" K with the column "vector" f. Moreover, if ν is a signed measure having the density g with respect to the same σ-finite measure μ, then

$$U_Q(\nu)(A) = \int \left(\int g(x)K(x,y)\mathrm{d}\mu(x) \right) 1_A(y)\mathrm{d}\mu(y),$$

the new density is $\int g(x)K(x,y)\mathrm{d}\mu(x)$. It is as if we multiply the "row vector" g with the same "matrix" K. In this particular case we also could denote the first operator as $f \mapsto K_\mu f$ and the other one as $\nu \mapsto \nu K_\mu$. Some authors call this function K to be the kernel and say that the (sub)markovian operator Q is *given by the kernel* K.

The Dobrushin coefficient. Suppose now that P is a Markovian kernel. The Dobrushin coefficient of P is defined (see [3], p. 88) by

$$\overline{\alpha}(P) = \frac{1}{2} \left\{ \|P_x - P_y\| : x, y \in E \right\}. \tag{3.1}$$

Recall that the norm of a signed measure ν is its variation, $\|\nu\| = |\nu|(E) = \nu_+(E) + \nu_-(E)$. It is easy to see that $\bar{\alpha}(P) \leq 1$, $\bar{\alpha}(PQ) \leqslant \bar{\alpha}(P)\bar{\alpha}(Q)$ ($\bar{\alpha}$ is submultiplicative) hence $\bar{\alpha}(P^n) \leq \bar{\alpha}(P)^n$. Moreover, if ν is a signed measure, then it is known that $\|\nu P\| \leqslant \bar{\alpha}(P)\|\nu\| + (1 - \bar{\alpha}(P))|\nu(E)|$ ([3], p. 91); as a particular case, if $\nu(E) = 0$ (as it is the case when ν is the difference between two probabilities), then $\|\nu P\| \leq \bar{\alpha}(P)\|\nu\|$.

The power of this Dobrushin coefficient is given in the following result – it should be well known, but we lack a precise reference:

LEMMA 3.1. *Let P be a Markovian kernel. If $\bar{\alpha}(P) < 1$ then there exists an invariant probability π such that $\|(P_x)^n - \pi\| \leq (\bar{\alpha}(P))^n \to 0$ as $n \to \infty$. Thus P^n has always a limit P_∞ which is a constant Markovian kernel* : $(P_\infty)_x = \pi$, $\forall x \in E$.

Proof. The sequence $(P_x)^n$ is Cauchy: $\|(P^n)_x-(P^{n+k})_x\| = \|(\delta_x-\delta_x P^k)\cdot P^n\| \le \bar{\alpha}(P^n)\|\delta_x - \delta_x P^k\| \le 2(\bar{\alpha}(P))^n < \varepsilon$ if n is great enough. Thus there exists a probability π_x such that $(P^n)_x$ converges in norm to π_x. Let $x, y \in E$ be arbitrary. Then

$$\|\pi_x-\pi_y\|=\left\|\lim_{n\to\infty}\delta_x P^n-\lim_{n\to\infty}\delta_y P^n\right\|=\left\|\lim_{n\to\infty}(\delta_x-\delta_y)P^n\right\|=\lim_{n\to\infty}\|(\delta_x-\delta_y)P^n\|.$$

As $\|(\delta_x - \delta_y)P^n\| \le 2\bar{\alpha}(P)^n$ the last limit equals 0; it means that the probability π does not depend on x. On the other hand, $\pi P = \lim(\delta_x P_n)\cdot P = \lim \delta_x P^{n+1} = \lim \delta_x P^n = \pi$.

Another fact: if the Markovian kernel has the form

$$Pf(x) = \int K(x,y)f(y)\mathrm{d}\mu(y),$$

then its Dobrushin coefficient is easier to compute: it is equal to

$$\begin{aligned}(3.2)\qquad \overline{\alpha}(P) &= \frac{1}{2}\sup\left\{\int |K(x,z)-K(y,z)|\,\mathrm{d}\mu(z) : x,y\in E\right\}\\ &:= \frac{1}{2}\sup\left\{\|K(x,\cdot)-K(y,\cdot)\|_1 : x,y\in E\right\}.\end{aligned}$$

Next fact provides a simple criterion to decide if $\bar{\alpha}(P) < 1$.

LEMMA 3.2. *Suppose that $(E,\mathcal{E})$ is a measurable space and μ is a probability on it.*

Let $K : E\times E \to [a,\infty)$, $a > 0$, be measurable bounded such that $\int K(x,y)\mathrm{d}\mu(y) = 1\ \forall x$ and

$$(3.3)\qquad Pf = \int K(x,y)f(y)\mathrm{d}\mu(y).$$

Then $\bar{\alpha}(P) \le 1 - a\mu(E) < 1$.

Proof. We know that $\bar{\alpha}(P) = \frac{1}{2}\sup\{\|K(x,\cdot)-K(y,\cdot)\|_1 : x,y\in E\}$. But

$$\begin{aligned}\int |K(x,z)-K(y,z)|\,\mathrm{d}\mu(z) &= \int |(K(x,z)-a)-(K(y,z)-a)|\,\mathrm{d}\mu(z) \leqslant\\ &\leqslant \int |K(x,z)-a|\,\mathrm{d}\mu(z)+\int |K(y,z)-a|\,\mathrm{d}\mu(z) =\\ &= \int (K(x,z)-a)d\mu(z)+\int(K(y,z)-a)\mathrm{d}\mu(z) = 2(1-a\mu(E)).\end{aligned}$$

Now we give the main result of this section.

THEOREM 3.3. *Let $(E,\mathcal{E})$ be a measurable space and μ be a finite measure on it.*

Let $K : E \times E \to [a, \infty)$, $a > 0$ be a measurable bounded function and

$$K_\mu f = \int K(x, y) f(y) \mathrm{d}\mu(y). \tag{3.4}$$

Let also B be the mapping defined for bounded measurable functions as

$$(Bf)(x) = \frac{(K_\mu f)(x)}{(K_\mu f)(y)}, \tag{3.5}$$

where $y \in E$ is fixed.

Suppose that there exists a bounded measurable function $h : E \to (0, \infty)$ and a positive constant $c > 0$ such that

$$cK_\mu h = h. \tag{3.6}$$

Then the sequence $(B^n f)_n$ converges to $f_\infty(x) = \frac{h(x)}{h(y)}$. The limit does not depend on f.

Proof. Consider the new kernel P defined by $(Pf)(x) = \frac{c}{h(x)}(K_\mu u)(x)$. As $(P1)(x) = \frac{c}{h(x)}(K_\mu(h))(x) = 1$, P is now a Markovian kernel. Its powers are given by

$$P^n f = \frac{c^n}{h}(K_\mu)^n(fh). \tag{3.7}$$

We can write it as $Pf(x) = \int \tilde{K}(x, y) f(y) \mathrm{d}\mu(y)$ with $\tilde{K}(x, y) = c\frac{h(y)}{h(x)}K(x, y)$. As h is bounded away from 0 ($h(x) = c \int K(x, y)h(y)\mathrm{d}\mu(y) \geq c\int ah(y)\mathrm{d}\mu(y) = ca \int h\mathrm{d}\mu > 0$) and bounded, the ratio $y \mapsto \frac{h(y)}{h(x)}$ is bounded away from 0, too by some $b > 0$. It follows that $\tilde{K}(x, y) \geq abc > 0$. According to Lemma 3.2, $\bar{\alpha}(P) < 1$. By Lemma 1, P^n converges in norm to a stationary measure π. Thus $P^n f \to \int f \mathrm{d}\pi$ for any bounded measurable f. By (3.7) that means, explicitly, that

$$\frac{c^n}{h(x)}(K_\mu)^n(fh)(x) \overset{n\to\infty}{\longrightarrow} \int f \mathrm{d}\mu.$$

To end the proof we write

$$(B^n f)(x) = \frac{(K_\mu)^n(f)(x)}{(K_\mu)^n(f)(y)} = \frac{c^n(K_\mu)^n(f)(x)/h(x)}{c^n(K_\mu)^n(f)(x)/h(y)} \cdot \frac{h(x)}{h(y)}$$
$$\overset{n\to\infty}{\longrightarrow} \frac{h(x)}{h(y)} \frac{\int f/h \mathrm{d}\pi}{\int f/h \mathrm{d}\pi} = \frac{h(x)}{h(y)}. \quad \square$$

We claim that Theorem 2.4 is a simple consequence of Theorem 3.3. We shall generalize a bit the operator defined by (2.6) replacing the uniform distribution on the interval $[0, T]$ by a continuous probability measure on $[0, \infty)$:

PROPOSITION 3.4. *Let F be a continuous distribution on $[0,\infty)$ and let $p \le 1$, $q = 1-p \ge 0$. Let B be the mapping $Bf(x) = 1 - p\frac{\int_0^x f(y)\mathrm{d}F(y)}{\int_0^\infty f(y)\mathrm{d}F(y)}$, defined for measurable bounded positive functions. Then $B^n f(x)$ converges to* to $q^{F(x)}$. *As a particular case, if $F = \mathrm{U}(0,T)$ is the uniform distribution on $[0,T]$, then the limit is $q^{\frac{x}{T}}$, with the convention that $0^0 = 1$.*

Moreover, if $q < 1$, the convergence is uniform.

Proof. Our operator will be

$$Bf(x) = \frac{q\int_0^x f(y)\mathrm{d}F(y) + \int_x^\infty f(y)\mathrm{d}F(y)}{\int_0^\infty f(y)\mathrm{d}F(y)}.$$

Now, the kernel K from (3.4) is

$$K(x,y) = q1_{[0,x]}(y) + 1_{[x,\infty]}(y). \tag{3.8}$$

As $K(x,y) \ge a := \min(1,q)$ the first condition from Theorem 3.3 is fulfilled if $q > 0$. The operator B can be written as $(Bf)(x) = \frac{(K_\mu f)(x)}{(K_\mu f)(0)}$. The equation $Bh = h$ has the solution $h(x) = q^{F(x)}$. Indeed

$$\begin{aligned} K_\mu h(x) &= q\int_0^x q^{F(y)}\mathrm{d}F(y) + \int_x^\infty q^{F(y)}\mathrm{d}F(y) \\ &= q\frac{q^{F(x)}-1}{\ln q} + \frac{q - q^{F(x)}}{\ln q} = \frac{q-1}{\ln q}q^{F(x)}. \end{aligned}$$

Thus the constant c from Theorem 3.3, (3.6) is $c = \frac{\ln q}{q-1}$ and $h(x) = q^{F(x)}$. The limit of $(B^n f)(x)$ is $h(x)/h(0) = h(x)$.

The case $p = 1 \Leftrightarrow q = 0$ is special. This time the kernel $K(x,y) = 1_{[x,\infty]}(y)$ vanishes for $x > y$. We shall use the brute force to check that $B^n f \to 1_{\{0\}}$.

Indeed, now $Bf(x) = \frac{\int_x^\infty f(y)\mathrm{d}F(y)}{\int_0^T f(y)\mathrm{d}F(y)}$. We have to prove that $B^n f(x)$ converges to 1 if $x = 0$ or to 0 if $x > 0$. The first assertion is obvious. Let $x > 0$. Firstly, remark that Bf is always decreasing and $(Bf)(0) = 1$. Replacing if necessary f by Bf, one can always accept that f is continuous, non-increasing, non-negative and $f(0) = 1$. Next, the reader is invited to check by induction that

$$B^n f(x) = \frac{\int_x^\infty (F(y)-F(x))^{n-1} f(y)\mathrm{d}F(y)}{\int_0^T F(y)^{n-1} f(y)\mathrm{d}F(y)}. \tag{3.9}$$

Let $\theta = \inf\{x : f(x) > 0\}$. Then

$$B^n f(x) = \frac{\int_x^\theta (F(y)-F(x))^{n-1} f(y)\mathrm{d}F(y)}{\int_0^\theta F(y)^{n-1} f(y)\mathrm{d}F(y)}.$$

Let $0 < x < \theta$ be fixed and let $\theta_0 < \theta$ be such that $F(\theta) - F(x) < F(\theta_0)$. Such a θ_0 does exist since F is continuous and $F(x) > 0$. Then, using the fact that $0 \le f(x) \le 1$ and f is non increasing, we have

$$B^n f(x) \leqslant \frac{\int_x^\theta (F(y)-F(x))^{n-1}\mathrm{d}F(y)}{\int_0^\theta F(y)^{n-1} f(y)\mathrm{d}F(y)} \le \frac{\int_x^\theta (F(y)-F(x))^{n-1}\mathrm{d}F(y)}{f(\theta_0)\int_0^\theta F(y)^{n-1}\mathrm{d}F(y)} = $$
$$= \left(\frac{F(\theta) - F(x)}{F(\theta_0)}\right)^n \cdot \frac{1}{f(\theta_0)}$$

and the last term obviously converges to 0.

To prove that the convergence is uniform if $p < 1$, notice that we can always assume that the functions $f_n = B^n f$ are non-decreasing, continuous, $f_n(0) = 1$, $f_n(\infty) = q := 1 - p$. Let $f(x) = q^{F(x)}$, $\varepsilon > 0$ be fixed and let δ be such that if $|x - x'| < \delta$ then $|f(x) - f(x')| < \varepsilon$. Let $\Gamma = \{0, \delta, 2\delta, \ldots, N\delta\}$ where N is chosen such that $f(N\delta) - f(\infty) < \varepsilon$. Let $n(\varepsilon)$ be such that $n > n(\varepsilon) \Rightarrow |f_n(k\delta) - f(k\delta)| < \varepsilon$, $\forall k = 0, 1, \ldots, N$. Let $x \in \mathcal{R}$ be any. There are two cases:

– either there exists a $k < N$ such that $k\delta \le x < (k+1)\delta$. Let $s = k\delta$, $t = (k+1)\delta$. As f_n are non-increasing, we have $(f(x) - f(s)) + (f(s) - f_n(s)) = f(x) - f_n(s) \le f(x) - f_n(x) \le f(x) - f_n(t) = (f(x) - f(t)) + (f(t) - f_n(t))$.

It follows that $|f(x) - f_n(x)| \le \max(|(f(x) - f(t)) + (f(t) - f_n(t))|, |(f(x) - f(s)) + (f(s) - f_n(s))|) \le \max |f(x) - f(t)| + |f(t) - f_n(t)|, |f(x) - f(s)| + |f(s) - f_n(s)| \le 2\varepsilon$

– or $x \ge N\delta$; in this case $2\varepsilon \le (f(\infty) - f(N\delta)) + (f(N\delta) - f_n(N\delta)) = f(\infty) - f_n(N\delta) \le f(x) - f_n(N\delta) \le f(x) - f_n(x) \le f(x) - f_n(\infty) \le f(N\delta) - q \le \varepsilon$.

Thus in both cases $n \ge n(\delta) \Rightarrow |f(x) - f_n(x)| \le 2\varepsilon$ hence the convergence is uniform. □

Remark and open problem. Is it necessary that the probability F from Proposition 3.4 be continuous? We think that the operator $B^n f$ *always has a limit which depends only on F, not on the chosen f*. The limit is not necessarily the exponential $q^{F(x)}$. The main problem is to decide if the operator B defined in Proposition 3.4 has a fixed point h, i.e., if there exists h such that $Bh = h$. Such a function h is necessarily non-increasing, $h(0) = 1$, $h(\infty) = 1 - p$. Were we able to prove the existence of h, then we could apply again Theorem 3.3.

Unfortunately, we are able to prove the conjecture only if F is a discrete probability with a discrete support, i.e., if $F = \sum_{k=1}^{\infty} \pi_k \delta_{a_k}$ with $0 < a_1 < a_2 < \cdots$. In this case B admits two versions: the right continuous one is

$$Bf(x) = 1 - p\frac{\int_{[0,x]} f(y)\mathrm{d}F(y)}{\int_0^\infty f(y)\mathrm{d}F(y)}$$

and the left continuous one is

$$Bf(x) = 1 - p\frac{\int_{[0,x)} f(y)\mathrm{d}F(y)}{\int_0^\infty f(y)\mathrm{d}F(y)}.$$

We focus on the first one. $\mu = \mu(f) = \int f\mathrm{d}F = \sum_{k=1}^{\infty} f(a_k)\pi_k$. Then

$$Bf(x) = \sum_{k=1}^{\infty}\left(1 - p\frac{\sum_{k:a_k\leqslant x} f(a_k)\pi_k}{\mu}\right)1_{[a_k,a_{k+1})}(x).$$

We claim that the equation $Bh = h$ admits a solution $h = 1_{[0,a_1)} + h_1 1_{[a_1,a_2)} + h_2 1_{[a_2,a_3)} + \cdots$. Indeed, if $x = a_1$ the equation becomes $h_1 = 1 - p\frac{h_1\pi_1}{\mu} \Leftrightarrow h_1 = \frac{\mu}{\mu+p\pi_1}$. If $x = a_2$, then the equation is $h_2 = 1 - p\frac{h_1\pi_1+h_2\pi_2}{\mu}$; replacing h_1 we find $h_2 = \frac{\mu^2}{(\mu+p\pi_1)(\mu+p\pi_2)}$. By induction, we see that

$$h_n = \frac{\mu^n}{(\mu+p\pi_1)(\mu+p\pi_2)\ldots(\mu+p\pi_n)} \text{ with } \mu = \mu(h) = \sum_{k=1}^{\infty} h_k\pi_k. \tag{3.10}$$

We have to prove that h is a solution, i.e., that the sum $\sum_{k=1}^{\infty} h_k\pi_k$ is indeed equally to μ. To this end, let $\sum_n = \sum_{k=1}^{n} h_k\pi_k$. One proves by induction that

$$\sum_n = \frac{\mu}{p}(1 - h_n) \tag{3.11}$$

and takes into account the fact that h_n , as constructed in (3.10) must tend to $1 - p$ if $n \to \infty$. It means that $\lim_{n\to\infty}\sum_n = \lim_{n\to\infty}\frac{\mu}{p}(1 - h_n) = \frac{\mu}{p}(1-\lim_{n\to\infty} h_n) = \frac{\mu}{p}(1-(1-p)) = \mu$, hence the function h defined by (3.10) is indeed, a solution of the equation $Bh = h$. For instance if $\pi_3 = \pi_4 = \ldots = 0$, the solution is

$$h = 1_{[0,a_1)} + \frac{q(\pi_1 - \pi_2) + \sqrt{g^2(\pi_1-\pi_2)^2 + 4q\pi_1\pi_2}}{2\pi_1}1_{[a_1,a_2)} + q1_{[a_1,\infty)}. \tag{3.12}$$

Notice that the equality $h = q^F$ still holds if $F = \mathrm{Uniform}(\{a_1, a_2, \ldots, a_N\})$, $N \geq 2$, since in that case the relation (3.10) becomes $h_n = (\frac{\mu}{\mu+\frac{p}{N}})^n$. For instance, for $N = 2$ and $\pi_1 = \pi_2$ the relation (3.12) becomes

$$h = 1_{[0,a_1)}+\frac{\sqrt{4q\pi_1\pi_2}}{2\pi_1}1_{[a_1,a_2)}+q1_{[a_2,\infty)} = q^0 1_{[0,a_1)}+q^{1/2}1_{[a_1,a_2)}+q^1 1_{[a_2,\infty)} = q^F.$$

Remark. One can always write a function $f \in Y$ (i.e., a decreasing function such that $f(0) = 1$, $f(T) = 1-p := q$) as $f = 1-pg$ where $g : [0,T] \to [0,1]$ is increasing, $g(0) = 0$ and $g(T) = 1$. Let X be the space of all the functions of this type. In that way the operator B defined at (2.6) becomes another operator $A : X \to X$ having the form

$$Ag(x) = \frac{\int_0^x (1 - pg(y))\mathrm{d}y}{\int_0^T (1 - pg(y))\mathrm{d}y}.$$

If we apply Proposition 3.4 we see that the iterates A^n, f converge to $h(x) = \frac{1-(1-p)^{\frac{x}{T}}}{p}$ for any $p < 1$.

4. ANOTHER APPROACH TO THEOREM 3.3 AND ITS APPLICATION

The readers which are already familiar with the theory of Markov chains may remark that the kernel considered in Theorem 3.3 is a very particular case in the R-theory for irreducible kernels, theory developed, in [7], to which we refer for terminology and properties involved in the sequel.

Let h be the c-invariant function for K_μ and let m, M denote its lower (upper) bound. Note the following relations, direct consequence of the assumptions in Theorem 3.3:

$$K_\mu 1_A(x) \geq a\mu(A) K_\mu 1_A(x) \quad \forall x \in E,\ A \in \mathcal{E} \tag{4.1}$$

and

$$K_\mu 1_A(x) \geq \frac{a}{M} h(x)\mu(A) \quad \forall x \in E,\ A \in \mathcal{E}. \tag{4.2}$$

Relation (4.1) implies that K_μ is μ-irreducible (i.e., $\mu(A) > 0 \Rightarrow K_\mu 1_A(x) > 0$, $\forall x \in E$), aperiodic and the whole space is a *small set.* Also relation (4.2) implies that h is a *small function.*

We summarize below the properties of K_μ which are relevant in our context

Proposition 4.1. (i) *The convergence parameter R of the kernel K_μ is c.*

(ii) *There exists a c-invariant measure π satisfying* $\pi(E) < \infty$, i.e., *the kernel is c-positive recurrent.*

(iii) *The kernel K_μ is c-uniformly ergodic, i.e.,*

$$\lim_{n\to\infty} \sup_{x\in E} \sup_{\{f:|f|\leq h\}} \left| \frac{c^n}{h(x)} K_\mu^n f(x) - \pi(f) \right| = 0. \tag{4.3}$$

Proof. (i) The very existence of a c-invariant function implies $c \leq R$. Next,

$$\sum_{n=1}^{\infty} c^n K_\mu^n 1(x) \geq \frac{1}{M} \sum_{n=1}^{\infty} K_\mu^n h(x) = \infty.$$

Now, for an arbitrary set A with $\mu(A) > 0$ we have by induction over n $K_\mu^{n+1}1_A(x) \geqslant a\mu(A)K_\mu^n 1(x)$, whence $\sum\limits_{n=1}^{\infty} c^n K_\mu^n 1(x)$ is also ∞, which actually means both the fact that $c \geq R$ and the c-recurrence of K_μ.

(ii) K_μ being c-recurrent there exists a c-invariant σ-finite measure π. We shall now prove that $\pi(h) < \infty$, which sets up the c-positive recurrence of K_μ. Let $B \in \mathcal{E}$ be a set for which $\pi(B) < \infty$. By (4.2) we have

$$\pi(B) = c(\pi K_\mu)(B) = c \int K_\mu 1_B(x)\mathrm{d}\pi(x) \geq$$
$$\geq \frac{ca}{M}\mu(B) \int h(x)\mathrm{d}\pi(x) = \frac{ca\,\mu(B)\pi(h)}{M}.$$

(iii) To show the claimed c-uniform ergodicity of K_μ, we invoke Corollary 6.12(ii) in [7] which states that this property is equivalent to the fact that the c-invariant function h is small. □

Coming now to the particular case which generated this discussion, i.e., to the kernel

$$K_\mu f(x) = q\int_0^x f(y)\mathrm{d}F(y) + \int_x^\infty f(y)\mathrm{d}F(y),$$

straightforward computations show that

$$h(x) = q^{F(x)} \quad \text{and} \quad \pi(f) = \int q^{-F(x)} f(x)\mathrm{d}F(x)$$

are respectively the c-invariant function and the c-invariant measure of K_μ, with $c = \frac{\ln q}{q-1}$.

Applying Proposition 4.1(iii) we get the uniform convergence

$$\lim_{n\to\infty} \sup_{x\in E} \left| \frac{c^n K_\mu^n f(x)}{h(x)} - \pi(f) \right| = 0$$

for any bounded f.

Acknowledgement. The authors thank to Marius Rădulescu for suggesting an improvement of Proposition 3.4 and to Ştefan Cruceanu for his contribution in the numerical study in the discrete case.

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Received 7 December 2009

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