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Research Report

Multi-period portfolio selection models with complementarity constraints

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1. Introduction

The standard portfolio model is one where an investor allocates his initial wealth among n risky assets. Assuming the assets returns are stochastic, Harry Markowitz (the 1990 Nobel laureate) in his pioneering work described a theory postulating that rational investors should select a portfolio from the set of all "feasible" portfolios which offers minimum risk for a given level of expected return and maximum expected return for a given level of risk. Since its introduction in the fifties, the model has dominated a great deal of the literature in portfolio analysis.
The classical Markowitz's portfolio selection model is a single period model which does not suppose the existence of initial holdings for the investor and does not takes into account the transaction costs. Pogue [29] had extended Markowitz's portfolio selection model to include transaction costs, short sales, leverage policies and taxes.
The mathematical problem of optimally managing a portfolio of securities in the presence of transaction costs has received considerable research attention in recent years.
Transaction costs consist of portfolio management fees and trade commissions. In literature are considered three types of transaction costs. The first term is a fixed fee for the basic services of a bank account. The second term summarizes the transaction commission. These fees can vary with different asset classes. The third term is referred to a fee which is proportional to the investor's wealth. It is some sort of management fee which is charged regularly for the management of a certain portfolio and it depends on the current wealth of the portfolio.
Models usually differ with respect to the design of the transaction cost function.
For the classical problem where the objective is to maximize expected utility of terminal wealth, most of the attention has been devoted to the case of proportional transaction costs, that is, to the case where the cost associated with a transaction is proportional to the amount of money that is shifted between the securities. Representative of work in this category are the papers by Cvitanic and Karatzas [7], Davis and Norman [9], Taksar et al. [37], Fleming et al. [15], and Shreve and Soner [34]. Typically, the optimal strategy is characterized by a

no-trade region, with trading that is essentially continuous on its boundary used to keep a certain process contained in the region.
Practical portfolio investment problems under uncertainty are multiperiod problems. They can be modeled well as multiperiod stochastic programs. For portfolio optimization problems, it is sufficient to deal with discrete time models since the investments are done at fixed time instants. The numerical optimization methods which need to be used to solve such problems seriously limit the level of detail in the uncertainty about future asset prices and returns which can be incorporated. There exist a large literature devoted to multiperiod portfolio optimization.
Early approaches to multiperiod portfolio selection problem can be found in [26], [8], [14],[16-18], [3], [4]. Important contributions in this direction were made in the eighties e.g. [5], [10], [19], [20], [27], [33].
An approach to multiperiod financial optimization based on optimal scenario tree generation can be found in [28]. For a recent bibliography connected with multiperiod models see [13], [22], [36]. All the contributions, together with the progress in numerical methods, software and computer technologies gave an impulse to the development of large scale real life applications.The multiperiod models can handle the difficult problems of asset liability management and the strategic decision making under risk.
In this paper we intend to extend Markowitz's portfolio selection model to multiperiod models which include proportional transaction costs in the presence of initial holdings for the investor. Our approach is new and is in the line of our previous papers [31], [32]. In the previous stage in order to reduce the amount of data (forecasts) from a series of experts a stationary assumption was made. In the present paper we shall consider the general model without the stationary assumption. Our model will work well only if in the considered time horizon no great changes will be forecasted by the financial experts.

2. Formulation of the multiperiod portfolio selection models

We shall study the problem of an investor who has some initial holdings and knows in a prescribed horizon of time the amount of payments (cash outflows) and the amount of cash inflows and the exact moments when these financial flows will take place.
In our model we suppose that:

- the horizon of time is divided in m - 1 units which we shall call periods.
- the investor has initial holdings in n assets.
- all the transaction are made only at the moments $t = 0,1,2,\ldots, m - 1$.
- the investor wants to withdraw at the initial moment $t = 0$ a sum of money equal to $W_{0,0}$. At moment $t = 1$ the investor wants to withdraw a sum of money equal to $W_{0,1}$, ..., at moment $t=m-1$ the investor wants to withdraw a sum of money equal to $W_{0,m-1}$. In the case when some $W_{0,t}$ is a nonnegative number we shall consider that the investor sells his assets in order to obtain an amount of cash equal to $W_{0,t}$. In the case that $W_{0,t}$ is a negative number we shall consider that the investor buys assets and the value of the assets is equal to $-W_{0,t} = |W_{0,t}|$.

- at period t=0 the initial holdings of the investor are described by the vector $\mathbf{b}=(b_1,b_2,...,b_n)$, that is the investor had purchased before the moment t=0 a quantity b_i of asset i, i=1,2.....,n. Of course all b_i are nonnegative numbers.
- the transaction costs for the assets are linear maps (usually they are piecewise linear) of the volume of assets purchased or sold. We shall denote by $q_{1,i}$ the fee (percent) paid by the investor for purchasing some assets from asset i. We shall denote by $q_{2,i}$ the fee (percent) paid by the investor for selling some assets from asset i. This means that if the investor has a sum M of money and if he purchases assets from asset i then he will pay as a commission a sum $q_{1i}M$. If the investor sells some of his assets from asset i of value M then he will pay as a commission the sum equal to $q_{2i}M$. We shall suppose that the vectors $\mathbf{q}_1=(q_{11},q_{12},...,q_{1n})$and $\mathbf{q}_2=(q_{21},q_{22},...,q_{2n})$ do not change during time horizon.
- a unit of time was chosen so that the rates of return for the n assets where the investor wants to make the investment at various moments $t\in\{0,1,2,...,m-1\}$can be computed. We shall denote by $\xi_{t,i}$ the rate of return for the asset $i\in\{1,2,...,n\}$ at moment $t\in\{1,2,...,m-1\}$. Obviously that $\xi_{t,i}$ are random variables. In order to be more rigorous we shall consider that a probability space $(\Omega,\mathcal{K},P)$ was defined and $\xi_{t,i}:\Omega\to\mathbf{R}$, i = 1,2,…, n, $t\in\{1,2,...,m-1\}$are random variables. We shall denote $d_i=\inf\{\xi_{t,i}(\omega):t\in\{0,1,2,...,m\text{-}1\},\omega\in\Omega\}$, i = 1,2,…, n. We shall suppose that $d_i\geq$ -1 for all i and that $\int_\Omega\left|\xi_{t,i}(\omega)\right|^m dP(\omega)<+\infty$.
- For every $r,s\in\{0,2,...,m-1\}$, $r<s$, $i\in\{1,2,...,n\}$ we consider the random variables
$$\eta_{r,s,i}=(1+\xi_{r,i})(1+\xi_{r+1,i})..(1+\xi_{s,i})$$
Let $\mu_{r,s,i}=E(\eta_{r,s,i})$ and consider the random vectors
$$\boldsymbol{\eta}_{r,s}=(\eta_{r,s,1},\eta_{r,s,2},...,\eta_{r,s,n})$$
and the vectors of means
$$\boldsymbol{\mu}_{r,s}=(\mu_{r,s,1},\mu_{r,s,2},...,\mu_{r,s,n})$$

Denote by **e** the n-dimensional vector of ones, that is $\mathbf{e}=(1,1,\ldots,1)^T$.

The investor makes rebalancing decisions at each of the moments $t\in\{0,1,2,...,m\text{-}1\}$, that is he purchases a portfolio $\mathbf{u}_t=(u_{t,1},u_{t,2},...,u_{t,n})$ and sells a portfolio $\mathbf{v}_t=(v_{t,1},v_{t,2},...,v_{t,n})$

The initial holding at moment t=0 in asset i before the rebalancing decision is $L_{0,i}=b_i$.

At moment t=0 the investor makes a rebalancing decision, that is he purchases a portfolio of assets $\mathbf{u}_0$=($u_{0,1}$, $u_{0,2}$,..., $u_{0,n}$) and sells a portfolio of assets $\mathbf{v}_0$=($v_{0,1}$, $v_{0,2}$,..., $v_{0,n}$). Here by $u_{0,i}$ we denoted the value of shares purchased from the i-th asset and by v_{0i} we denoted the value of shares sold from the i-th asset. After the rebalancing decision the holdings of the investor in asset i are $R_{0,i}=b_i+u_{0,i}-v_{0,i}$. The holdings in asset i at moment t=1 before the rebalancing decision are $L_{1,i}=(1+\xi_{0,i})(b_i+u_{0,i}-v_{0,i})$.

The holdings in asset i at moment t=1 after the rebalancing decision are
$R_{1,i}=(1+\xi_{0,i})(b_i+u_{0,i}-v_{0,i})+u_{1,i}-v_{1,i}$

The holdings in asset i at moment t=2 before the rebalancing decision are

$$L_{2,i} = (1+\xi_{0,i})(1+\xi_{1,i})(b_i + u_{0,i} - v_{0,i}) + (1+\xi_{1,i})(u_{1,i} - v_{1,i})$$

The holdings in asset i at moment t=2 after the rebalancing decision are

$$L_{2,i} = (1+\xi_{0,i})(1+\xi_{1,i})(b_i + u_{0,i} - v_{0,i}) + (1+\xi_{1,i})(u_{1,i} - v_{1,i}) + u_{2,i} - v_{2,i}$$

The holdings in asset i at moment t before the rebalancing decision is

$$L_{t,i} = \eta_{0,t-1,i} b_i + \sum_{s=0}^{t-1} \eta_{s,t-1,i} (u_{s,i} - v_{s,i})$$

The holdings in asset i at moment t=m before the rebalancing decision are

$$L_{m,i} = \eta_{0,m-1,i}(b_i + u_{0,i} - v_{0,i}) + \eta_{1,m-1,i}(u_{1,i} - v_{1,i}) + ... + \eta_{m-1,m-1,i}(u_{m-1,i} - v_{m-1,i})$$

The return at moment t before the rebalancing decision is

$$L_t(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} L_{t,i} = \boldsymbol{\eta}_{0,t-1}^T \mathbf{b} + \sum_{s=0}^{t-1} \boldsymbol{\eta}_{s,t-1}^T (\mathbf{u}_s - \mathbf{v}_s)$$

The expected return at moment t before the rebalancing decision is

$$E(L_t(\mathbf{u}, \mathbf{v})) = \boldsymbol{\mu}_{0,t-1}^T \mathbf{b} + \sum_{s=0}^{t-1} \boldsymbol{\mu}_{s,t-1}^T (\mathbf{u}_s - \mathbf{v}_s)$$

The total transaction costs at moment t is equal to

$$C_t(\mathbf{u}, \mathbf{v}) = \mathbf{q}_1^T \left(\sum_{s=0}^{t-1} \mathbf{u}_s \right) + \mathbf{q}_2^T \left(\sum_{s=0}^{t-1} \mathbf{v}_s \right)$$

Expected net return at moment t is equal to

$$N_t(\mathbf{u}, \mathbf{v}) = E(L_t(\mathbf{u}, \mathbf{v})) - C_t(\mathbf{u}, \mathbf{v})$$

For every $t \in \{0,1,2,...,m-1\}$ let $W_{0,t}$ be the sum of money withdrawn by the investor. Then the following budget constraints hold

$$(\mathbf{e} - \mathbf{q}_2)^T \mathbf{v}_t - (\mathbf{e} - \mathbf{q}_1)^T \mathbf{u}_t = W_{0,t} \qquad t \in \{0,1,2,...,m-1\}$$

The financial risk will be measured by the variance of the return.

For every $t \in \{0,1,2,...,m-1\}$ let $\mathrm{Var}(L_t(\mathbf{u}, \mathbf{v}))$ be the variance of the return at moment t.

The constraints

$$\mathbf{u}_t^T \mathbf{v}_t = 0,\ \mathbf{u}_t \geq 0,\ \mathbf{v}_t \geq 0,\ t \in \{0,1,2,...,m-1\}$$

are called complementarity constraints. They show that at each moment t the investor do not buy and sell assets of the same type.

The multi-objective multiperiod portfolio selection problem

The investor wants to find how to rebalance the portfolio at each moment $t \in \{0,1,2,...,m-1\}$ such that the financial risks of the investment should be minimized and the maximum expected net return should be maximized.

$$\begin{cases} \min\{\mathrm{Var}(L_t(\mathbf{u},\mathbf{v}))\}, & t\in\{0,1,2,...,m-1\} \\ \max\{N_t(\mathbf{u},\mathbf{v})\}, & t\in\{0,1,2,...,m-1\} \\ (\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}_t-(\mathbf{e}-\mathbf{q}_1)^T\mathbf{u}_t=W_{0,t}, & t\in\{0,1,2,...,m-1\} \\ \mathbf{u}_t^T\mathbf{v}_t=0,\ \mathbf{u}_t\geq 0,\ \mathbf{v}_t\geq 0, & t\in\{0,1,2,...,m-1\} \end{cases}$$

The minimum risk multiperiod portfolio selection problem

The investor wants to find how to rebalance the portfolio at each moment $t\in\{0,1,2,...,m-1\}$ so that to minimize the financial risks taking into account the budget constraints, complementarity constraints and the performance constraints

$$\begin{cases} \min\{\mathrm{Var}(L_t(\mathbf{u},\mathbf{v}))\}, & t\in\{0,1,2,...,m-1\} \\ N_t(\mathbf{u},\mathbf{v})\geq W_{1,t}, & t\in\{0,1,2,...,m-1\} \\ (\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}_t-(\mathbf{e}-\mathbf{q}_1)^T\mathbf{u}_t=W_{0,t}, & t\in\{0,1,2,...,m-1\} \\ \mathbf{u}_t^T\mathbf{v}_t=0,\ \mathbf{u}_t\geq 0,\ \mathbf{v}_t\geq 0, & t\in\{0,1,2,...,m-1\} \end{cases}$$

The minimum risk multiperiod portfolio selection problem is equivalent to a mixed binary problem.

$$\begin{cases} \min\{\mathrm{Var}(L_t(\mathbf{u},\mathbf{v}))\}, & t\in\{0,1,2,...,m-1\} \\ N_t(\mathbf{u},\mathbf{v})\geq W_{1,t}, & t\in\{0,1,2,...,m-1\} \\ (\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}_t-(\mathbf{e}-\mathbf{q}_1)^T\mathbf{u}_t=W_{0,t}, & t\in\{0,1,2,...,m-1\} \\ 0\leq\mathbf{u}_t\leq M_t\mathbf{w}_t, \quad 0\leq\mathbf{v}_t\leq M_t(\mathbf{e}-\mathbf{w}_t) & \\ \mathbf{u}_t\geq 0,\ \mathbf{v}_t\geq 0, \quad \mathbf{w}_t\in\{0,1\}^n, & t\in\{0,1,2,...,m-1\} \end{cases}$$

3. Numerical example

In the following we shall analyze two numerical examples for 5 assets and 3 periods for the risk minimization model.

In the first example we considered the case $q_1=0.08$ and $q_2=0.09$. In the second example the case $q_1=0.01$ and $q_2=0.02$ is studied.

In table 1 is displayed vector **b** of initial holdings. In table nr. 2 is displayed the sequence of portfolios (optimal portfolios for the risk minimization model) purchased and sold in the case $q_1=0.08$ and $q_2=0.09$. On the last row are displayed the sum of the entries from the corresponding column. One can see that at moment $t = 0$ the investor purchases only assets 1 and 4 and sells assets 2,3 and 5. At moments $t = 1$ and $t = 2$ the investor do not purchase assets. The value of the assets purchased by the investor during the whole time horizon is 83.112. The value of the assets sold by the investor during the whole time horizon is equal to 373.999.

In table nr. 3 is displayed the sequence of purchased and sold portfolios in the case $q_1=0.01$ and $q_2=0.02$. One can see that there a similarity between table 2 and table 3. If at some moment, in example 1, the investor does not purchase some asset, then this is also true in example 2.

The value of the assets (in example 2) purchased by the investor during the whole time horizon is 90.671. The value of the assets sold by the investor during the whole time horizon is 348.558.

One can easily see from the above results that if the transaction costs decrease, then the value of the assets purchased (in the optimal portfolio) increase and the value of the assets sold decrease.

Asset	Vector b
1	500
2	1500
3	1200
4	2026
5	2500
Total	7726

Table 1. Vector **b** of the initial holdings

Asset	u0	u1	u2	v0	v1	v2
1	79.835	0.000	0.000	0.000	0.000	12.209
2	0.000	0.000	0.000	13.272	49.955	31.700
3	0.000	0.000	0.000	75.024	75.353	39.225
4	3.278	0.000	0.000	0.000	39.527	26.756
5	0.000	0.000	0.000	10.978	0.000	0.000
Total	83.113	0.000	0.000	99.275	164.835	109.890

Table 2. Sequence of portfolios purchased and sold in the case $q_1 = 0.08$ and $q_2 = 0.09$.

Asset	u0	u1	u2	v0	v1	v2
1	87.617	0.000	0.000	0.000	0.000	11.338
2	0.000	0.000	0.000	12.748	46.388	29.436
3	0.000	0.000	0.000	70.099	69.976	36.425
4	3.054	0.000	0.000	0.000	36.697	24.842
5	0.000	0.000	0.000	10.609	0.000	0.000
Total	90.671	0.000	0.000	93.456	153.061	102.040

Table 3. Sequence of portfolios purchased and sold in case $q_1 = 0.01$ and $q_2 = 0.02$.

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