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Research Report

Single period and multi-period portfolio selection models with complementarity constraints

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Chapter 1

Single period portfolio selection models with complementarity constraints

The ideas of this chapter represent a continuation of the ideas presented in the Research Report 2014.

1. Algorithms for solving the minimum risk problem

In this section we shall give some estimations for the optimal value of the minimum risk problem with proportional transaction costs.

Since the feasible set of the minimum risk problem is not convex one cannot use the standard algorithms from nonlinear programming theory.

We shall give two heuristic algorithms in order to provide suboptimal solutions to problem (7). These algorithms can provide good starting points for a third algorithm.

The application of the third algorithm supposes the resolution of a sequence of nonlinear programming problems. The set of feasible solutions of each such problem is compact and convex. This fact allows the application of standard algorithms from nonlinear programming.

However, the resolution of problem (7) requires a great computational effort.

Denote

$$Z = \left\{(\mathbf{u},\mathbf{v}) \in \mathbf{R}^{2n} : \mathbf{u} \geq \mathbf{0}, \mathbf{0} \leq \mathbf{v} \leq \mathbf{b}\right\}$$

$$X = \left\{(\mathbf{u},\mathbf{v}) \in Z : W_0 = (\mathbf{e}-\mathbf{q}_2)^T \mathbf{v} - \mathbf{q}_3^T \mathbf{u}\,,\ \boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v}) - \mathbf{q}_2^T \mathbf{v} - \mathbf{q}_1^T \mathbf{u} \geq W_1\right\}$$

$$Y = \left\{(\mathbf{u},\mathbf{v}) \in X : \mathbf{u}^T \mathbf{v} = 0\right\}$$

Note that Y is the feasible set for the model (7), and X is the feasible set for the same model without the complementarity restriction.

Denote:

$$f(\mathbf{u},\mathbf{v}) = (\mathbf{b}+\mathbf{u}-\mathbf{v})^T \mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v}) - \mathbf{q}_2^T \mathbf{v} - \mathbf{q}_1^T \mathbf{u}, \quad (\mathbf{u},\mathbf{v}) \in X$$

$$\gamma_0 = \min\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Y\}$$

$$\gamma_1 = \min\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in X\}$$

Without any loss of generality we may suppose that the components of the vector b of initial holdings satisfy the following condition:

$$b_1 \geq b_2 \geq \ldots \geq b_k > b_{k+1} = b_{k+1} = \ldots = b_n = 0$$

Thus k is the number of assets where the investor has initial holdings.

Denote $I = \{1,2,\ldots,n\}$, $I_0 = \{1,2,\ldots,k\}$, $I_0' = I - I_0$, $\mathrm{P}(I_0)$ the family of all subsets of I_0. For every $K \in \mathrm{P}(I_0)$ we denote

$$V(K) = \left\{(\mathbf{u},\mathbf{v}) \in X : u_i = 0, i \in K \bigcup I_0', v_j = 0, j \in (I_0 - K) \bigcup I_0'\right\}$$

Note that all the sets $V(K)$ are convex and $Y = \bigcup_{K \in \mathrm{P}(I_0)} V(K)$

One can easily see that the optimal solution of the problem (7) can be found by solving 2^k standard quadratic programming problems (SQPP). Here by a SQPP we understand a quadratic programming problem with linear restrictions. More precisely, for every $K \in \mathrm{P}(I_0)$ we solve the following SQPP: $\min\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in V(K)\}$

If we denote by $(\mathbf{u}(K), \mathbf{v}(K))$ a solution of the above problem, then

$$\gamma_0 = \min\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Y\} = \min\{f(\mathbf{u}(K), \mathbf{v}(K)) : K \in \mathrm{P}(I_0)\}.$$

If k is small then it will be no problem to find the optimal solution of problem (7). If k is large we have to use heuristic algorithms in order to obtain suboptimal solutions.

Of considerable interest is to find lower estimates for γ_0, that are greater than γ_1. In order to do that, for every $i, j \in I$ we shall define the sets:

$$A_i = \{(\mathbf{u},\mathbf{v}) \in X : u_i = 0\}, \quad B_i = \{(\mathbf{u},\mathbf{v}) \in X : v_i = 0\}, \quad C_{ij} = \left(A_i \bigcup B_i\right) \bigcap \left(A_j \bigcup B_j\right).$$

One can easily see that

$$Y = \bigcap_{i=1}^{n} \left(A_i \bigcup B_i\right) = \bigcap_{1 \le i < j \le n}^{n} C_{ij}$$

Since for every $i \in I_0'$ we have $B_i = X$ it follows that for $k \ge 2$ we have

$$Y = \bigcap_{i=1}^{k} \left(A_i \bigcup B_i\right) = \bigcap_{1 \le i < j \le n}^{k} C_{ij}$$

Consequently

$$\inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Y\} \ge \max_{1 \le i < j \le k} \inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in C_{ij}\} \ge$$

$$\ge \max_{1 \le i \le k} \inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_i \bigcup B_i\right\} \ge \inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in X\}$$

whence $\gamma_1 \le \gamma_2 \le \gamma_3 \le \gamma_0$

We denoted

$$\gamma_2 = \max_{1 \le i \le k} \inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_i \bigcup B_i\right\}$$

$$\gamma_3 = \max_{1 \le i < j \le k} \inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in C_{ij}\}.$$

We note that γ_2 can be computed by solving $2k$ SQPP**.** More precisely for every $i \in I_0$ we have to solve the problems

$$\inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_i\}, \quad \inf\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in B_i\}$$

We note that γ_3 can be computed by solving $2k^2 - 2k$ SQPP**.** More precisely for every $i, j \in I_0$, $i < j$ we have to solve the problems

$$\inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_i \bigcap A_j\right\}, \quad \inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_i \bigcap B_j\right\},$$

$$\inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in A_j \bigcap B_i\right\}, \quad \inf\left\{f(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in B_i \bigcap B_j\right\}.$$

In the following we shall describe two heuristic algorithms for solving problem (7).

Heuristic algorithm 1.

We shall present a simple heuristic method for reducing the number of variables in the minimum risk model.

In the first step we find a solution $(\mathbf{u}^*, \mathbf{v}^*)$ of the following SQPP:

$$\min\{f(\mathbf{u},\mathbf{v}):(\mathbf{u},\mathbf{v})\in X\} \qquad (8)$$

Denote $I_1=\{i\in I: u_i^*=0\}$, $I_2=\{i\in I: v_i^*=0\}$, $I_3=I_1\cup I_2$, $I_4=I-I_3$, m=card(I_4),
$T=\{(\mathbf{u},\mathbf{v})\in X, u_i=0, i\in I_1, v_i=0, i\in I_2, u_i v_i=0, i\in I_4\}$.

In the second step we solve the problem

$$\min\{f(\mathbf{u},\mathbf{v}):(\mathbf{u},\mathbf{v})\in T\} \qquad (9)$$

It is clear that the solution of the above model is complementary.

In order to obtain the optimal solution of the problem (9) we have to solve 2^m SQPP. More precisely, for every $K\in \mathsf{P}(I_4)$ we solve the following SQPP:

$$\min\{f(\mathbf{u},\mathbf{v}):(\mathbf{u},\mathbf{v})\in X, u_i=0, i\in I_1\bigcup K, v_j=0, j\in I_2\bigcup(I_4-K)\}$$

Heuristic algorithm 2.

Choose a n-dimensional vector $\mathbf{v}^{(0)}$ with the following properties: $\mathbf{0}\le\mathbf{v}^{(0)}\le\mathbf{b}$ and there exists a vector $\mathbf{u}\ge 0$ such that $(\mathbf{u},\mathbf{v}^{(0)})\in X$.

Solve the following SQPP:

(P_0) $\min\{f(\mathbf{u},\mathbf{v}^{(0)}):(\mathbf{u},\mathbf{v}^{(0)})\in X, \mathbf{u}^T\mathbf{v}^{(0)}=0\}$

Let $u^{(0)}$ be a solution of (P_0). Solve the following SQPP:

(P_0') $\min\{f(\mathbf{u}^{(0)},\mathbf{v}):(\mathbf{u}^{(0)},\mathbf{v})\in X, \mathbf{v}^T\mathbf{u}^{(0)}=0\}$

If $\mathbf{v}^{(1)}$ is a solution of (P_0') then we solve the following SQPP:

(P_1) $\min\{f(\mathbf{u},\mathbf{v}^{(1)}):(\mathbf{u},\mathbf{v}^{(1)})\in X, \mathbf{u}^T\mathbf{v}^{(1)}=0\}$ and denote by $\mathbf{u}^{(1)}$ a solution of (P_1).

If we continue the procedure we obtain a sequence $\{(\mathbf{u}^{(k)},\mathbf{v}^{(k)})\}_{k\ge 0}$ from Y such that $f((\mathbf{u}^{(k+1)},\mathbf{v}^{(k+1)}))\le f((\mathbf{u}^{(k)},\mathbf{v}^{(k)}))$ for every $k\ge 0$.

We can continue the above procedure until the improvement of the value of the objective function at $(\mathbf{u}^{(k)}, \mathbf{v}^{(k)})$, (that is $\delta_k = f((\mathbf{u}^{(k+1)}, \mathbf{v}^{(k+1)})) - f((\mathbf{u}^{(k)}, \mathbf{v}^{(k)}))$) become very small and the distance between γ_1 and $f((\mathbf{u}^{(k)}, \mathbf{v}^{(k)}))$ decreases under some prescribed number. If this does not happen after a number of steps, one can choose at random another vector $\mathbf{v}^{(0)}$ such that $\mathbf{0} \le \mathbf{v}^{(0)} \le \mathbf{b}$, and repeat the algorithm.

Combinations of the above heuristic algorithms allow us to obtain better solutions.

The third algorithm.

For every $r \in [\gamma_1, \infty)$ consider the set $A(r) = \{(\mathbf{u}, \mathbf{v}) \in X : f(\mathbf{u}, \mathbf{v}) \le r\}$. Note that all the sets $A(r)$ are compact and convex and the function $\phi(r) = \min\{\mathbf{u}^T\mathbf{v} : (\mathbf{u}, \mathbf{v}) \in A(r)\}$, $r \in [\gamma_1, \infty)$ is decreasing and has the following properties:

(i) $\phi(r) = 0$ for every $r \in [\gamma_0, \infty)$

(ii)If $\gamma_1 < \gamma_0$ then $\phi(r) > 0$ for every $r \in [\gamma_1, \gamma_0)$

Let $(\mathbf{u}_1, \mathbf{v}_1) \in Y$ be a suboptimal solution of the problem (7). Eventually it can be found using heuristic algorithms 1 and 2. If $(\mathbf{u}_1, \mathbf{v}_1)$ is not optimal then $f(\mathbf{u}_1, \mathbf{v}_1) > \gamma_0$. Denote $r_1' = \gamma_1$, $r_1'' = f(\mathbf{u}_1, \mathbf{v}_1)$. Note that $r_1' < \gamma_0 < r_1''$.

If $\phi\left(\frac{r_1' + r_1''}{2}\right) = 0$ denote $r_2'' = \frac{r_1' + r_1''}{2}$, $r_2' = r_1'$.

If $\phi\left(\frac{r_1' + r_1''}{2}\right) > 0$ denote $r_2' = \frac{r_1' + r_1''}{2}$, $r_2'' = r_1''$.

Note that $r_2' < \gamma_0 \le r_2''$, $\phi(r_2') > 0$, $\phi(r_2'') = 0$. We iterate the above argument. Suppose that $r_s' < \gamma_0 \le r_s''$, $\phi(r_s') > 0$, $\phi(r_s'') = 0$.

If $\phi\left(\frac{r_s' + r_s''}{2}\right) = 0$ denote $r_{s+1}'' = \frac{r_s' + r_s''}{2}$, $r_{s+1}' = r_s'$.

If $\phi\left(\frac{r_s' + r_s''}{2}\right) > 0$ denote $r_{s+1}' = \frac{r_s' + r_s''}{2}$, $r_{s+1}'' = r_s''$.

One can easily see that $r_{s+1}' < \gamma_0 \le r_{s+1}''$. We obtain two sequences $(r_s')_{s\ge1}$, $(r_s'')_{s\ge1}$ with the following properties: $r_s' < \gamma_0 \le r_s''$, $(r_s')_{s\ge1}$ is increasing and convergent to γ_0, $(r_s'')_{s\ge1}$ is decreasing and convergent to γ_0 and for $s \ge 1$ $|r_s' - r_s''| = \frac{|f(\mathbf{u}_1, \mathbf{v}_1) - \gamma_1|}{2^{s-1}}$. For every $s \ge 2$ denote by $(\mathbf{u}_s, \mathbf{v}_s)$ a solution of the

problem $\min\{\mathbf{u}^T\mathbf{v} : (\mathbf{u},\mathbf{v}) \in A(r_s'')\}$. Then $\{f(\mathbf{u}_s,\mathbf{v}_s)\}_{s\geq 1}$ is a decreasing sequence which is convergent to γ_0.

2. Formulation of the minimum risk model as a mixed-binary model.

Suppose that $b_1 \geq b_2 \geq \ldots b_k > b_{k+1} = b_{k+2} = \ldots = b_n = 0$.

Consider the vector $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)$ where

$$\tilde{b}_i = \begin{cases} 1 & \text{if} \quad i \in \{1,2,\ldots,k\} \\ 0 & \text{if} \quad i \in \{k+1,k+2,\ldots,n\} \end{cases}$$

Let $\mathbf{D} = \text{diag}\left((1+q_{11})^{-1}, (1+q_{12})^{-1}, \ldots, (1+q_{1n})^{-1}\right)$, $\mathbf{B}$=diag($\mathbf{b}$).

The minimum risk problem can be formulated as a mixed binary problem as follows

$$\begin{cases} \min\left[(\mathbf{b}+\mathbf{u}-\mathbf{v})^T\mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v})\right] \\ W_0 = (\mathbf{e}-\mathbf{q}_2)^T\mathbf{v} - (\mathbf{e}+\mathbf{q}_1)^T\mathbf{u} \\ \boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v}) - \mathbf{q}_2{}^T\mathbf{v} - \mathbf{q}_1\mathbf{u} \geq W_1 \\ \mathbf{0} \leq \mathbf{u} \leq \left[(\mathbf{e}-\mathbf{q}_2)^T\mathbf{b} - W_0\right]\mathbf{D}(\mathbf{e}-\mathbf{w}) \\ \mathbf{0} \leq \mathbf{v} \leq \mathbf{B}\mathbf{w},\ \mathbf{w} \leq \tilde{\mathbf{b}},\ \mathbf{w} \in \{0,1\}^n \end{cases}$$

The decision variables are the vectors **u,v,w**. **u** and **v** are vectors in $\mathbb{R}^n$ and **w** is a vector with binary entries.

Note that the number of variables in the mixed binary problem is greater than the number of variables in the initial model.

Note that the objective function in the mixed binary problem is quadratic and the constraints are linear. For such a model there exist software packages that can solve it efficiently.

3. Numerical results

We consider the risk minimization model in the case of proportional transaction costs. We shall suppose that the transaction cost rate for purchasing (respectively selling) assets does not depend on the asset. We denote by p_1 (respectively p_2) the transaction cost rate for purchasing (respectively selling) assets. That is $c_{1i}(s) = p_1 s$, $c_{2i}(s) = p_2 s$ for all i=1,2,…,n. Then $c_{3i}(s) = \dfrac{s}{1-p_1} = p_3 s$ for all i.

Let n=10, W_0=100, W_1=300. In table 1 are displayed the optimal portfolios for the cases: p_1=p_2=0.01 and p_1=p_2=0.07. More precisely:

– On columns 3 and 4 are displayed the assets purchased (vector **u**(1)) and sold (vector **v**(1)) in the case $p_1 = p_2 = 0.01$.

– On column 5 is displayed the vector **x**(1) = **b** + **u**(1) - **v**(1)

– On columns 6 and 7 are displayed the assets purchased (vector **u**(2)) and sold (vector **v**(2)) in the case $p_1 = p_2 = 0.07$.

– On column 8 is displayed the vector **x**(2) = **b** + **u**(2) - **v**(2)

On the last row are displayed the sum of the entries from the corresponding column.

Table 1. The trade decisions for different transaction costs rates

Assets	Vector **b**	Vector **u**(1)	Vector **v**(1)	Vector **u**(2)	Vector **v**(2)
1	2500	0,00	443,96	0,00	437,01
2	1300	0,00	477,92	0,00	479,40
3	1000	522,41	0,00	524,22	0,00
4	2500	2685,90	0,00	2691,36	0,00
5	1500	0,00	1319,69	0,00	1317,75
6	1100	0,00	1100,00	0,00	1100,00
7	1200	0,00	1200,00	0,00	1200,00
8	2900	0,00	1178,59	0,00	1176,73
9	1400	2009,45	0,00	1343,38	0,00
10	700	289,57	0,00	287,39	0,00
Sum	16100	5507,33	5720,16	4846,34	5710,88

4. Simplified minimum risk model with proportional transaction costs. Dimension reduction

Let $\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$, $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$. $\mathbf{C}_{11}$ is a $k \times k$ matrix, $\mathbf{C}_{12} = \mathbf{C}_{21}^T$ is a $k \times (n-k)$ matrix and $\mathbf{C}_{22}$ is a $(n-k) \times (n-k)$. $B_1 = \mathrm{diag}(b_1, b_2, ..., b_k)$

$$\begin{cases} \min\left[(\mathbf{b}_1+\mathbf{u}_1-\mathbf{v}_1)^T\mathbf{C}_{11}(\mathbf{b}_1+\mathbf{u}_1-\mathbf{v}_1)+2(\mathbf{b}_1+\mathbf{u}_1-\mathbf{v}_1)^T\mathbf{C}_{12}\mathbf{u}_2+\mathbf{u}_2\mathbf{C}_{22}\mathbf{u}_2\right] \\ W_0=(1-q_2)\mathbf{e}_1^T\mathbf{v}_1-(1+q_1)\mathbf{e}^T\mathbf{u} \\ \boldsymbol{\mu}_1^T(\mathbf{b}_1+\mathbf{u}_1-\mathbf{v}_1)+\boldsymbol{\mu}_2^T\mathbf{u}_2-q_2\mathbf{e}_1^T\mathbf{v}_1-q_1\mathbf{e}^T\mathbf{u}\geq W_1 \\ \mathbf{0}\leq\mathbf{u}_1\leq\dfrac{1}{1+q_1}\left[(1-q_2)\mathbf{e}_1^T\mathbf{b}_1-W_0\right](\mathbf{e}_1-\mathbf{w}_1) \\ \mathbf{0}\leq\mathbf{u}_2\leq\dfrac{1}{1+q_1}\left[(1-q_2)\mathbf{e}_1^T\mathbf{b}_1-W_0\right]\mathbf{e}_2 \\ \mathbf{0}\leq\mathbf{v}_1\leq\mathbf{B}_1\mathbf{w}_1,\ \mathbf{w}_1\in\{0,1\}^k \end{cases}$$

Decision variables are

- $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2)\in\mathbf{R}^k\times\mathbf{R}^{n-k}$
- $\mathbf{v}_1\in\mathbf{R}^k$
- $\mathbf{w}_1\in\{0,1\}^k$

The number of decision variables is equal to $n+2k$.
$n+k$ variables are continuous variables (vector u and vector v_1)
k variables are binary (vector w_1)
The objective function is quadratic and the constraints are linear.
The model can be solved with the MINLP solver from GAMS.

5. Minimum risk model with proportional transaction costs and non-smooth constraints

Let **x** be the portfolio after the revision**.**

Denote $a_+=\max(a,0)=\dfrac{a+|a|}{2}$.

Note that for every asset i $(x_i-b_i)_+$ is the amount from the i-asset the investor purchases $(b_i-x_i)_+$ is the amount from the i-asset the investor sells.
The minimum risk model with proportional transaction costs is the following

$$\begin{cases} \min\left[\sum_{i=1}^n\sum_{j=1}^n c_{ij}x_ix_j\right] \\ W_0=(1-q_2)\left(\sum_{i=1}^n(b_i-x_i)_+\right)-(1+q_1)\left(\sum_{i=1}^n(x_i-b_i)_+\right) \\ \sum_{i=1}^n\mu_ix_i-q_1\left(\sum_{i=1}^n(x_i-b_i)_+\right)-q_2\left(\sum_{i=1}^n(b_i-x_i)_+\right)\geq W_1 \\ x_i\geq 0,\quad i=1,2,\ldots,n \end{cases}$$

6. Comparison between efficient frontiers of minimum risk models with proportional transaction cost

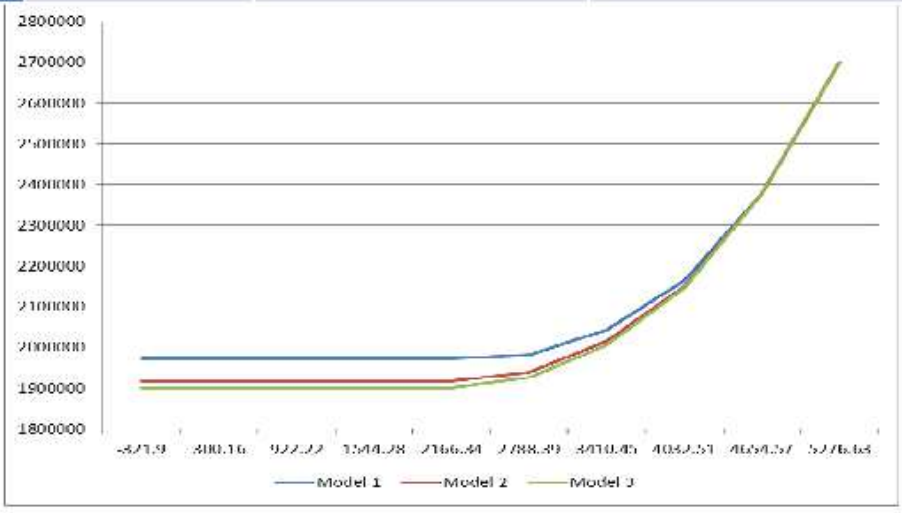
The main risk minimization models are

- Minimum risk model with proportional transaction costs and non-smooth constraints
- Minimum risk model with proportional transaction costs and complementarity constraints
- Minimum risk model with proportional transaction costs and binary variables

Comparison between efficient frontiers of

- Minimum risk model with proportional transaction costs and binary variables
- Minimum risk model with proportional transaction costs without complementarity constraints
- Minimum risk model with proportional transaction costs and relaxed binary variables

Nr. Crt.	Parameter W_1	Minimum risk for the Model 1	Minimum risk for the Model 2	Minimum risk for the Model 3
1	-344.42	1972139.1	1917026.29	1901381.29
2	273.64	1972139.1	1917026.29	1901381.29
3	891.71	1972139.1	1917026.29	1901381.29
4	1509.78	1972139.1	1917026.29	1901381.29
5	2127.85	1972139.1	1917026.29	1901381.29
6	2745.91	1980011.5	1936932.71	1921846.08
7	3363.98	2035834.3	2007247.25	1993876.36
8	3982.05	2152305.5	2136231.66	2126125.54
9	4600.12	2356610	2353944.93	2349686.74
10	5218.19	2665349.9	2664665.56	2661442.53
11	5836.25	4721326.6	4719782.74	4712508.87



7. Conclusions

We have presented several single period portfolio selection models in the presence of initial holdings and transaction costs. Since the set of feasible solutions is not convex the objective map may present several points of local minimum (maximum).
An efficient approach in the attempt to find a global minimum (maximum) of the objective map is a combination of computer simulation with heuristic algorithms.
We gave conditions under which one can find analytical solutions to our problems and we presented some heuristic algorithms for obtaining suboptimal solutions and an algorithm for finding the optimal solution.
A formulation of the minimum risk problem as a mixed binary problem was given. Extensions of our models in order to treat the discrete cases and to include some other measures of risk are underway.
In the next chapter a minimum risk model for the multi-period case will be a formulated.

Chapter 2

Multi-period portfolio selection models with complementarity constraints

In this chapter we extend Markowitz's portfolio selection model to multi-period models which include proportional transaction costs in the presence of initial holdings for the investor. Our approach is new and is in the line of our previous chapter. In order to reduce the amount of data (forecasts) from a series of experts a stationary assumption is made.

1. Introduction

The standard portfolio model is one where an investor allocates his initial wealth among n risky assets. Assuming the assets returns are stochastic, Harry Markowitz (the 1990 Nobel laureate) in his pioneering work described a theory postulating that rational investors should select a portfolio from the set of all "feasible" portfolios which offers minimum risk for a given level of expected return and maximum expected return for a given level of risk. Since its introduction in the fifties, the model has dominated a great deal of the literature in portfolio analysis. For references concerning portfolio theory see: [23], [24], [30].

The classical Markowitz's portfolio selection model is a single period model which does not suppose the existence of initial holdings for the investor and does not takes into account the transaction costs. Pogue [29] had extended Markowitz's portfolio selection model to include transaction costs, short sales, leverage policies and taxes.

The mathematical problem of optimally managing a portfolio of securities in the presence of transaction costs has received considerable research attention in recent years (see [1], [2], [6], [7], [9], [11], [12], [15], [22],[31],[32],[34]). Transaction costs consist of portfolio management fees and trade commissions. In literature are considered three types of transaction costs. The first term is a fixed fee for the basic services of a bank account. The second term summarizes the transaction commission. These fees can vary with different asset classes. The third term is referred to a fee which is proportional to the investor's wealth. It is some sort of management fee which is charged regularly for the management of a certain portfolio and it depends on the current wealth of the portfolio.

Models usually differ with respect to the design of the transaction cost function. A discussion on the transaction costs can be found in [35].

For the classical problem where the objective is to maximize expected utility of terminal wealth, most of the attention has been devoted to the case of proportional transaction costs, that is, to the case where the cost associated with a transaction is proportional to the amount of money that is shifted between the securities. Representative of work in this category are the papers by Cvitanic and Karatzas [7], Davis and Norman [9], Taksar et al. [37], Fleming et al. [15], and Shreve and Soner [34]. Typically, the optimal strategy is characterized by a no-trade region, with trading that is essentially continuous on its boundary used to keep a certain process contained in the region.

Other authors assumed the transaction cost has a fixed component, thereby precluding the optimality of continuous trading. Morton and Pliska [25] assumed that the transaction cost is proportional to the current value of the portfolio. They showed that the optimal strategy is characterized by a fixed vector of portfolio proportions together with an optimal stopping rule, a pair which can be computed by solving a free boundary problem.

There exists a large literature in this direction. For further references on this subject see [31], [32].

Practical portfolio investment problems under uncertainty are multiperiod problems. They can be modelled well as multiperiod stochastic programs. For portfolio optimization problems, it is sufficient to deal with discrete time models since the investments are done at fixed time instants. The numerical optimization methods which need to be used to solve such problems seriously limit the level of detail in the uncertainty about future asset prices and returns which can be incorporated. There exist a large literature devoted to multiperiod portfolio optimization.

Early approaches to multiperiod portfolio selection problem can be found in [26], [8], [14],[16-18], [3], [4]. Important contributions in this direction were made in the eighties e.g. [5], [10], [19], [20], [27], [33].

An approach to multiperiod financial optimization based on optimal scenario tree generation can be found in [28]. For a recent bibliography connected with multiperiod models see [13], [22], [36]. All the contributions, together with the progress in numerical methods, software and computer technologies gave an impulse to the development of large scale real life applications.The multiperiod models can handle the difficult problems of asset liability management and the strategic decision making under risk.

In this paper we intend to extend Markowitz's portfolio selection model to multiperiod models which include proportional transaction costs in the presence of initial holdings for the investor. Our approach is new and is in the line of our previous papers [31], [32]. In order to reduce the amount of data (forecasts) from a series of experts a stationarity assumption is made. Our model will work well only if in the considered time horizon no great changes will be forecasted by the financial experts.

2. Formulation of the multiperiod portfolio selection models

We shall study the problem of an investor who has some initial holdings and knows in a prescribed horizon of time the amount of payments (cash outflows) and the amount of cash inflows and the exact moments when these financial flows will take place.

In our model we suppose that:

- the horizon of time is divided in m - 1 units which we shall call periods.
- the investor has initial holdings in n assets.
- all the transaction are made only at the moments $t = 0,1,2,\ldots, m - 1$.
- the investor wants to withdraw at the initial moment $t = 0$ a sum of money equal to W_0. At moment $t = 1$ the investor wants to withdraw a sum of money equal to W_1, ..., at moment $t=m$-1 the investor wants to withdraw a sum of money equal to W_{m-1}. In the case when some W_i is a nonnegative number we shall consider that the investor sells his assets in order to obtain an amount of cash equal to W_i. In the case that W_i is a negative number we shall consider that the investor buys assets and the value of the assets is equal to - $W_i = |W_i|$.
- at period t=0 the initial holdings of the investor are described by the vector $b = (b_1, b_2, ..., b_n)$, that is the investor had purchased before period t=0 a quantity b_i of asset i, i=1,2.....,n. Of course all b_i are nonnegative numbers.
- the transaction costs for the assets are linear maps (usually they are piecewise linear) of the volume of assets purchased or sold. We shall denote by q_1 the fee (percent) paid by the investor for purchasing some assets. We shall denote by q_2 the fee (percent) paid by the investor for selling some assets. This means that if the investor has a sum M of money and if he purchases assets then he will pay as a commission a sum $q_1 M$. If the investor sells some of his assets of value M then he will pay as a commission the sum equal to $q_2 M$. We shall suppose that q_1 and q_2 do not change during time horizon.
- a unit of time was chosen so that the rates of return for the n assets where the investor wants to make the investment can be computed. We shall denote by ξ_i the rate of return for the asset i, i=1,2,…,n. Obviously that ξ_i are random variables. In order to be more rigorous we shall consider that a probability space $(\Omega, \mathcal{K}, P)$ was defined and $\xi_i : \Omega \to \mathbf{R}$, i = 1,2,…, n. We shall denote $d_i = \inf\{\xi_i(\omega) : \omega \in \Omega\}$, i = 1,2,…, n. We shall suppose that $d_i \geq -1$ for all i and that $\int_\Omega |\xi_i(\omega)|^m \, dP(\omega) < +\infty$.
- we shall denote by μ_i the expected value of ξ_i. Let $\xi=(\xi_1,\xi_2,....,\xi_n)$ be the random vector of the rates of return for the n assets and $\mu=(\mu_1, \mu_2,....,\mu_n)$ the vector of means(expected rates of return). At each moment in the horizon of time, the random vector of the rates of return for the n assets is the same, that is it not depends of time. Of course this stationary hypothesis is not very realistic but it will save us from collecting a large amount of data (forecasts) from a series of experts. Of course our model will work well only if in the considered time horizon no great changes will be forecasted by the financial experts.
- the risk of the investment is defined as the variance of the total return.

For every $r \in \{1,2,...,m\}, i \in \{1,2,...,n\}$ we consider the random variables: $\eta_{ri} = (1+\xi_i)^r$ and the real numbers $a_{ri} = (1+d_i)^r$. We shall denote by v_{ri} the expected value of η_{ri}, that is $v_{ri} = E(\eta_{ri})$ For every $r \in \{1,2,...,m\}$ we define the random vector: $\eta_r = (\eta_{r1}, \eta_{r2}, ..., \eta_{rn})$ and the vector of real numbers $a_r = (a_{r1}, a_{r2}, \ldots, a_{rn})$. Let $v_r = (v_{r1}, v_{r2}, ..., v_{rn})$, $r=1,2,...,m$ the vectors whose coordinates are the expected values of the random variables η_{ri}.

Suppose that at moment $t = 0$ the investor purchases a portfolio of assets $u_0=(u_{01}, u_{02},..., u_{0n})$ and sells a portfolio of assets $v_0=(v_{01}, v_{02},..., v_{0n})$. Here by u_{0i} we denoted the value of shares purchased from the i-th asset and by v_{0i} we denoted the value of shares sold from the i-th asset.

Consequently the sum of money used by the investor to purchase assets from portfolio u_0 will be equal to $\frac{1}{1-q_1}\sum_{i=1}^{n} u_{0i}$ and the sum of money obtained by the investor after selling assets from portfolio v_0 will be equal to $(1-q_2)\sum_{i=1}^{n} v_{0i}$. Of course the following restrictions must be satisfied:

$$(1-q_2)\sum_{i=1}^{n} v_{0i} - \frac{1}{1-q_1}\sum_{i=1}^{n} u_{0i} = W_0$$

$$b_i + u_{0i} - v_{0i} \geq 0, \quad i = 1,2,...,n.$$

At moment $t = 1$, before the transaction, the value of the shares from the i-th asset will be

$$\zeta'_{1i} = (1+\xi_i)(b_i + u_{0i} - v_{0i})$$

We suppose that at moment $t = 1$, the investor purchases a portfolio of assets $u_1=(u_{11}, u_{12},..., u_{1n})$ and sells a portfolio of assets $v_1=(v_{11}, v_{12},..., v_{1n})$. At moment $t = 1$, after the transactions, the value of the shares from i-th asset will be

$$\zeta''_{1i} = \zeta'_{1i} + u_{1i} - v_{1i} = (1+\xi_i)(b_i + u_{0i} - v_{0i}) + u_{1i} - v_{1i}.$$

Since the sum withdrawn by the investor at moment $t = 1$ is W_1 the following equality must hold:

$$(1-q_2)\sum_{i=1}^{n} v_{1i} - \frac{1}{1-q_1}\sum_{i=1}^{n} u_{1i} = W_1.$$

Since ζ''_{1i} must be nonnegative, the following inequality must hold

$$(1+d_i)(b_i + u_{0i} - v_{0i}) + u_{1i} - v_{1i} \geq 0.$$

This is equivalent to $a_{1i}(b_i + u_{0i} - v_{0i}) + u_{1i} - v_{1i} \geq 0.$

At moment $t = 2$, before the transaction, the value of the shares from the i-th asset will be:

$$\zeta'_{2i} = (1+\xi_i)\,\zeta'_{1i} = (1+\xi_i)^2(b_i + u_{0i} - v_{0i}) + (1+\xi_i)(u_{1i} - v_{1i}).$$

We suppose that at moment $t = 2$, the investor purchases a portfolio of assets $u_2=(u_{21}, u_{22},..., u_{2n})$ and sells a portfolio of assets $v_2=(v_{21}, v_{22},..., v_{2n})$. At moment $t = 2$, after the transactions, the value of the shares from i-th asset will be $\zeta''_{2i} = \zeta'_{2i} + u_{2i} - v_{2i} = (1+\xi_i)^2(b_i + u_{0i} - v_{0i}) + (1+\xi_i)(u_{1i} - v_{1i}) + u_{2i} - v_{2i}$.

Since ζ''_{2i} must be nonnegative, the following inequality must hold

$$(1+d_i)^2\,(b_i + u_{0i} - v_{0i}) + (1+d_i)(u_{1i} - v_{1i}) + u_{2i} - v_{2i} \geq 0.$$

This is equivalent to $a_{2i}(b_i + u_{0i} - v_{0i}) + a_{1i}(u_{1i} - v_{1i}) + u_{2i} - v_{2i} \geq 0.$

Since the sum of money withdrawn by the investor at moment $t = 2$ is W_2 the following equality must hold:

$$(1-q_2)\sum_{i=1}^{n} v_{2i} - \frac{1}{1-q_1}\sum_{i=1}^{n} u_{2i} = W_2.$$

By the repeated application of the above arguments we obtain that for every $i \in \{1,2,...,n\}$ the following equalities hold:

$$\zeta'_{0i} = b_i \qquad \zeta'_{ri} = (1+\xi_i)\,\zeta''_{r-1,i}, \text{ for every } r \in \{1,2,...,m\}$$

$$\zeta''_{r,i} = \zeta'_{r,i} + u_{r,i} - v_{r,i} \text{ for every } r \in \{0,1,2,...,m-1\}$$

Hence for every $i \in \{1,2,...,n\}$ and $r \in \{0,1,2,...,m-1\}$ we have

$$\zeta'_{ri} = (1+\xi_i)\,\zeta''_{r-1,i} = (1+\xi_i)(\zeta'_{r-1,i} + u_{r-1,i} - v_{r-1,i}) =$$
$$= \eta_{ri}(b_i + u_{0i} - v_{0i}) + \eta_{r-1,i}(u_{1i} - v_{1i}) + \eta_{r-2,i}(u_{2i} - v_{2i}) + ... + \eta_{1,i}(u_{r-1,i} - v_{r-1,i})$$

For every $r \in \{0,1,2,...,m-1\}$ we have $(1-q_2)\sum_{i=1}^{n} v_{ri} - \frac{1}{1-q_1}\sum_{i=1}^{n} u_{ri} = W_r.$

A sufficient condition that $\zeta''_{ri} \geq 0$ for every $i \in \{1,2,...,n\}$ and $r \in \{0,1,2,...,m-1\}$ is

$$b_i + u_{0i} - v_{0i} \geq 0, \quad i = 1,2,...,n \quad \text{and for every } i \in \{1,2,...,n\} \text{ and } r \in \{0,1,2,...,m-1\}$$

$$a_{ri} b_i + u_{ri} - v_{ri} + \sum_{s=1}^{r-1} a_{r-s,i}(u_{si} - v_{si}) \geq 0$$

The value of the shares from i-th asset owned by the investor at moment $t = m$ will be

$$\zeta'_{mi} = (1+\xi_i)\zeta''_{m-1,i} = b_i\eta_{mi} + \sum_{j=0}^{m-1}\eta_{m-j,i}(u_{ji} - v_{ji}).$$

The value of the assets in the investor's portfolio at moment $t = m$ is equal to:

$$e^T\zeta'_m = \sum_{i=1}^{n}\zeta'_{mi} = \sum_{i=1}^{n}b_i\eta_{mi} + \sum_{i=1}^{n}\sum_{j=0}^{m-1}\eta_{m-j,i}(u_{ji} - v_{ji}) = \eta_m^T b + \sum_{j=0}^{m-1}\eta_{m-j}^T(u_j - v_j).$$

and the expected value of it is equal to

$$E(e^T\zeta'_m) = E(\eta_m^T b) + \sum_{j=0}^{m-1}E(\eta_{m-j}^T(u_j - v_j)) = \nu_m^T b + \sum_{j=0}^{m-1}\nu_{m-j}^T(u_j - v_j).$$

We denoted by e the n-dimensional vector of ones, that is $e = (1,1,\ldots,1)^T$.

The risk of the investment will be measured by the variance of the random variable $e^T\zeta'_m$.

One can easily see that:

$$D^2(e^T\zeta'_m) = E\left[(e^T\zeta'_m - E(e^T\zeta'_m))^T(e^T\zeta'_m - E(e^T\zeta'_m))\right] =$$

$$= E\left[\left((\eta_m - \nu_m)^T b + \sum_{j=0}^{m-1}(\eta_{m-j} - \nu_{m-j})^T(u_j - v_j)\right)^T \cdot \left((\eta_m - \nu_m)^T b + \sum_{k=0}^{m-1}(\eta_{m-k} - \nu_{m-k})^T(u_k - v_k)\right)\right] =$$

$$= E\left[b^T(\eta_m - \nu_m)(\eta_m - \nu_m)^T b + 2\sum_{j=0}^{m-1}b^T(\eta_m - \nu_m)(\eta_{m-j} - \nu_{m-j})^T(u_j - v_j) + \right.$$

$$\left. + \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}(u_j - v_j)^T(\eta_{m-j} - \nu_{m-j})(\eta_{m-k} - \nu_{m-k})^T(u_k - v_k)\right] =$$

$$= b^T C_{m,m} b + 2\sum_{j=0}^{m-1}b^T C_{m,m-j}(u_j - v_j) + \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}(u_j - v_j)^T C_{m-j,m-k}(u_k - v_k).$$

We made the following notations: $C_{j,k} = E\left[(\eta_j - \nu_j)(\eta_k - \nu_k)^T\right]$, $j,k \in \{1,2,\ldots,m\}$.

In the portfolio selection models we are going to study we shall suppose that the investor is rational, that is at the moments $t = 0,1,\ldots, m$ - 1 when the investor is rebalancing his portfolio, he will not purchase assets which he will sell at the same moment. In this case we can write that:

(*) $$u_{ri}v_{ri} = 0, \quad \text{for every} \quad i \in \{1,2,\ldots,n\} \text{ and } r \in \{0,1,2,\ldots,m-1\}$$

Since $u_{ri} \geq 0, v_{ri} \geq 0$, for every $i \in \{1,2,\ldots,n\}$ and $r \in \{0,1,2,\ldots,m-1\}$ it follows that the condition (*) may be written in vector form as:

(**) $$u_r^T v_r = 0 \qquad \text{for every} \quad r \in \{0,1,2,\ldots,m-1\}$$

We shall call the condition (**) the investor's rationality condition.

3. The risk minimization model

In this model the investor seeks to:

- withdraw for consumption at moments $t = 0, t = 1, \ldots, t = m - 1$, the following sums of money W_0, $W_1,\ldots, W_{m-1}$ from his portfolio. To do this he will rebalance his portfolio at moments $t = 0, t = 1, \ldots, t = m - 1$ by purchasing and selling assets of his portfolio.
- to obtain at moment $t = m$ an expected return greater or equal than W_m
- to obtain a minimum risk for the return at moment $t = m$.

Denote by e the n-dimensional vector of ones, that is $e = (1,1,\ldots,1)^T$.

The equations of the risk minimization model are:

$$\begin{cases} \min\left[\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}(u_j - v_j)^T C_{m-j,m-k}(u_k - v_k) + 2\sum_{j=0}^{m-1} b^T C_{m,m-j}(u_j - v_j)\right] \\ e^T\left[(1-q_2)v_r - \frac{1}{1-q_1}u_r\right] = W_r,\ r = 0,1,2,\ldots,m-1 \\ \nu_m^T b + \sum_{j=0}^{m-1}\nu_{m-j}^T(u_j - v_j) \geq W_m \qquad a_{ri}b_i + u_{ri} - v_{ri} + \sum_{j=0}^{r-1} a_{r-j,i}(u_{ji} - v_{ji}) \geq 0,\ r = 1,2,\ldots,m-1, \quad i = 1,2,\ldots,n \\ b + u_0 - v_0 \geq 0 \qquad u_j{}^T v_j = 0, \qquad u_j \geq 0,\ v_j \geq 0,\ j = 0,1,2,\ldots,m-1. \end{cases}$$

We note that the input data of the model are:

- the covariance matrices $C_{jk}, j,k \in \{0,1,2,\ldots,m\}$
- vectors of means ν_r, $r = 1,2,\ldots,m$
- the nonnegative vector $b=(b_1, b_2,\ldots,b_n)$ of initial holdings.
- the transaction commissions (percent) for purchasing q_1 and selling q_2.
- the matrix $A = (a_{ri})$

Parameters whose values are selected by the investor are:

- the length of the time unit (period). The estimation of the random vector ξ is based on this length.
- the length of the time horizon
- the sums of money $W_0, W_1,\ldots,W_{m-1}$ which are withdrawn at the moments $t = 0,1,2,\ldots,m-1$
- the sum of money W_m which represents a lower bound for the expected return of the investor at moment $t = m$. Of course when the investor chooses W_m he must take into account the sums of money $W_0, W_1,\ldots,W_{m-1}$.

The decision variables are:

- The sequence of portfolios $u_0, u_1,\ldots,u_{m-1}$ purchased by the investor
- The sequence of portfolios $v_0, v_1,\ldots,v_{m-1}$ sold by the investor

The aim of the investor is to find the sequence of portfolios u_j, v_j, $j = 0,1,2,...,m$-1 which minimizes the objective map.
The multiperiod portfolio selection model lead us to a special quadratic programming problem in which are present complementarity restrictions (**). Unfortunately, the models presented here cannot be solved efficiently at the moment. A portfolio with a few (two or three) risky assets and ten periods might be tractable. A realistic portfolio with 30 or more assets and two or three periods will need too much time to be solved.
Taking the transaction cost portfolio optimisation model we can analyse how the results change compared to the case when we disregard transaction costs.

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