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Research Report

Single period portfolio selection models with complementarity constraints

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Abstract. In this paper we define a class of mathematical programming problems with complementarity constraints. This type of constraints increase the difficulty of the problems, which now enter in the category of combinatorial optimization problems. The set of feasible solutions for the problems from the above mentioned class is the union of a set of convex sets but it is no longer convex.

We give an algorithm for finding the solution of problems with complementarity constraints which supposes the resolution of a sequence of mathematical programming problems with convex set of feasible solutions. The algorithm requires a great computational effort.

Applications to portfolio selection models which include transaction costs and initial holdings for the investor are given. For the case of linear transaction costs we show that our portfolio selection models are equivalent to mathematical programming problems with nonsmooth constraints.

1. Introduction

Mathematical programming problems with complementarity constraints arise in many domains, such as engineering, mathematical economy, mathematical physics etc. For references and discussions on applications of the optimizations problems with complementarity constraints see Ferris and Pang [8] and Luo et. al.[18]. The complementarity constraints increase the difficulty of the problems, which now enter in the category of combinatorial problems.

There are many approaches to solving mathematical programming problems with complementarity constraints. See for example Fukushima et al [10], Fukushima and Pang [11], Fukushima and Tseng [12], Jeroslow [13], Jiang and Ralph [14], Liu and Zhang [16], Luo et al [17], Mangasarian and Pang [19], Outrata [24], Ralph [29], Tajima et al [33].

Optimality conditions for optimization problems with complementarity constraints were studied in Scheel and Scholtes [30] and Ye [35].

In this paper we introduce the following class of mathematical programming problems with complementarity constraints.

$$\text{(A)} \quad \begin{cases} \min f(u,v) \\ (u,v) \in X \\ u^T v = 0, \quad u \geq 0, v \geq 0 \end{cases}$$

Here X is a compact convex set of $\mathbf{R}^n \times \mathbf{R}^n$ and $f : X \to \mathbf{R}$ is a convex continuous function. Note that the set of feasible solutions of the problem (A) is not convex.

In order to solve the above problem we consider the problem

$$\text{P(r)} \quad \begin{cases} \min u^T v \\ f(u,v) \leq r \\ (u,v) \in X \\ u \geq 0, v \geq 0 \end{cases}$$

Note that the set of feasible solutions of the problem (P(r)) is convex. This gives us the possibility to apply classical local search algorithms for the resolution of the problem (P(r)).

Let γ be the optimal value of the objective map of the problem (A). If $r > \gamma$ then one can find a solution (u_1, v_1) of P(r) such that $u_1^T v_1 = 0$. In this case we decrease r and we solve again (P(r)). If the optimal solution satisfies the complementarity constraint we decrease r and we solve again

(P(r)). If the optimal solution does not satisfy the complementarity constraint we increase r and we solve again (P(r)). We obtain with this procedure a sequence (r_k) which is convergent to γ.

In the following we formulate a few portfolio selection models which include transaction costs and initial holdings for the investor. The portfolio selection models have complementarity constraints.

The standard portfolio model is one where an investor allocates his initial wealth among n risky assets. Assuming the assets returns are stochastic, Harry Markowitz (the 1990 Nobel laureate) in his pioneering work [20] described a theory postulating that rational investors should select a portfolio from the set of all "feasible" portfolios which offers minimum risk for a given level of expected return and maximum expected return for a given level of risk. Such a portfolio is said to be efficient. The objective of the portfolio selection problem is then to determine the set of efficient portfolios.

Risk is measured in terms of the variance of the total return. The above model of Markowitz is known as the mean-variance model. Since its introduction in the fifties, the model has dominated a great deal of the literature in portfolio analysis.

The classical Markowitz's portfolio selection model is a single period model which does not suppose the existence of initial holdings for the investor and does not takes into account the transaction costs. It is static and assumes a frictionless market. Pogue [23] had extended Markowitz's portfolio selection model to include transaction costs, short sales, leverage policies and taxes.

The transaction costs are an increasing function of the size of the transaction. There are transaction costs for purchasing assets and for selling assets. Transaction costs consist of portfolio management fees and trade commissions. In literature are considered three types of transaction costs. The first term is a fixed fee for the basic services of a bank account. The second term summarizes the transaction commission. These fees can vary with different asset classes. The third term is referred to a fee, which is proportional to the investor's wealth. It is some sort of management fee which is charged regularly for the management of a certain portfolio and it depends on the current wealth of the portfolio. The mathematical problem of optimally managing a portfolio of securities in the presence of transaction costs has received considerable research attention in recent years (see [1]-[7], [9], [15], [22], [25]-[28],[31]). Models usually differ with respect to the design of the transaction cost function. An interesting discussion on the transaction costs can be found in [32].

For the classical problem where the objective is to maximize expected utility of terminal wealth, most of the attention has been devoted to the case of proportional transaction costs, that is, to the

case where the cost associated with a transaction is proportional to the amount of money that is shifted between the securities. Representative of work in this category are the papers by Cvitanic and Karatzas [4], Davis and Norman [5], Taksar et al. [34], Fleming et al. [9], and Shreve and Soner [31]. Other authors assumed the transaction cost has a fixed component, thereby precluding the optimality of continuous trading. Morton and Pliska [21] assumed that the transaction cost is proportional to the current value of the portfolio. There exists a large literature in this direction. For further references on this subject see [27], [28].

In this paper we intend to extend Markowitz's portfolio selection model to a single period model, which include transaction costs in the presence of initial holdings for the investor. Our approach is new. The investor purchases and sells assets at the same time. The fact that there is no asset which is purchased and sold at the same time introduces in the model constraints called complementarity constraints. Our aim is to obtain an optimal portfolio which has a minimum risk or a maximum return. The portfolio selection models developed in this paper are mathematical programming problems with complementarity constraints.

In the following we shall suppose that:

- There exists a single period. The beginning of the period is the moment $t = 0$ and the end of the period is the moment $t = 1$. There are n assets were the investor allocates his wealth.
- At moment $t = 0$ the initial holdings of the investor are described by the vector $\mathbf{b} = (b_1, b_2, ..., b_n)$, that is the investor had purchased before moment $t = 0$ a quantity b_i of asset i, i=1,2.....,n.
- The transaction costs for the assets are nonlinear maps (usually they are piecewise linear) of the volume of assets purchased or sold. We shall suppose that transaction costs depend on the traded asset and they are represented by fees paid by the investor when the assets are traded. We shall denote by $c_{1i}(M)$ the fee paid by the investor for purchasing a quantity M of asset i. We shall denote by $c_{2i}(M)$ the fee paid by the investor for selling a quantity M of asset i. Denote by $c_{3i}(M)$ the sum of money the investor must pay for purchasing a quantity of asset i with net price M. One can easily see that $c_{3i}(M) - c_{ti}(c_{3i}(M)) = M$ for all $i \in \{1,2,...,n\}$. Note that $0 = c_{3i}(0) = c_{ji}(0) \le c_{ji}(s) < s \le c_{3i}(s)$ for every $s \ge 0, i \in \{1,2,...,n\}, j \in \{1,2\}$ and all the maps c_{1i}, c_{2i}, c_{3i} are non-decreasing.
- We shall denote by ξ_i the rate of return for the asset i, i=1,2, ...,n. Obviously that ξ_i are random variables. In order to be more rigorous we shall consider that a probability space $(\Omega, \mathcal{K}, P)$ was defined and $\xi_i : \Omega \to \mathbf{R}$, i = 1,2,…, n. We shall suppose that for all i we have $\int_{\Omega} |\xi_i(\omega)|^2 \, dP(\omega) < +\infty$.

– We shall denote by μ_i the expected value of ξ_i. Let $\boldsymbol{\xi}=(\xi_1,\xi_2,....,\xi_n)$ be the random vector of the rates of return for the n assets and $\boldsymbol{\mu}=(\mu_1, \mu_2,....,\mu_n)$ the vector of means (expected rates of return).

At moment $t = 0$, the investor wants to make a revision of his portfolio, that is he purchases and sells assets in order to obtain a sum W_0 . If W_0 is positive, the investor will put it in his pocket (that is the investor will use the money for his personal consumption). If W_0 is negative the investor will use the sum $|W_0|=-W_0$ from his own pocket to purchase assets. To do this the investor will purchase a portfolio of assets $\mathbf{u}=(u_1,u_2,...,u_n)$ and will sell a portfolio of assets $\mathbf{v}=(v_1,v_2,...,v_n)$. Let $\mathbf{x}=\mathbf{b}+\mathbf{u}-\mathbf{v}$ be the investor' s portfolio at moment $t = 0$ after the revision. To purchase the portfolio u the investor will need a sum of money equal to $\sum_{i=1}^{n} c_{3i}(u_i)$. When the investor sells the portfolio v, he will obtain a sum of money equal to

$$\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} c_{2i}(v_i) \qquad (1)$$

Consequently we can write the equality:

$$W_0 = \sum_{i=1}^{n} v_i - \sum_{i=1}^{n} c_{2i}(v_i) - \sum_{i=1}^{n} c_{3i}(u_i) \qquad (1)$$

.

Let $\mathbf{c}_j : \mathbf{R}_+^n \to \mathbf{R}_+^n$, j = 1,2,3 be defined as follows:

$\mathbf{c}_j(v_1, v_2, \ldots, v_n) = (c_{j1}(v_1), c_{j2}(v_1), \ldots, c_{jn}(v_1))$ for every $(v_1, v_2, \ldots, v_n) \in \mathbf{R}_+^n$.

Let $\mathbf{e}=(1,1,...,1)^{\mathrm{T}}$ be the n dimensional vector having all coordinates equal to 1.

Then equation (2) can be written in the vectorial form as follows:

$$W_0 = \mathbf{e}^T(\mathbf{v} - \mathbf{c}_2(\mathbf{v}) - \mathbf{c}_3(\mathbf{u})) \qquad (3)$$

Obviously we must have:

$b_i + u_i - v_i \geq 0$, i=1,2,...n. (4)

At moment $t = 1$ the investor's return will be:

$$\xi^T\mathbf{x} - \mathbf{e}^T\left(\mathbf{c}_2(\mathbf{v}) + \mathbf{c}_1(\mathbf{u})\right) = \sum_{i=1}^{n} \xi_i x_i - \mathbf{e}^T\left(\mathbf{c}_2(\mathbf{v}) + \mathbf{c}_1(\mathbf{u})\right) = \sum_{i=1}^{n} \xi_i\left(b_i + u_i - v_i\right) - \left(c_{2i}(\mathbf{v}) + c_{1i}(\mathbf{u})\right) \qquad (2)$$

The expected value of portfolio return will be:

$$\mathrm{E}\left(\boldsymbol{\xi}^T\mathbf{x}\right)-\mathbf{e}^T\left(\mathbf{c}_2(\mathbf{v})+\mathbf{c}_1(\mathbf{u})\right)=\sum_{i=1}^{n}\mathrm{E}(\xi_i)x_i-\mathbf{e}^T\left(\mathbf{c}_2(\mathbf{v})+\mathbf{c}_1(\mathbf{u})\right)= \tag{3}$$

$$=\boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v})-\mathbf{e}^T\left(\mathbf{c}_2(\mathbf{v})+\mathbf{c}_1(\mathbf{u})\right)$$

In the portfolio selection models we are going to study we shall suppose that the investor is rational, that is at moment $t = 0$, when the investor is rebalancing his portfolio consisting of his initial holdings, he will not purchase assets which he will sell at the same moment.

In this case we can write that:

$$u_i v_i = 0, \quad \text{for every} \quad i \in \{1,2,\ldots,n\} \tag{4}$$

Since $u_i \geq 0, v_i \geq 0$, for every $i \in \{1,2,\ldots,n\}$ it follows that the condition (7) may be written in vectorial form as:

$$\mathbf{u}^T\mathbf{v} = 0 \tag{5}$$

We shall call the condition (8) the investor's rationality condition or the complementarity constraint. From (4) and (7) we obtain that $0 \leq v_i \leq b_i$ for all *i*.

2. The risk minimization model

We shall define the investment risk as the variance of the random variable return. Let $\sigma_{ij} = E\left((\xi_i - \mu_i)(\xi_j - \mu_j)\right), \quad 1 \leq i, j \leq n$ and $\mathbf{C}=(\sigma_{ij})$. Note that σ_{ij} represents the covariance of the random variables ξ_i and ξ_j. The matrix $\mathbf{C}$ is the covariance matrix for the random vector $\boldsymbol{\xi}$. One can easily see that the variance of the return is equal to:

$$D^2\left(\boldsymbol{\xi}^T\mathbf{x}\right)=E\left(\left(\boldsymbol{\xi}^T\mathbf{x}-\boldsymbol{\mu}^T\mathbf{x}\right)^T\left(\boldsymbol{\xi}^T\mathbf{x}-\boldsymbol{\mu}^T\mathbf{x}\right)\right)=$$

$$=E\left(\mathbf{x}^T(\boldsymbol{\xi}-\boldsymbol{\mu})(\boldsymbol{\xi}-\boldsymbol{\mu})^T\mathbf{x}\right)=\mathbf{x}^T E\left((\boldsymbol{\xi}-\boldsymbol{\mu})(\boldsymbol{\xi}-\boldsymbol{\mu})^T\right)\mathbf{x}=\mathbf{x}^T\mathbf{C}\mathbf{x}$$

In this model the investor seeks to:

- withdraw for consumption at moment $t = 0$ a sum of money equal to W_0 from his portfolio. To do this he will rebalance his portfolio by purchasing and selling assets of his portfolio (initial holdings).
- obtain with minimum risk, at moment $t = 1$ an expected net return at least equal to W_1.

In the vectorial form the risk minimization model can be written as follows:

$$\begin{cases} \min\left[(\mathbf{b}+\mathbf{u}-\mathbf{v})^T \mathbf{C} (\mathbf{b}+\mathbf{u}-\mathbf{v}) \right] \\ W_0 = \mathbf{e}^T \left(\mathbf{v} - \mathbf{c}_2(\mathbf{v}) - \mathbf{c}_3(\mathbf{u}) \right) \\ \boldsymbol{\mu}^T (\mathbf{b}+\mathbf{u}-\mathbf{v}) - \mathbf{e}^T \left(\mathbf{c}_2(\mathbf{v}) + \mathbf{c}_1(\mathbf{u}) \right) \geq W_1 \\ \mathbf{u}^T \mathbf{v} = 0, \mathbf{u} \geq \mathbf{0}, \mathbf{0} \leq \mathbf{v} \leq \mathbf{b} \end{cases} \qquad (6)$$

Note that the input data in the model is represented by:

– the covariance matrix of the rates of return $\mathbf{C}=(\sigma_{ij})$

– the vector of the mean (expected) rate of return $\boldsymbol{\mu}=(\mu_1, \mu_2,\ldots,\mu_n)$

– the vector $\mathbf{b}=(b_1,b_2,\ldots,b_n)$, of investor's initial holdings, that is the investor portfolio at moment $t = 0$.

The user's parameters are represented by the numbers:

– W_0 = the sum the investor wish to withdraw for consumption at moment $t = 0$.

– W_1 = the inferior limit for the return the investor wish to obtain at moment $t = 1$. (Obviously W_1 does not contain W_0 in its definition).

The decision variables are represented by the portfolios **u** and **v** the investor purchases and sells. The purpose of the model is to find the vectors **u** and **v**, which minimize the objective map.

One obtains a quadratic programming problem. We note that not all the restrictions which define the set of feasible solutions are given by linear maps. Therefore the set of feasible solutions is not convex. This is due to the complementarity restrictions (7). One can easily see that the set of feasible solutions, in the case of linear transaction costs, is the union of 2^n convex sets and these sets are not disjoint. In order to compute the exact solution of problem (9), an exhaustive combinatorial search can be performed. We shall show that the above number of convex sets can be reduced from 2^n to 2^k where k is the number of assets where the investor has initial holdings.

We can solve a quadratic programming problem with linear restrictions for each convex component of the set of feasible solutions. Because of the computational effort this approach becomes difficult for k larger than 10 and certainly intractable for k larger than 25. The exhaustive search can be improved by using branch and bound methods, which can reduce the required computational effort. These methods require a procedure for computing upper and lower bounds for the objective map.

Another approach for solving the problem (9) is by using computer simulation.

An important problem for the investor is to find the restrictions he might take into account when he chooses the parameters W_0 and W_1. We shall consider that the investor first chooses the parameter W_0 and then the parameter W_1 . Denote

$\mathbf{c}_4(\mathbf{v}) = \mathbf{v} - \mathbf{c}_2(\mathbf{v}), \mathbf{v} \geq 0$.

If all the maps c_{4i} are nondecreasing then the parameter W_0 can be chosen everywhere in the interval $\left(-\infty, \mathbf{e}^T \mathbf{c}_4(\mathbf{b})\right]$.

Denote $M = \left\{(\mathbf{u}, \mathbf{v}) \in \mathbf{R}^{2n} : \mathbf{u}^T \mathbf{v} = 0, \mathbf{u} \geq \mathbf{0}, \mathbf{0} \leq \mathbf{v} \leq \mathbf{b}\right\} Q(W_0) = \left\{(\mathbf{u}, \mathbf{v}) \in M : W_0 = \mathbf{e}^T (\mathbf{v} - \mathbf{c}_2(\mathbf{v}) - \mathbf{c}_3(\mathbf{u}))\right\}$,
$L(\mathbf{u}, \mathbf{v}) = \boldsymbol{\mu}^T (\mathbf{b} + \mathbf{u} - \mathbf{v}) - \mathbf{e}^T (\mathbf{c}_2(\mathbf{v}) + \mathbf{c}_1(\mathbf{u}))$

$W_1' = \min\{L(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Q(W_0)\}$, $W_1'' = \max\{L(\mathbf{u},\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Q(W_0)\}$

The investor can choose the parameter W_1 everywhere in the closed interval $\left[W_1', W_1''\right]$. In this case the set of feasible solutions for problem (9) is not empty.

3. The maximization of the average return model

In the framework of this model the investor seeks to purchase and sell assets so that:

– he will maximize his average net return at the moment $t = 1$ keeping the risk under a prescribed level r.

– he will withdraw for consumption a sum W_0 at moment $t = 0$.

The equations for the maximization of the average return model are:

$$\begin{cases} \max\left[\boldsymbol{\mu}^T (\mathbf{b} + \mathbf{u} - \mathbf{v}) - \mathbf{e}^T (\mathbf{c}_2(\mathbf{v}) + \mathbf{c}_1(\mathbf{u}))\right] \\ W_0 = \mathbf{e}^T (\mathbf{v} - \mathbf{c}_2(\mathbf{v}) - \mathbf{c}_3(\mathbf{u})) \\ (\mathbf{b} + \mathbf{u} - \mathbf{v})^T \mathbf{C} (\mathbf{b} + \mathbf{u} - \mathbf{v}) \leq r \\ \mathbf{u}^T \mathbf{v} = 0, \mathbf{u} \geq \mathbf{0}, \mathbf{0} \leq \mathbf{v} \leq \mathbf{b} \end{cases} \quad (7)$$

The investor can choose the parameter W_0 taking into account the same restrictions as in the preceding model. Denote:

$r_1 = \min\left\{(\mathbf{b} + \mathbf{u} - \mathbf{v})^T \mathbf{C} (\mathbf{b} + \mathbf{u} - \mathbf{v}) : (\mathbf{u}, \mathbf{v}) \in Q(W_0)\right\}$

and

$$r_2 = \max\left\{(\mathbf{b}+\mathbf{u}-\mathbf{v})^T \mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v}) : (\mathbf{u},\mathbf{v}) \in Q(W_0)\right\}.$$

Then the investor can choose the parameter r everywhere in the closed interval $[r_1, r_2]$.

4. The mean-variance model

This model studies the case of an investor who is characterized by a risk aversion coefficient. We shall denote the risk aversion coefficient by θ and we shall suppose that $\theta \in [0,1]$. The case when θ takes small values, that is near 0, corresponds to the case the investor has a great aversion towards risk, that is safety is essential for the investor and the amount of the return is not so important. The case when θ takes great values, that is near 1, corresponds to the case of a risk lover investor, that is the amount of the return is essential and the safety is not so important.

The objective map of the model is a linear combination between the risk and the expected value of the portfolio return. It depends on the investor's risk aversion coefficient θ. The aim of the investor is to minimize the objective map and to withdraw for consumption at moment $t = 0$ a sum W_0.

Let

$$f(\mathbf{u},\mathbf{v}) = (1-\theta)(\mathbf{b}+\mathbf{u}-\mathbf{v})^T \mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v}) - \theta\left(\boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v}) - \mathbf{e}^T(\mathbf{c}_2(\mathbf{v})+\mathbf{c}_1(\mathbf{u}))\right)$$

The equations for the mean-variance model are:

$$\begin{cases} \min\left[f(\mathbf{u},\mathbf{v})\right] \\ W_0 = \mathbf{e}^T\left(\mathbf{v} - \mathbf{c}_2(\mathbf{v}) - \mathbf{c}_3(\mathbf{u})\right) \\ \mathbf{u}^T\mathbf{v} = 0,\ \mathbf{u} \geq \mathbf{0},\ \mathbf{0} \leq \mathbf{v} \leq \mathbf{b} \end{cases} \quad (8)$$

Since the above models, in the general case are very complex, it is necessary to use computer simulation in order to solve them. We used (see [25]) Monte Carlo simulation to solve the optimization portfolio selection problems of the above mentioned type and some multiperiod portfolio models. The experiments using Monte Carlo simulation aim to obtain an optimal portfolio, which has a minimum risk or a maximum return. We built a portfolio simulation module (PORTOPT) that simulates the income produced by the portfolio during the horizon period and computes market value at the end of the horizon. Total return, capital gains and losses and income are all computed.

Our simulation module is integrated in a risk management decision support system.

5. The case of proportional transaction costs

In this section we shall formulate the minimum risk model in the case the cost for each transaction is proportional to the amount traded. If we denote by q_{ji} the transaction cost rate for c_{ji} then $c_{ji}(s)=q_{ji}s$, $s\geq 0$, j=1,2, i=1,2,…,n. One can easily see that $q_{ji}\in[0,1)$ and $c_{3i}(s)=q_{3i}s$, $s\geq 0$, where $q_{3i}=\dfrac{1}{1-q_{1i}}$ i=1,2,…,n , j=1,2. Let $\mathbf{q}_j=(q_{j1},q_{j2},\ldots,q_{jn})$, j=1,2,3.

The minimum risk model in the case of proportional transaction costs is:

$$\begin{cases}\min\left[(\mathbf{b}+\mathbf{u}-\mathbf{v})^T\mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v})\right]\\ W_0=(\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}-\mathbf{q}_3^T\mathbf{u}\\ \boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v})-\mathbf{q}_2^T\mathbf{v}-\mathbf{q}_1^T\mathbf{u}\geq W_1\\ \mathbf{u}^T\mathbf{v}=0,\ \mathbf{u}\geq\mathbf{0},\ \mathbf{0}\leq\mathbf{v}\leq\mathbf{b}\end{cases}\qquad(9)$$

If a is a real number we shall denote

$$a_+=\max(a,0)=\frac{|a|+a}{2},\quad a_-=\max(-a,0)=\frac{|a|-a}{2}.$$

Recall that $\mathbf{x}=\mathbf{b}+\mathbf{u}-\mathbf{v}$ is the investor's portfolio at moment $t=0$ after the revision.

Then the following equalities hold: $u_i=(x_i-b_i)_+$, $v_i=(x_i-b_i)_-=(b_i-x_i)_+$. Note that equality (8) holds. If we denote

$\alpha_i=\dfrac{1-q_{2i}-q_{3i}}{2}$ and $\beta_i=\dfrac{1-q_{2i}+q_{3i}}{2}$ then

$$W_0=(\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}-\mathbf{q}_3^T\mathbf{u}=\sum_{i=1}^{n}(v_i-q_{2i}v_i-q_{3i}u_i)=$$

$$=\sum_{i=1}^{n}((b_i-x_i)_+-q_{2i}(b_i-x_i)_+-q_{3i}(x_i-b_i)_+)=$$

$$=\sum_{i=1}^{n}\left[(1-q_{2i})\frac{|b_i-x_i|+b_i-x_i}{2}-q_{3i}\frac{|b_i-x_i|+x_i-b_i}{2}\right]=$$

$$=\sum_{i=1}^{n}\alpha_i|x_i-b_i|-\sum_{i=1}^{n}\beta_i(x_i-b_i)$$

Thus an alternative formulation of the minimum risk model is the following:

$$\begin{cases} \min\left(\mathbf{x}^T\mathbf{C}\mathbf{x}\right) \\ \sum_{i=1}^{n}\alpha_i\left|x_i-b_i\right|-\sum_{i=1}^{n}\beta_i\left(x_i-b_i\right)=W_0 \\ \boldsymbol{\mu}^T\mathbf{x}-\mathbf{q}_2^T\mathbf{v}-\mathbf{q}_1^T\mathbf{u}\geq W_1 \\ \mathbf{x}\geq 0 \end{cases} \tag{10}$$

The main difficulty in solving the above model is the first restriction, which contains a nonsmooth function.
In the special case when the system:

$$\begin{cases} \mathbf{C}\mathbf{x}=0 \\ \sum_{i=1}^{n}\alpha_i\left|x_i-b_i\right|-\sum_{i=1}^{n}\beta_i\left(x_i-b_i\right)=W_0 \\ \boldsymbol{\mu}^T\mathbf{x}-\mathbf{q}_2^T\mathbf{v}-\mathbf{q}_1^T\mathbf{u}\geq W_1 \\ \mathbf{x}\geq 0 \end{cases} \tag{11}$$

has at least one solution one can obtain all the solutions of the minimum risk model. Indeed, one can easily see that all the solutions $(\mathbf{u},\mathbf{v})$ of the system (14) are optimal solutions of (13).

6. Formulation of the minimum risk model as a mixed-binary model.

Suppose that $b_1\geq b_2\geq\ldots b_k>b_{k+1}=b_{k+2}=\ldots=b_n=0$.
Consider the vector $\tilde{\mathbf{b}}=\left(\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_n\right)$ where

$$\tilde{b}_i=\begin{cases}1 & \text{if} \quad i\in\{1,2,\ldots,k\}\\ 0 & \text{if} \quad i\in\{k+1,k+2,\ldots,n\}\end{cases}$$

Let $\mathbf{D}=\operatorname{diag}\left(\left(1+\mathrm{q}_{11}\right)^{-1},\left(1+\mathrm{q}_{12}\right)^{-1},\ldots,\left(1+\mathrm{q}_{1n}\right)^{-1}\right)$, $\mathbf{B}$=diag($\mathbf{b}$).
The minimum risk problem can be formulated as a mixed binary problem as follows

$$\begin{cases} \min\left[(\mathbf{b}+\mathbf{u}-\mathbf{v})^T\mathbf{C}(\mathbf{b}+\mathbf{u}-\mathbf{v})\right] \\ W_0=(\mathbf{e}-\mathbf{q}_2)^T\mathbf{v}-(\mathbf{e}+\mathbf{q}_1)^T\mathbf{u} \\ \boldsymbol{\mu}^T(\mathbf{b}+\mathbf{u}-\mathbf{v})-\mathbf{q}_2{}^T\mathbf{v}-\mathbf{q}_1\mathbf{u}\geq W_1 \\ \mathbf{0}\leq\mathbf{u}\leq\left[(\mathbf{e}-\mathbf{q}_2)^T\mathbf{b}-W_0\right]\mathbf{D}(\mathbf{e}-\mathbf{w}) \\ \mathbf{0}\leq\mathbf{v}\leq\mathbf{B}\mathbf{w},\ \mathbf{w}\leq\tilde{\mathbf{b}},\ \mathbf{w}\in\{0,1\}^n \end{cases}$$

The decision variables are the vectors **u,v,w**. **u** and **v** are vectors in $\mathbb{R}^n$ and **w** is a vector with binary entries.

7. Conclusions

We have presented several single period portfolio selection models in the presence of initial holdings and transaction costs. Since the set of feasible solutions is not convex the objective map may present several points of local minimum (maximum).

An efficient approach in the attempt to find a global minimum (maximum) of the objective map is a combination of computer simulation with heuristic algorithms.

We gave conditions under which one can find analytical solutions to our problems and we presented some heuristic algorithms for obtaining suboptimal solutions and an algorithm for finding the optimal solution. A formulation of the minimum risk problem as a mixed binary problem was given. Extensions of our models in order to treat the discrete cases, the multi-period cases and to include some other measures of risk are underway.

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