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Mathematical Models of Diffusion in Nonhomogeneous Porous Media

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Contents

1. Physical context and mathematical hypotheses

From the hydraulic point of view, the problems we shall study are related to a Darcian flow of an incompressible fluid in an isotropic, nonhomogeneous non-deformable porous medium with a variable porosity and with no hysteresis development.

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The general boundary value problem. Assume that the flow domain Ω is an open bounded subset of $\mathbf{R}^N$ ($N = 1, 2, 3$), and the time runs within the finite time interval $(0, T)$. The boundary of Ω is denoted by Γ and it is considered piecewise smooth. The vector of space variables is denoted by $x = (x_1, x_2, x_3) \in \Omega$ and the time by $t \in (0, T)$.

We consider the Richards' equation describing the water infiltration into an isotropic, nonhomogeneous, unsaturated porous medium with a variable porosity, with initial data and various boundary conditions (see [7])

$$\frac{\partial(m(x)S_w(h))}{\partial t} - \nabla \cdot (k(h)\nabla h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q = \Omega \times (0, T), \tag{1.1}$$

$$h(x, 0) = h_0(x) \text{ in } \Omega, \tag{1.2}$$

$$\textit{boundary conditions for } h \text{ on } \Sigma = \Gamma \times (0, T). \tag{1.3}$$

The unknown in Richards' equation is the *capillary pressure* $h(x, t)$ (or *pressure head*, or *water pressure* in the unsaturated soil), S_w is the *water saturation* in pores, m is the medium porosity and $\theta = m(x)S_w$ is the *volumetric water content* or *soil moisture.* In this work the dependence of m on x models the nonhomogeneity of the medium. The function k is the *hydraulic conductivity*, $f(x, t)$ is a source (or sink) in the flow domain and h_0 is the initial pressure distribution in the domain, f and h_0 being given. In general $m \in (0, 1)$ but a limit case with m tending to 0 may have a physical relevance. The properties of the dependence of S_w and k on h will be specified.

In particular, we shall exemplify the theory for the case of the medium having a part of the boundary, Γ_α semipermeable, allowing a water flux across it and the other part Γ_u at which the pressure will be given. Here, Γ_u and Γ_α are disjoint and $\Gamma = \Gamma_u \cup \overline{\Gamma_\alpha}$. In infiltration problems, we can often meet the situation in which water ponds on the soil surface (let it be Γ_u). This happens when the rainfall rate is greater than the soil conductivity at saturation and the soil begins to saturate from the surface, or when the soil surface is in contact with an open water body, for example the bottom of a lake. In consequence the boundary conditions we shall consider are

$$h(x, t) = h_u(x, t) \geq 0 \text{ on } \Sigma_u = \Gamma_u \times (0, T), \tag{1.4}$$

$$q \cdot \nu = f_\alpha \text{ on } \Sigma_\alpha = \Gamma_\alpha \times (0, T), \tag{1.5}$$

where q is the water flux defined by

$$q(x, t) = k(h)i_3 - k(h)\nabla h, \tag{1.6}$$

ν is the outer normal vector at the boundary and i_3 is the unit vector of the Ox_3 axis, downwards directed.

We can reverse the boundary conditions by considering that Γ_α is the soil surface and Γ_u is the underground boundary. Thus we can interpret that the flux through the soil surface, is provided by a water supply as a rain or irrigation and that the lower part of the porous medium is in contact with the phreatic aquifer.

Description of the hydraulic model. The behaviour of an unsaturated soil, i.e., partially filled with water, is completely known from the hydraulic point of view if two functions are given: one is the *retention curve*

$$S_w = \widetilde{C}^*(h), \tag{1.7}$$

linking the *water saturation* S_w, to the pressure head h, and the other is the *hydraulic conductivity*

$$k = k(h), \tag{1.8}$$

both depending nonlinearly on h. For an isotropic soil the latter is a scalar function.

Since we study the nonhysteretic case, the retention curve and the hydraulic conductivity are assumed single-valued functions of the pressure.

In soil sciences, the unsaturated pressure is considered negative ($h < 0$) and the saturation is characterized by $h = 0$. Also, it is considered that the process of infiltration-drainage (opposite to infiltration) takes place between two limits of h. The lowest limit is denoted h_r and at this pressure head the soil is considered dry even if some water still resides in the pores and the hydraulic conductivity is still positive. The corresponding water saturation is denoted S_r and the volumetric water content θ_r is called *residual moisture* (see [7]). The upper limit is $h = 0$ where saturation is reached and water saturation becomes equal to 1. However, we shall denote this value by S_s. At saturation, moisture attains its *saturation value* θ_s equal to the medium porosity at this point (if the porosity is not constant). The parts of the medium where $h > 0$ are completely saturated. We define the derivative of the water saturation with respect to the pressure

$$\widetilde{C}(h) = \frac{\mathrm{d}S_w}{\mathrm{d}h}(h). \tag{1.9}$$

For the saturated flow, when $h \geq 0$, the previously functions take constant values.

Generally, the hydraulic models raise a difficult mathematical problem. When the pressure head in the unsaturated soil comes close to the saturation value, $\widetilde{C}$ vanishes and Richards' equation degenerates. Correspondingly, the diffusion coefficient expressed as a function of moisture exhibits a blow-up development around saturation. In soil sciences the model which reflects this behaviour is the strongly nonlinear Green-Ampt limit model, see [10]. The situation in which $\widetilde{C}(0) > 0$ corresponds to a less nonlinear hydraulic behaviour, the typical model for this class being the Burgers' model, see [10], too. Depending on the particularities of the hydraulic functions which are determined by the soil pore structure, the models of water infiltration range between these two limit models (see [44]).

Previous theoretical results. In the most mathematical literature devoted to this subject the blow-up of the diffusivity in the diffusive form of Richards' equation was avoided, by considering a finite-valued diffusivity, or studying the problem only in the pressure form (see [2], [4], [12], [19], [20], [25], [26], [27], [37], [38]). More recently, in the paper [9] a model of the saturated-unsaturated flow lying on a special definition of the boundary conditions that changes during the phenomenon evolution, has been developed also for a finite value of the diffusivity at saturation (which was implied by the assumption that $\widetilde{C}(0) > 0$). Following the technique presented in [20] the model was reduced to systems in class of Stefan-like problems of high-order, see [19].

However, apart from specific infiltration problems, previous existence and uniqueness studies for solutions to the elliptic-parabolic equation

$$\frac{\partial(b(u))}{\partial t} + \nabla \cdot (a(\nabla u, b(u))) + f(b(u)) = 0 \text{ in } \Omega \times (0,T)$$

have been presented in the literature especially using a technique inspired by the method of entropy solutions introduced by S.N. Krushkov in [28]. Originally, this method was devoted to prove L^1-contraction for entropy solutions for scalar conservation laws, i.e., generalized solutions in the sense of distributions satisfying admissibility conditions similar to those of entropy growth in gas dynamics (see also [8]). J. Carillo applied Krushkov's method to second order equations (see [13], [14], [15], [16]). F. Otto (see [35], [36]) proved a L^1-contraction principle and uniqueness of solutions for this type of equation by applying Krushkov's technique only to the time variable. H.W. Alt and S. Luckhaus showed in [1] that the natural solution space for this equation is given by all functions u of finite energy in the sense that

$$\sup_{t\in(0,T)}\int_\Omega \Psi(b(u(t)))\mathrm{d}x + \int_Q |\nabla u|^r \,\mathrm{d}x\mathrm{d}t < \infty,$$

where Ψ is the Legendre transform of the primitive of b.

We also mention the results of J.L. Vázquez regarding the fast diffusion equations (see [18], [40], [41], [42], [17] and the book [43]).

Concerning the degenerate evolution equations, extensive studies have been performed for linear operators, relying on the properties of the resolvent of an appropriate multivalued linear operator accounting for the multiplication by the function m (see [21], [23] and the monograph [22]). We mention also the paper [24] related to a similar topic in which a degenerate model with homogeneous Dirichlet boundary conditions and no transport was studied.

The analysis of the well-posedness of the diffusive form of Richards' equation in the unsaturated case ($\theta < \theta_s$) with the porosity m constant, was developed in the papers [6], [29], [30], [31] within a functional approach. The existence results which were deduced showed that solutions reaching saturation can be obtained but only on zero-measure subsets of Q. Somehow, this was expected because the unsaturated model reflects a behaviour of a particular soil only and not the general feature of the process which includes the possible soil saturation.

In the paper [32] a rigorous mathematical model able to describe the saturation occurrence (with the blow-up of the diffusivity) was introduced for a homogeneous porous medium (with m constant) in the diffusive form and developed then in [33].

In the first part of this chapter we introduce the diffusive models of water flow in saturated-unsaturated media characterized by a space variation of the porosity. Then we analyze a model with mixed boundary conditions involving a flux on a part of the boundary and a nonhomogeneous Dirichlet condition corresponding to a singular situation on another part of the domain boundary. The model will be degenerate because we shall assume that porosity can vanish on a subset of Ω.

2. Diffusion models in nonhomogeneous porous media

We intend first to reveal how the particular character of the hydraulic models is determined by the behaviour of the functions $\widetilde{C}^*$ and k around 0.

Mathematical hypotheses. For the unsaturated flow, where $h < 0$, we assume the following:

(m_1) $\widetilde{C}^* : [h_r, 0) \to [S_r, S_s)$ is single-valued, positive, differentiable on $[h_r, 0)$, monotonically increasing ;

(m_2) $k : [h_r, 0) \to [K_r, K_s)$ is single-valued, positive, differentiable on $[h_r, 0)$, monotonically increasing and satisfies the property $k'(h_r) = 0$;

(m_3) $\widetilde{C} : [h_r, 0) \to (\widetilde{C}_0, \widetilde{C}_r]$ is single-valued, non-negative, differentiable on $[h_r, 0)$ monotonically decreasing and satisfies $\widetilde{C}'(h_r) = 0$;

In the saturated flow we have

(m_4) $\widetilde{C}^*(h) = S_s$, $k(h) = K_s$ and $\widetilde{C}(h) = 0$ for $h \geq 0$.

We denote

$$S_s = (\widetilde{C}^*)(0) > 0, \tag{2.1}$$

$$\widetilde{C}_0 = (\widetilde{C}^*)'(0) = \widetilde{C}(0) \geq 0, \tag{2.2}$$

$$K_s = k(0) > 0, \tag{2.3}$$

$$K_0' = \lim_{h \nearrow 0} k'(h), \ K_0' \in [0, \infty). \tag{2.4}$$

Therefore, the unsaturated flow is characterized either by $h < 0$ or $S_w \in [S_r, S_s)$ while the saturated one is indicated by $h \geq 0$ or $S_w = S_s$.

The positive values S_r, S_s and their corresponding conductivities K_r, K_s are soil characteristics and they are known for each type of soil apart. The properties $k'(h_r) = 0$ and $\widetilde{C}'(h_r) = 0$ were put into evidence by experiments (see [10]).

We notice that the functions $\widetilde{C}^*$ and k are continuous on $[h_r, \infty)$, and h_r is the maximum point for $\widetilde{C}$. Also $\widetilde{C}$ is continuous on $[h_r, \infty)$, except possibly at the point 0.

We stress the fact that these properties are verified by the empirical hydraulic models set up in the last decades (see e.g., [44]).

We emphasize that the main role is played by the increase rate of the functions $\widetilde{C}^*$ and k around 0, the significant contribution being given by the behaviour of the retention curve $\widetilde{C}^*$.

2.1. Strongly nonlinear saturated-unsaturated diffusive model

Let us assume $(m_1) - (m_4)$ and

$$\widetilde{C}_0 = 0$$

which is the main characteristic of this case. It follows then that $\widetilde{C}$ is continuous on $[h_r, \infty)$ and we can write $\widetilde{C}^* : [h_r, \infty) \to [S_r, S_s]$, as

$$\widetilde{C}^*(h) = \begin{cases} S_r + \int_{h_r}^{h} \widetilde{C}(\zeta)\mathrm{d}\zeta, \ h < 0, \\ S_s, \ h \geq 0. \end{cases} \tag{2.5}$$

Strongly nonlinear hydraulic conductivity. This situation corresponds to $K_0' \in \mathbf{R}_+ = (0, \infty)$.

We define a primitive of K by

$$K^*(h) = \begin{cases} K_r^* + \int_{h_r}^{h} k(\zeta)\mathrm{d}\zeta, \ h < 0, \\ K_s^* + K_s h, \ h \geq 0, \end{cases} \tag{2.6}$$

where $K^* : [h_r, \infty) \to [K_r^*, \infty)$ and

$$K_s^* = K^*(0) > 0. \tag{2.7}$$

The function K^* is differentiable, monotonically increasing on $[h_r, \infty)$ and with these notations Richards' equation (1.1) becomes

$$\frac{\partial(m(x)S_w)}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q. \tag{2.8}$$

By the initial condition (1.2) we obtain

$$S_w(x, 0) = S_{w0}, \ S_{w0} = \widetilde{C}^*(h_0).$$

We can also consider the initial condition

$$m(x)S_w(x, 0) = \theta_0(x) \text{ in } \Omega, \text{ where } \theta_0 = m(x)\widetilde{C}^*(h_0) \tag{2.9}$$

and corresponding replacements should be made in the boundary conditions (1.4)–(1.5).

Since it is more convenient to work with the variable S_w, we introduce from (2.5) the inverse of $\widetilde{C}^*$, $(\widetilde{C}^*)^{-1} : [S_r, S_s] \to [h_r, +\infty)$, by

$$(\widetilde{C}^*)^{-1}(S_w) = \begin{cases} (\widetilde{C}^*)^{-1}(S_w), \ S_w \in [S_r, S_s), \\ [0, +\infty), \ S_w = S_s, \end{cases} \tag{2.10}$$

which is multivalued at $S_w = \theta_s$ and continuous and monotonically increasing on $[S_r, S_s)$. Then, we replace it all over in (1.1)–(1.5).

Thus, instead of the conductivity written in function of pressure, we obtain the conductivity expressed in terms of water saturation

$$\widetilde{K} : [S_r, S_s] \to [K_r, K_s],\ \widetilde{K}(S_w) = (k \circ \widetilde{C}^*)^{-1}(S_w),\ S_w \in [S_r, S_s], \tag{2.11}$$

function that preserves some of the properties of k, i.e., it is positive, differentiable (except at S_s) and monotonically increasing, since for any $S_w \in [S_r, S_s)$ we have that

$$\widetilde{K}'(S_w) = k'((\widetilde{C}^*)^{-1}(S_w)) \cdot ((\widetilde{C}^*)^{-1})'(S_w) = \frac{k'((\widetilde{C}^*)^{-1}(S_w))}{\widetilde{C}((\widetilde{C}^*)^{-1}(S_w))} > 0. \tag{2.12}$$

We notice also that

$$\widetilde{K}'(S_r) = 0 \tag{2.13}$$

and

$$\lim_{S_w \nearrow S_s} \widetilde{K}'(S_w) = +\infty. \tag{2.14}$$

However, for $S_w \in [S_r, S_l]$ with $S_l < S_s$ the derivative of $\widetilde{K}$ is bounded, so that $\widetilde{K}$ follows to be Lipschitz on intervals strictly included in $[S_r, S_s)$

$$\left|\widetilde{K}(S_w) - \widetilde{K}(\overline{S_w})\right| \le M_l \left|S_w - \overline{S_w}\right|,\ \forall S_w, \overline{S_w} \in [S_r, S_l],\ S_l < S_s, \tag{2.15}$$

where

$$M_l = \max_{S_w \in [S_r, S_l]} \frac{k'((\widetilde{C}^*)^{-1}(S_w))}{\widetilde{C}((\widetilde{C}^*)^{-1}(S_w))} < \infty. \tag{2.16}$$

Plugging (2.10) in (2.6) we get the function

$$\widetilde{\beta}^*(S_w) = \begin{cases} (K^* \circ (\widetilde{C}^*)^{-1})(S_w),\ S_w \in [S_r, S_s), \\ [K_s^*, +\infty),\ S_w = S_s \end{cases} \tag{2.17}$$

that is multivalued at $S_w = S_s$ but is continuous from the left at this point

$$\lim_{S_w \nearrow S_s} \widetilde{\beta}^*(S_w) = K_s^*. \tag{2.18}$$

For $S_w \in [S_r, S_s)$ the function $(\widetilde{C}^*)^{-1}$ is monotonically increasing, so that we can calculate $\widetilde{\beta}^*(S_w)$ by changing the variable in the integral (2.6) and denoting $\zeta = (\widetilde{C}^*)^{-1}(\xi)$. In this way we get

$$\widetilde{\beta}^*(S_w) = K_r^* + \int_{S_r}^{S_w} \beta(\xi) \mathrm{d}\xi,\ \text{for } S_w \in [S_r, S_s),$$

where

$$\widetilde{\beta}(S_w) = \frac{k((\widetilde{C}^*)^{-1}(S_w))}{\widetilde{C}((\widetilde{C}^*)^{-1}(S_w))}, \text{ for } S_w \in [S_r, S_s). \tag{2.19}$$

In this way we have rigorously recovered the definition of the *water diffusivity* function.

We notice that $\widetilde{\beta}$ has two important properties

$$\widetilde{\beta}(S_w) \geq \widetilde{\rho} = \widetilde{\beta}(S_r) = \frac{K_r}{\widetilde{C}_r} > 0, \ \forall S_w \in [S_r, S_s) \tag{2.20}$$

and

$$\lim_{S_w \nearrow Ss} \widetilde{\beta}(S_w) = +\infty. \tag{2.21}$$

Moreover, by the hypotheses made upon the functions $\widetilde{C}$ and k it follows that $\widetilde{\beta}$ is monotonically increasing, i.e.,

$$\widetilde{\beta}' = \frac{k'\widetilde{C} - k\widetilde{C}'}{\widetilde{C}^3} \geq 0, \text{ on } [S_r, S_s), \tag{2.22}$$

$$\widetilde{\beta}'(S_r) = 0. \tag{2.23}$$

Hence, $\widetilde{\beta}^*$ is twice differentiable and strictly monotonically increasing on $[S_r, S_s)$ and as a matter of fact we can write

$$\widetilde{\beta}^*(S_w) = \begin{cases} K_r^* + \int_{S_r}^{S_w} \widetilde{\beta}(\xi)\mathrm{d}\xi \text{ for } S_w \in [S_r, S_s), \\ [K_s^*, +\infty) \text{ for } S_w = S_s. \end{cases} \tag{2.24}$$

Moreover, by (2.20) and (2.24) we deduce that the function $\widetilde{\beta}^*$ satisfies the inequality

$$(\widetilde{\beta}^*(S_w) - \widetilde{\beta}^*(\overline{S_w}))(S_w - \overline{S_w}) \geq \rho(S_w - \overline{S_w})^2, \forall S_w, \overline{S_w} \in [S_r, S_s]. \tag{2.25}$$

In conclusion we can set

Model 1. Let us assume $(m_1) - (m_4)$, $\widetilde{C}_0 = 0$ and $K_0' \in \mathbf{R}_+$. Then, the diffusive model of the *strongly nonlinear saturated-unsaturated infiltration with a strongly nonlinear hydraulic conductivity* is given by

$$\frac{\partial(m(x)S_w)}{\partial t} - \Delta\widetilde{\beta}^*(S_w) + \frac{\partial \widetilde{K}(S_w)}{\partial x_3} = f \text{ in } Q, \tag{2.26}$$

$$m(x)S_w(x,0) = \theta_0(x) \text{ in } \Omega, \tag{2.27}$$

$$\text{boundary conditions in } S_w \text{ on } \Sigma, \tag{2.28}$$

where $\widetilde{\beta}^*$ is the multivalued function defined by (2.24), $\widetilde{\beta}$ is given by (2.19) and $\widetilde{K}$ is the single-valued function (2.11). Moreover, $\widetilde{\beta}^*$ is strongly monotone, $\widetilde{\beta}$ satisfies (2.20)–(2.23) and $\widetilde{K}$ has the properties (2.13)–(2.16).

The boundary conditions (1.4)–(1.5) become

$$S_w(x,t) = S_s \text{ on } \Sigma_u, \tag{2.29}$$

$$\left(\widetilde{K}(S_w)i_3 - \nabla\widetilde{\beta}^*(S_w)\right) \cdot \nu = f_\alpha \text{ on } \Sigma_\alpha. \tag{2.30}$$

The qualifier of *strongly nonlinear* is implied by the property of the function β which evolves highly nonlinear around the saturation point, S_s. This is justified by the fact that the typical representative for this behaviour (correlated with that of its primitive $\widetilde{\beta}^*$ which is finite at this point) is of the form

$$\widetilde{\beta}(S_w) = \frac{1}{(S_s - S_w)^{1-p}} \text{ for } 0 < p < 1.$$

We notice that this form of the diffusivity function reveals the character of *fast diffusion* of this process (see the review of diffusion-type processes in [3]).

However, $\widetilde{\beta}^*$ is multivalued and the sign equal (=) in (2.26) is not properly used. The appropriate symbol should be $\ni$. Also, we shall specify later the exact meaning of the solutions to (2.26)-(2.30). The fact that equation (2.26) is multivalued must not be surprising if one takes into account that it models a free boundary problem. This means that, at each time t, the domain Ω can be decomposed into two regions: the saturated one, $\{x;\ S_w(x,t) = S_s\}$ and the unsaturated one $\{x;\ S_w(x,t) < S_s\}$, separated by a free boundary. The extension of a nonlinear function arising in such a problem to a multivalued one is common in the theory of nonlinear differential equations with discontinuous coefficients as well as in that modelling free boundary processes.

Thus, equation (2.26) represents an extension of Richards' equation (written for the unsaturated infiltration) to the simultaneous saturated-unsaturated flow.

Weakly nonlinear hydraulic conductivity. A strongly nonlinear model, but with a weaker nonlinear behaviour of the conductivity may be obtained under conditions that lead to $\lim_{S_w \nearrow S_s} \widetilde{K}'(S_w) < \infty$. To reach such a situation we have to impose just from the beginning a stronger condition for k, namely that there exists $M > 0$, such that

$$k'(h) \leq M\widetilde{C}(h),\ \forall h \in [h_r, 0], \tag{2.31}$$

which implies that

$$K_0' = 0,\ \lim_{h \nearrow 0} \frac{k'(h)}{\widetilde{C}(h)} = M. \tag{2.32}$$

In this way $\widetilde{K}$ turns out to be Lipschitz on $[S_r, S_s]$ with the constant M. We observe that the functions $\widetilde{\beta}$ and $\widetilde{K}$ remain monotonically increasing. This situation is put into evidence e.g., in the van Genuchten model (see [39]) for the model parameter m close to 1. This case can be resumed in

Model 2. Let us assume $(m_1) - (m_4)$, $\widetilde{C}_0 = 0$ and (2.31)–(2.32). Then, the diffusive model of *strongly nonlinear saturated-unsaturated infiltration with a weakly nonlinear hydraulic conductivity* is given by (2.26)–(2.28), where the functions $\widetilde{\beta}$ and $\widetilde{\beta}^*$ have the properties specified in Model 1 except for $\widetilde{K}$ which is given by (2.11), with

$$\lim_{S_w \nearrow S_s} \widetilde{K}'(S_w) = M < \infty.$$

2.2. Weakly nonlinear saturated-unsaturated diffusive model

For some hydraulic models the diffusivity is finite at $S_w = S_s$. We intend to reveal which properties of the functions $\widetilde{C}^*$ and k can provide such a value. Let us suppose that the retention curve increases from the left to its maximum value with a nonzero rate at the left of zero,

$$\widetilde{C}_0 > 0,$$

but very close to 0. In this case $\widetilde{C}^*$ is not differentiable at $h = 0$ and the function

$$\widetilde{C} : [h_r, \infty) \to [0, \widetilde{C}_r],\ \widetilde{C}(h) = \begin{cases} \dfrac{\mathrm{d}S_w}{\mathrm{d}h}(h), & h < 0 \\ 0, & h \geq 0 \end{cases} \tag{2.33}$$

is no longer continuous at $h = 0$, having the jump $\left|\widetilde{C}_0\right| = \lim_{h \nearrow 0} \frac{\mathrm{d}S_w}{\mathrm{d}h}$.

The functions $\widetilde{K}$ and $\widetilde{\beta}^*$ and $\widetilde{\beta}$ will be defined in the same way as before, but in this case the value of $\widetilde{\beta}$ at $S_w = S_s$ exists and it is

$$\lim_{S_w \nearrow S_s} \widetilde{\beta}(S_w) = \frac{K_s}{\widetilde{C}_0} < \infty. \tag{2.34}$$

However, the function $\widetilde{\beta}^*(S_w)$ will be extended in a multivalued way, by $\widetilde{\beta}^*(S_w) = K_s^*$ at S_s.

Weakly nonlinear hydraulic conductivity. Assume that the derivative of k at $h = 0$, has a finite value, $K_0' < \infty$. Hence, $\widetilde{K}$ is Lipschitz with the constant

$$M = \max_{S_w \in [S_r, S_s]} \frac{k'((\widetilde{C}^*)^{-1}(S_w))}{\widetilde{C}((\widetilde{C}^*)^{-1}(S_w))} \leq \frac{K_0'}{\widetilde{C}_0}, \tag{2.35}$$

so that we can settle

Model 3. Let us assume $(m_1) - (m_4)$, $\widetilde{C}_0 > 0$ and $K_0' < \infty$. Then, the diffusive model of *weakly saturated-unsaturated infiltration with a weakly nonlinear hydraulic conductivity* is given by (2.26)-(2.28), where $\widetilde{\beta}^*$ is the multivalued function defined by (2.24), $\widetilde{\beta}$ is given by (2.19) and $\widetilde{K}$ is the single-valued function (2.11) with $\widetilde{K}'(S_w)$ finite on $[S_r, S_s]$. Moreover, $\widetilde{\beta}^*$ is strongly monotone, (2.25), $\widetilde{\beta}$ satisfies (2.20), (2.22)-(2.23) with

$$\lim_{S_w \nearrow S_s} \widetilde{\beta}(S_w) < +\infty \tag{2.36}$$

and K is Lipschitz on $[S_r, S_s]$, i.e., there exists $M > 0$ such that

$$\left|\widetilde{K}(S_w) - \widetilde{K}(\overline{S_w})\right| \leq M \left|S_w - \overline{S_w}\right|, \ \forall S_w, \overline{S_w} \in [S_r, S_s]. \tag{2.37}$$

It is obvious that this situation which is illustrated by nonsingular diffusivities including also power functions

$$\widetilde{\beta}(S_w) = S_w^p, \text{ with } p > 1,$$

is related to a *slow diffusion* and to the well-known *porous media equation* (see [3]).

We write the model in the dimensionless form, introducing for example

$$S_w^{\dim} = \frac{S_w - S_r}{S_s - S_r}, \ \widetilde{K}^{\dim}(S_w^{\dim}) = \frac{\widetilde{K}(S_w) - K_r}{K_s - K_r}, \ \widetilde{\beta}^{\dim}(S_w) = \frac{\widetilde{\beta}(S_w)}{\beta_d},$$

where β_d is a characteristic value for the diffusivity. Without entering into details we specify that the dimensionless model has the same form as (2.26)–(2.28). The dimensionless $S_{wr}^{\dim} = 0$ and $K_r = 0$ and for convenience, we shall extend $\widetilde{\beta}$ and $\widetilde{K}$ at the left of $S_{wr}^{\dim}$ by the constant values $\widetilde{\rho}$ and 0 (for all these details see [34]). For simplicity, further we shall no longer indicate dimensionless by the superscript $^{\dim}$.

3. Analysis of the porosity-degenerate model

In this part we shall approach Model 2 given by (2.26)–(2.27), (2.29)–(2.30) corresponding to the strongly nonlinear saturated-unsaturated case with a weakly nonlinear hydraulic conductivity. We shall study a limit case letting m to vanish on a subset Ω_0 strictly included in Ω, see Fig. 1. This characterizes the existence of possible solid intrusions in the soil and we shall call this model *porosity-degenerate.*

In fact we intend to treat a little more general mathematical problem, in which we shall consider that the function conductivity depends both on the space variables and the solution. Therefore the model reads

$$\frac{\partial(m(x)S_w)}{\partial t} - \Delta\widetilde{\beta}^*(S_w) + \frac{\partial \widetilde{K}(x,S_w)}{\partial x_3} \ni f \text{ in } Q, \tag{3.1}$$

$$m(x)S_w(x,0) = S_{w0}(x) \text{ in } \Omega, \tag{3.2}$$

$$S_w(x,t) = S_s \text{ on } \Sigma_u, \tag{3.3}$$

$$\left(\widetilde{K}(x,S_w)i_3 - \nabla\widetilde{\beta}^*(S_w)\right)\cdot\nu \ni f_\alpha \text{ on } \Sigma_\alpha. \tag{3.4}$$

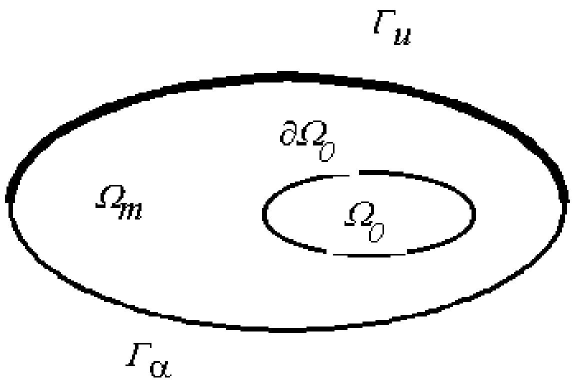

Fig. 1: The domain Ω.

At the points where m vanishes the equation degenerates. The function m is supposed to be essentially bounded, $m \in L^\infty(\Omega)$ with $0 \le m(x) \le 1$ a.e. $x \in \Omega$. However, we shall see that this assumption is not sufficient to get the solution existence and a stronger hypothesis upon m is required. We specify once again the hypotheses made for the problem parameters, i.e.,

$$\widetilde{\beta}(r) \geq \widetilde{\rho} \text{ for } r < S_s,\ \widetilde{\beta}(r) = \widetilde{\rho} \text{ for } r \leq 0,\ \lim_{r \nearrow S_s} \widetilde{\beta}(r) = +\infty, \tag{3.5}$$

$$\widetilde{\beta}^*(r) = \begin{cases} \int_0^r \widetilde{\beta}(\xi)\mathrm{d}\xi,\ r < S_s \\ [K_s^*, +\infty),\ r = S_s, \end{cases} \tag{3.6}$$

$$\lim_{r\to-\infty} \widetilde{\beta}^*(r) = -\infty,\ \lim_{r \nearrow S_s} \widetilde{\beta}^*(r) = \widetilde{K}_s^* > 0, \tag{3.7}$$

$$(\widetilde{\beta}^*(r) - \widetilde{\beta}^*(\overline{r}))(r - \overline{r}) \geq \widetilde{\rho}(r - \overline{r})^2, \forall r, \overline{r} \in (-\infty, S_s]. \tag{3.8}$$

In what concerns $\widetilde{K}$ we assume that it has the form

$$\widetilde{K}(x,r) = \begin{cases} \widetilde{K}_0(x) \text{ on } \{x;\ m(x) = 0\} \\ \widetilde{K}_m(r) \text{ otherwise,} \end{cases} \tag{3.9}$$

$$\widetilde{K}(x,r) = 0 \text{ for } r \leq 0 \text{ and } \widetilde{K}(x,r) = \widetilde{K}_s \text{ for } r \geq S_s, \tag{3.10}$$

where $\widetilde{K}_s = \widetilde{K}(x, S_s) > 0$.
Moreover, we assume that $\widetilde{K}_0 \in H^1(\Omega_0)$ and $\widetilde{K}$ is Lipschitz with respect to r, uniformly with respect to x, i.e., there exists $M > 0$, such that
(i_K) $\left|\widetilde{K}(x,r) - \widetilde{K}(x,\overline{r})\right| \leq M\left|r - \overline{r}\right|$, $\forall r,\ \overline{r} \in \mathbf{R}$, $\forall x \in \Omega$.
Finally we shall impose that

$$m \in C^1(\overline{\Omega}),\ 0 \leq m(x) \leq 1. \tag{3.11}$$

Functional framework. We perform a function replacement by denoting

$$w = S_w - S_s, \tag{3.12}$$

so that we are led to the system

$$\frac{\partial(m(x)w)}{\partial t} - \Delta\widetilde{\beta}^*(w + S_s) + \frac{\partial \widetilde{K}(x, w + S_s)}{\partial x_3} \ni f \text{ in } Q, \tag{3.13}$$

$$m(x)w(x,0) = v_0(x) \text{ in } \Omega, \tag{3.14}$$

$$w(x,t) = 0 \text{ on } \Sigma_u, \tag{3.15}$$

$$\left(\widetilde{K}(x, w + S_s)i_3 - \nabla\widetilde{\beta}^*(w + S_s)\right) \cdot \nu \ni f_\alpha \text{ on } \Sigma_\alpha, \tag{3.16}$$

which we are going to study. Here $v_0(x) = S_{w0} - m(x)S_s$. We shall indicate the value of w at saturation by w_s (actually, by (3.12) it is equal to zero, but we shall keep the notation w_s in order to put into evidence the behaviour of the solution at this point).

We consider the spaces $L^2(\Omega)$ with the standard norm denoted $\|\cdot\|$,

$$V = \{w \in H^1(\Omega);\ w = 0 \text{ on } \Gamma_u\}, \tag{3.17}$$

with the norm

$$\|\psi\|_V = \left(\int_\Omega |\nabla\psi|^2 \, dx\right)^{1/2}, \tag{3.18}$$

and its dual V' on which we introduce the scalar product by

$$(w, \overline{w})_{V'} = \langle w, \psi \rangle_{V',V},$$

where ψ is the solution to the boundary value problem

$$-\Delta\psi = \overline{w},\ \psi = 0 \text{ on } \Gamma_u,\ \nabla\psi \cdot \nu = 0 \text{ on } \Gamma_\alpha. \tag{3.19}$$

Let $f_\alpha \in L^2(0,T; L^2(\Gamma_\alpha))$. We define the functional $f_{\Gamma_\alpha} \in L^2(0,T;V')$ by

$$f_{\Gamma_\alpha}(t)(\psi) = -\int_{\Gamma_\alpha} f_\alpha(t)\psi d\sigma \text{ for any } \psi \in V \tag{3.20}$$

and notice that

$$\|f_{\Gamma_\alpha}(t)\|_{V'} \le c_{tr} \|f_\alpha(t)\|_{L^2(\Gamma_\alpha)}$$

where c_{tr} is the constant provided by the trace theorem.

For the further mathematical developments it is more convenient to work with the multivalued function

$$\beta^*(r) = \widetilde{\beta}^*(r + S_s) - \widetilde{K}_s^*. \tag{3.21}$$

DEFINITION **3.1** *Let*

$$\begin{aligned} m &\in C^1(\overline{\Omega}),\ f \in L^2(0,T;V'),\ f_\alpha \in L^2(0,T;L^2(\Gamma_\alpha)), \\ v_0 &\in L^2(\Omega),\ \frac{v_0}{m} \in L^2(\Omega),\ \frac{v_0}{m} \le w_s,\ a.e.\ x \in \Omega. \end{aligned} \tag{3.22}$$

We call w a solution to (3.13)-(3.16) *if*

$$\begin{aligned} w &\in L^2(0,T;V), \\ \zeta &\in L^2(0,T;V),\ \zeta \in \beta^*(w(x,t))\ a.e.\ on\ Q, \\ mw &\in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V'), \end{aligned} \tag{3.23}$$

satisfies the equation

$$\left\langle \frac{\mathrm{d}(m(x)w)}{\mathrm{d}t}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla\zeta(t)\cdot\nabla\psi - \widetilde{K}(x, w(t)+S_s)\frac{\partial\psi}{\partial x_3} \right) \mathrm{d}x =$$
$$= \langle f(t), \psi\rangle_{V',V} + \langle f_{\Gamma_\alpha}(t), \psi\rangle_{V',V}, \ \text{a.e. } t\in(0,T), \ \forall\psi\in V, \quad (3.24)$$

the initial condition $m(x)w(0) = v_0$ *and the property*

$$w \le w_s, \ \text{a.e. } (x,t)\in Q. \quad (3.25)$$

Eq. (3.24) can be written also in the equivalent form

$$\int_0^T \left\langle \frac{\mathrm{d}(m(x)w)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t \quad (3.26)$$
$$+ \int_Q \left(\nabla\zeta\cdot\nabla\phi - \widetilde{K}(x, w+S_s)\frac{\partial\phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t)\rangle_{V',V}\,\mathrm{d}t, \ \forall\phi\in L^2(0,T;V).$$

Replacing S_w from (3.12) we get that S_w satisfies

$$\begin{aligned} S_w &\in L^2(0,T;H^1(\Omega)), \\ \widetilde{\zeta} &\in L^2(0,T;H^1(\Omega)), \ \widetilde{\zeta}\in\widetilde{\beta}^*(S_w(x,t)) \text{ a.e. on } Q, \\ mS_w &\in C([0,T];L^2(\Omega))\cap W^{1,2}(0,T;V'). \end{aligned}$$

We set

$$D(A) = \{\theta\in L^2(\Omega); \ \exists\eta\in V, \ \eta(x)\in\beta^*(\theta(x)) \text{ a.e. } x\in\Omega\}$$

and we introduce the multivalued operator $A : D(A)\subset V'\to V'$ by

$$\langle A\theta, \psi\rangle_{V',V} = \int_\Omega \left(\nabla\eta\cdot\nabla\psi - \widetilde{K}(x,\theta+S_s)\frac{\partial\psi}{\partial x_3} \right) \mathrm{d}x,$$

for any $\psi\in V$, where $\eta\in\beta^*(\theta)$ a.e. $x\in\Omega$. Thus, we can write the problem

$$\begin{aligned} \frac{\mathrm{d}(m(x)w)}{\mathrm{d}t} + Aw &\ni f + f_{\Gamma_\alpha}, \ \text{a.e. } t\in(0,T) \\ m(x)w(0) &= v_0. \end{aligned} \quad (3.27)$$

We consider now the multiplication operator

$$M : D(A)\to L^2(\Omega), \ Mw = mw, \quad (3.28)$$

whose inverse is multivalued and denoting

$$v(x,t) = m(x)w(x,t), \tag{3.29}$$

we can rewrite (3.27) in terms of v as

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} + A_M v &\ni f + f_{\Gamma_\alpha}, \text{ a.e. } t \in (0,T) \\ v(0) &= v_0, \end{aligned} \tag{3.30}$$

where $A_M v = AM^{-1}v = A\left(\frac{v}{m}\right)$ for any $v \in D(A_M)$, where

$$D(A_M) = \left\{ v \in L^2(\Omega); \frac{v}{m} \in L^2(\Omega), \exists \eta \in V,\ \eta \in \beta^*\left(\frac{v}{m}\right) \text{ a.e. } x \in \Omega \right\}.$$

We see that $v \in D(A_M)$ implies $\frac{v}{m} \in D(A)$. Conversely, if $w = \frac{v}{m} \in D(A)$, then $v = mw \in D(A_M)$.
We still define $\widetilde{j} : \mathbf{R} \to (-\infty, +\infty]$ by

$$\widetilde{j}(r) = \begin{cases} \int_0^r \widetilde{\beta}^*(\xi)\mathrm{d}\xi, & r \le S_s \\ +\infty, & r > S_s, \end{cases}$$

where the left limit of $\widetilde{\beta}^*$ at S_s is specified in (3.7). This function is proper, convex, lower semicontinuous and

$$\partial\widetilde{j}(r) = \begin{cases} \widetilde{\beta}^*(r), & r < S_s, \\ [\widetilde{K}_s^*, +\infty), & r = S_s, \\ \varnothing, & r > S_s. \end{cases} \tag{3.31}$$

(The proof is similar to that done for a slightly different function in [34], Sect. 5.3.)

3.1. Approximating problem

Since the operator A_M is multivalued, in order to prove the existence for (3.27) we introduce an approximating problem replacing m by

$$m_\varepsilon(x) = m(x) + \varepsilon, \text{ for } \varepsilon > 0$$

and $\widetilde{\beta}^*$ by the single-valued continuous function

$$\widetilde{\beta}_\varepsilon^*(r) = \begin{cases} \widetilde{\beta}^*(r), & r < S_s - \varepsilon \\ \widetilde{\beta}^*(S_s - \varepsilon) + \frac{\widetilde{K}_s^* - \widetilde{\beta}^*(S_s - \varepsilon)}{\varepsilon}\left[r - (S_s - \varepsilon)\right], & r \ge S_s - \varepsilon. \end{cases}$$

Then we define

$$\beta_\varepsilon^*(r) = \widetilde{\beta}_\varepsilon^*(r + S_s) - \widetilde{K}_s^* \tag{3.32}$$

and the single valued operator

$$A_\varepsilon : D(A_\varepsilon) \subset V' \to V',$$

$$\langle A_\varepsilon \theta, \psi \rangle_{V',V} = \int_\Omega \left(\nabla \beta_\varepsilon^*(\theta) \cdot \nabla \psi - \widetilde{K}(x, \theta + S_s) \frac{\partial \psi}{\partial x_3} \right) \mathrm{d}x,\ \forall \psi \in V,$$

with

$$D(A_\varepsilon) = \{\theta \in L^2(\Omega); \beta_\varepsilon^*(\theta) \in V\}.$$

We can write the approximating Cauchy problem (corresponding to (3.27))

$$\begin{aligned} \frac{\mathrm{d}(m_\varepsilon w_\varepsilon)}{\mathrm{d}t} + A_\varepsilon w_\varepsilon &= f + f_{\Gamma_\alpha},\ \text{a.e. } t \in (0,T), \\ m_\varepsilon w_\varepsilon(0) &= v_{0\varepsilon}, \end{aligned} \tag{3.33}$$

where

$$v_{0\varepsilon} = m_\varepsilon \frac{v_0}{m}. \tag{3.34}$$

DEFINITION **3.2** *Let* $\varepsilon > 0$ *and*

$$\begin{aligned} m &\in C^1(\overline{\Omega}),\ f \in L^2(0,T;V'),\ f_\alpha \in L^2(0,T;L^2(\Gamma_\alpha)), \\ v_0 &\in L^2(\Omega),\ \frac{v_0}{m} \in L^2(\Omega),\ \frac{v_0}{m} \le w_s. \end{aligned}$$

A solution to (3.33) *is a function* w_ε *that satisfies*

$$\begin{aligned} w_\varepsilon &\in C([0,T];L^2(\Omega)) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V'), \\ \beta_\varepsilon^*(w_\varepsilon) &\in L^2(0,T;V), \end{aligned}$$

$$\begin{aligned} &\int_0^T \left\langle \frac{\mathrm{d}(m_\varepsilon w_\varepsilon)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t \\ &+ \int_Q \left\{ \nabla \beta_\varepsilon^*(w_\varepsilon) \cdot \nabla \phi - \widetilde{K}(x, w_\varepsilon + S_s) \frac{\partial \phi}{\partial x_3} \right\} \mathrm{d}x\,dt \\ = &\int_0^T \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t) \rangle_{V',V} \mathrm{d}t,\ \forall \phi \in L^2(0,T;V), \end{aligned} \tag{3.35}$$

and the initial condition $m_\varepsilon w_\varepsilon(0) = v_{0\varepsilon}$.

Then denoting

$$v_\varepsilon(x,t) = m_\varepsilon(x) w_\varepsilon(x,t), \tag{3.36}$$

we can write problem (3.33) in the equivalent form (corresponding to (3.30))

$$\begin{aligned} \frac{dv_\varepsilon}{dt} + B_\varepsilon v_\varepsilon &= f, \text{ a.e. } t \in (0,T), \\ v_\varepsilon(0) &= v_{0\varepsilon}. \end{aligned} \tag{3.37}$$

The operator $B_\varepsilon : D(B_\varepsilon) \subset V' \to V'$ is single-valued, has the domain

$$D(B_\varepsilon) = \left\{ \theta \in L^2(\Omega);\ \beta_\varepsilon^* \left(\frac{\theta}{m_\varepsilon} \right) \in V \right\}$$

and is given by

$$\langle B_\varepsilon \theta, \psi \rangle_{V',V} = \int_\Omega \left(\nabla \beta_\varepsilon^* \left(\frac{\theta}{m_\varepsilon} \right) \cdot \nabla \psi - \widetilde{K} \left(x, \frac{\theta}{m_\varepsilon} + S_s \right) \frac{\partial \psi}{\partial x_3} \right) dx,\ \forall \psi \in V.$$

Then (3.37) can be still written

$$\begin{aligned} & \int_0^T \left\langle \frac{dv_\varepsilon}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \\ & + \int_Q \left\{ \nabla \beta_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon} \right) \cdot \nabla \phi - \widetilde{K} \left(x, \frac{v_\varepsilon}{m_\varepsilon} + S_s \right) \frac{\partial \phi}{\partial x_3} \right\} dx dt = \\ = & \int_0^T \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t) \rangle_{V',V} dt,\ \forall \phi \in L^2(0,T;V), \end{aligned} \tag{3.38}$$

which is in fact (3.35).

For a later use we define $\widetilde{j}_\varepsilon : \mathbf{R} \to \mathbf{R}$,

$$\widetilde{j}_\varepsilon(r) = \int_0^r \widetilde{\beta}_\varepsilon^*(\xi) d\xi,$$

and notice that

$$\partial \widetilde{j}_\varepsilon(r) = \widetilde{\beta}_\varepsilon^*(r),\ \forall r \in \mathbf{R}. \tag{3.39}$$

First we shall prove that (3.37) has, for each $\varepsilon > 0$, a unique solution, v_ε in appropriate functional spaces.

3.2. Existence for the approximating problem

PROPOSITION **3.1** *Let*

$$\begin{aligned} m &\in C^1(\overline{\Omega}),\ 0 \le m \le 1, \\ f &\in L^2(0,T;V'),\ f_\alpha \in L^2(0,T;L^2(\Gamma_\alpha)), \\ v_0 &\in L^2(\Omega),\ \frac{v_0}{m} \in L^2(\Omega),\ \frac{v_0}{m} \le w_s \ \textit{a.e. on } \Omega. \end{aligned}$$

Then, the Cauchy problem (3.37) *has, for each* $\varepsilon > 0$, *a unique solution*

$$v_\varepsilon \in C([0,T];L^2(0,T)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V) \tag{3.40}$$

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^2(0,T;V), \tag{3.41}$$

$$\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^\infty(0,T;L^1(\Omega)), \tag{3.42}$$

that satisfies the estimates

$$\begin{aligned} &\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}(x,t) + S_s\right)\mathrm{d}x + \int_0^t \left\|\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau)\right\|_{V'}^2 \mathrm{d}\tau + \\ &+ \int_0^t \left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \le \\ \le\ &\beta_0\left(\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t + \int_0^T \|f_\alpha(t)\|_{L^2(\Gamma_\alpha)}^2\,\mathrm{d}t + 1\right), \end{aligned} \tag{3.43}$$

$$\left\|\sqrt{m_\varepsilon}\left(\frac{v_\varepsilon}{m_\varepsilon}(t)\right)\right\| \le c_0,\ \forall t \in [0,T], \tag{3.44}$$

$$\|v_\varepsilon(t)\| \le c_1,\ \forall t \in [0,T], \tag{3.45}$$

where β_0, c_0 *and* c_1 *do not depend on* ε.

Moreover, if v_ε *and* $\overline{v_\varepsilon}$ *are two solutions corresponding to the pairs of data* f, f_{Γ_α}, v_0 *and* $\overline{f}$, $\overline{f_{\Gamma_\alpha}}$, $\overline{v_0}$, *we have the estimate*

$$\begin{aligned} &\|v_\varepsilon(t) - \overline{v_\varepsilon}(t)\|_{V'}^2 + \int_0^t \|v_\varepsilon(\tau) - \overline{v_\varepsilon}(\tau)\|^2\,\mathrm{d}\tau \le \\ \le\ &\alpha_0(\varepsilon)\Big(\|v_0 - \overline{v_0}\|_{V'}^2 + \\ &+ \int_0^T \left\|f(t) - \overline{f}(t)\right\|_{V'}^2 \mathrm{d}t + \int_0^T \left\|f_\alpha(t) - \overline{f_\alpha}(t)\right\|_{L^2(\Gamma_\alpha)}^2 \mathrm{d}t\Big). \end{aligned} \tag{3.46}$$

Proof. The proof is based on the quasi m-accretivity of the operator B_ε which is proved below. To show the quasi monotony we compute

$$\begin{aligned}&\left((\lambda I + B_\varepsilon)\theta - (\lambda I + B_\varepsilon)\overline{\theta}, \theta - \overline{\theta}\right)_{V'} = \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \\ &+ \int_\Omega \nabla\left(\beta_\varepsilon^*\left(\frac{\theta}{m_\varepsilon}\right) - \beta_\varepsilon^*\left(\frac{\overline{\theta}}{m_\varepsilon}\right)\right)\cdot\nabla\psi \mathrm{d}x - \\ &- \int_\Omega \left(\widetilde{K}(x, \frac{\theta}{m_\varepsilon} + S_s) - \widetilde{K}(x, \frac{\overline{\theta}}{m_\varepsilon} + S_s)\right)\frac{\partial\psi}{\partial x_3}\mathrm{d}x,\end{aligned}$$

where $-\Delta\psi = \theta - \overline{\theta}$, $\nabla\psi\cdot\nu = 0$ on Γ_α and $\psi = 0$ on Γ_u. Hence

$$\begin{aligned}&\left((\lambda I + B_\varepsilon)\theta - (\lambda I + B_\varepsilon)\overline{\theta}, \theta - \overline{\theta}\right)_{V'} \geq \\ \geq\ & \lambda\left\|\theta - \overline{\theta}\right\|_{V'}^2 + \widetilde{\rho}\left\|\frac{\theta - \overline{\theta}}{\sqrt{m_\varepsilon}}\right\|^2 - M\left\|\frac{\theta - \overline{\theta}}{m_\varepsilon}\right\|\left\|\theta - \overline{\theta}\right\|_{V'} \geq \\ \geq\ & \left(\lambda - \frac{M^2}{2\widetilde{\rho}\varepsilon}\right)\left\|\theta - \overline{\theta}\right\|_{V'}^2 + \frac{\widetilde{\rho}}{2}\left\|\frac{\theta - \overline{\theta}}{\sqrt{m_\varepsilon}}\right\|^2 > 0\end{aligned}$$

for $\lambda > \frac{M^2}{2\widetilde{\rho}\varepsilon}$. Here we used the fact that $\varepsilon \leq m_\varepsilon(x) \leq 1 + \varepsilon$.

Next we have to prove that

$$R(I + B_\varepsilon) = V',$$

i.e., to show that the equation

$$v_\varepsilon + B_\varepsilon v_\varepsilon = g \tag{3.47}$$

has a solution $v_\varepsilon \in D(B_\varepsilon)$ for any $g \in V'$. Recall that ε is fixed.

If we denote $\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) = \zeta \in V$, due to the fact that β_ε^* is continuous and monotonically increasing on $\mathbf{R}$ and $R(\beta_\varepsilon^*) = (-\infty, \infty)$ it follows that its inverse

$$G_\varepsilon(\zeta) = m_\varepsilon(\beta_\varepsilon^*)^{-1}(\zeta) \tag{3.48}$$

is continuous from V to $L^2(\Omega)$ because

$$\begin{aligned}&\left\|G_\varepsilon(\zeta) - G_\varepsilon(\overline{\zeta})\right\| = \\ =\ & \left\|m_\varepsilon\left((\beta_\varepsilon^*)^{-1}(\zeta) - (\beta_\varepsilon^*)^{-1}(\overline{\zeta})\right)\right\| \leq \\ \leq\ & \frac{1+\varepsilon}{\widetilde{\rho}}\left\|\zeta - \overline{\zeta}\right\| \leq \frac{(1+\varepsilon)c_\Omega}{\widetilde{\rho}}\left\|\zeta - \overline{\zeta}\right\|_V,\ \forall\zeta, \overline{\zeta} \in V.\end{aligned} \tag{3.49}$$

Here we used (3.8) and Poincaré's inequality (with the constant c_Ω). So, (3.47) can be rewritten as

$$G_\varepsilon(\zeta) + B_0^\varepsilon\zeta = g \tag{3.50}$$

with $B_0^\varepsilon : V \to V'$ defined by

$$\langle B_0^\varepsilon \zeta, \psi \rangle_{V',V} = \int_\Omega \left(\nabla \zeta \cdot \nabla \psi - \widetilde{K}\left(x, \frac{G_\varepsilon(\zeta)}{m_\varepsilon} + S_s \right) \frac{\partial \psi}{\partial x_3} \right) \mathrm{d}x, \ \forall \psi \in V. \tag{3.51}$$

The operator $G_\varepsilon + B_0^\varepsilon$ is monotone, continuous and coercive for $\lambda > \frac{M^2}{2\widetilde{\rho}\varepsilon}$, hence it is surjective. Therefore (3.50) has a solution $\zeta \in V$, implying that (3.47) has a solution $v_\varepsilon \in D(B_\varepsilon)$.

a) Now we assume that $f \in W^{1,1}(0,T;V')$, $f_\alpha \in W^{1,1}(0,T;L^2(\Omega))$ and $\frac{v_0}{m} \in V$ which is equivalent to $v_{0\varepsilon} \in D(B_\varepsilon)$.

Therefore, the existence of a unique solution to (3.37)

$$v_\varepsilon \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;D(B_\varepsilon))$$

follows from the general theorems for evolution equations with m-accretive operators, hence $\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^\infty(0,T;V)$. Since the inverse of β_ε^* is Lipschitz we deduce that $\frac{v_\varepsilon}{m_\varepsilon} \in L^\infty(0,T;V)$.

It follows that (3.33) has a solution

$$w_\varepsilon = \frac{v_\varepsilon}{m_\varepsilon}$$

in the same spaces.

To prove estimate (3.43) we test (3.37) at $\beta_\varepsilon^*(v_\varepsilon)$ and integrate over $(0,t)$. Taking into account (3.36) and (3.32) we have

$$\begin{aligned}
&\int_0^t \left\langle \frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau), \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\rangle_{V',V} \mathrm{d}\tau + \int_0^t \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \\
\leq\ & \int_0^t \left\| \widetilde{K}\left(\cdot, \frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\| \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V \mathrm{d}\tau \\
&+ \int_0^t \|f(\tau)\|_{V'} \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V \mathrm{d}\tau + \int_0^t \|f_{\Gamma_\alpha}(\tau)\|_{V'} \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V \mathrm{d}\tau \\
\leq\ & \frac{1}{2}\int_0^t \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)(\tau)\right)\right\|_V^2 \mathrm{d}\tau + C_0,
\end{aligned}$$

where we have used the boundedness of $\widetilde{K}$ and

$$C_0 = \frac{3}{2}\left\{ \widetilde{K}_s^2 T \mathrm{meas}(\Omega) + \int_0^T \|f(\tau)\|_{V'}^2 \mathrm{d}\tau + c_{tr}^2 \int_0^T \|f_\alpha(\tau)\|_{L^2(\Gamma_\alpha)}^2 \mathrm{d}\tau \right\}.$$

Next, we take into account that

$$
\begin{aligned}
&\int_0^t \left\langle \frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau), \beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\rangle_{V',V} \mathrm{d}\tau \\
= \; &\int_0^t \left\langle \frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau), \widetilde{\beta}_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau) + S_s\right) - \widetilde{K}_s^*\right\rangle_{V',V} \mathrm{d}\tau \\
= \; &\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)\mathrm{d}x - \int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_0}{m}(x) + S_s\right)\mathrm{d}x \\
&-\int_\Omega \widetilde{K}_s^* v_\varepsilon(x,t)\mathrm{d}x + \int_\Omega \widetilde{K}_s^* v_{0\varepsilon}\mathrm{d}x
\end{aligned}
$$

and obtain that

$$
\begin{aligned}
&\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)\mathrm{d}x + \frac{1}{2}\int_0^t \left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \leq \\
\leq \; &\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_0}{m}(x) + S_s\right)\mathrm{d}x + \int_\Omega \widetilde{K}_s^* v_\varepsilon(t)\mathrm{d}x + C_1,
\end{aligned} \tag{3.52}
$$

where

$$
C_1 = \frac{1}{2}\widetilde{K}_s^{*2}\mathrm{meas}(\Omega) + \frac{1}{2}\left\|\frac{v_0}{m}\right\|^2 + C_0.
$$

Since

$$
\widetilde{j}_\varepsilon(r) \geq \frac{\widetilde{\rho}}{2}r^2, \;\; \forall r \in \mathbf{R},
$$

we have

$$
\begin{aligned}
&\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)\mathrm{d}x \geq \\
\geq \; &\frac{\widetilde{\rho}}{2}\int_\Omega m_\varepsilon(x)\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)^2 \mathrm{d}x \geq \frac{\widetilde{\rho}}{2}\int_\Omega m_\varepsilon\left\{\frac{1}{2}\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon}\right)^2 - S_s^2\right\}\mathrm{d}x.
\end{aligned}
$$

On the other hand we recall that $\frac{v_0}{m} \leq w_s = 0$ and notice that

$$
\begin{aligned}
&\widetilde{j}_\varepsilon\left(\frac{v_{0\varepsilon}}{m_\varepsilon} + S_s\right) = \int_0^{\frac{v_0}{m}+S_s} \widetilde{\beta}_\varepsilon^*(r)\mathrm{d}r \leq \int_0^{S_s} \widetilde{\beta}_\varepsilon^*(r)\mathrm{d}r = \\
= \; &\lim_{\delta \nearrow 0}\int_0^{S_s-\delta} \widetilde{\beta}_\varepsilon^*(r)\mathrm{d}r = \lim_{\delta \nearrow 0}\int_0^{S_s-\delta} \widetilde{\beta}^*(r)\mathrm{d}r \leq \widetilde{K}_s^* S_s.
\end{aligned}
$$

Thus we obtain by (3.52) that

$$\frac{\widetilde{\rho}}{4}\int_{\Omega} m_\varepsilon(x)\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon}\right)^2 \mathrm{d}x + \int_0^t \left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \leq \tag{3.53}$$
$$\leq 2\widetilde{K}_s^* S_s \mathrm{meas}(\Omega) + \int_{\Omega} \widetilde{K}_s^* m_\varepsilon \left(\frac{v_\varepsilon}{m_\varepsilon}(t)\right) \mathrm{d}x + C_1 + \frac{\widetilde{\rho}}{2} S_s^2 \int_{\Omega} m_\varepsilon(x)\mathrm{d}x \leq$$
$$\leq C_2 + \frac{\widetilde{\rho}}{8}\int_{\Omega} m_\varepsilon(x)\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon}\right)^2 \mathrm{d}x + \frac{4}{\widetilde{\rho}}\widetilde{K}_s^{*2}\mathrm{meas}(\Omega).$$

We have used several times that $m_\varepsilon \leq 1+\varepsilon \leq 2$. We can conclude that

$$\left\|\sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}(t)\right\| \leq c_0,\ \forall t \in [0,T]. \tag{3.54}$$

Next, from the relation

$$v_\varepsilon(t) = \sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}(t)\sqrt{m_\varepsilon} \tag{3.55}$$

we get that

$$\|v_\varepsilon(t)\|^2 = \int_{\Omega}\left(\sqrt{m_\varepsilon(x)}\frac{v_\varepsilon(t)}{m_\varepsilon}\right)^2 m_\varepsilon(x)\mathrm{d}x \leq 2\left\|\sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}(t)\right\|^2$$

and therefore

$$\|v_\varepsilon(t)\| \leq c_1,\ \forall t \in [0,T] \tag{3.56}$$

where c_0, c_1, C_0, C_1, C_2 are independent of ε. Replacing this in (3.52) we deduce

$$\int_{\Omega} m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)\mathrm{d}x + \int_0^t \left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \leq \tag{3.57}$$
$$\leq C_2\left(\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t + \int_0^T \|f_\alpha(t)\|_{L^2(\Gamma_\alpha)}^2\,\mathrm{d}t + 1\right). \tag{3.58}$$

Then we multiply (3.37) scalarly in V' by $\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}t}(t)$, integrate over $(0,t)$ and proceeding as before we get

$$\int_{\Omega} m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right)\mathrm{d}x + \int_0^t \left\|\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau)\right\|_{V'}^2 \mathrm{d}\tau \leq \tag{3.59}$$
$$\leq C_2\left(\int_0^T \|f(t)\|_{V'}^2\,\mathrm{d}t + \int_0^T \|f_\alpha(t)\|_{L^2(\Gamma_\alpha)}^2\,\mathrm{d}t + 1\right).$$

Adding this relation with (3.58) we obtain

$$\begin{aligned} &\int_\Omega m_\varepsilon(x)\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}(x,t)+S_s\right)\mathrm{d}x+\int_0^t\left\|\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}\tau}(\tau)\right\|_{V'}^2\mathrm{d}\tau+ \\ &+\int_0^t\left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}(\tau)\right)\right\|_V^2\mathrm{d}\tau\le \\ \le\ &\beta_0\left(\int_0^T\|f(t)\|_{V'}^2\,\mathrm{d}t+\int_0^T\|f_\alpha(t)\|_{L^2(\Gamma_\alpha)}^2\,\mathrm{d}t+1\right), \end{aligned} \tag{3.60}$$

with β_0 independent of ε.

To show the estimate (3.46) we write two equations (3.37) corresponding to different pairs of data, subtract them, multiply the difference scalarly in V' by $v_\varepsilon-\overline{v_\varepsilon}$ and integrate over $(0,t)$. We get

$$\begin{aligned} &\frac{1}{2}\|v_\varepsilon(t)-\overline{v_\varepsilon}(t)\|_{V'}^2+\frac{\widetilde{\rho}}{2}\int_0^t\int_\Omega\frac{1}{m_\varepsilon}(v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau))^2\mathrm{d}\tau\mathrm{d}x\le \\ \le\ &\frac{1}{2}\|v_0-\overline{v_0}\|_{V'}^2+\frac{M^2}{2\widetilde{\rho}\varepsilon}\int_0^t\|v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau)\|_{V'}^2\,\mathrm{d}\tau+ \\ &+\int_0^t\left\|f(\tau)-\overline{f}(\tau)\right\|_{V'}^2\|v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau)\|_{V'}\,\mathrm{d}\tau+ \\ &+c_{tr}^2\int_0^t\left\|f_\alpha(\tau)-\overline{f_\alpha}(\tau)\right\|_{L^2(\Gamma_\alpha)}^2\|v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau)\|_{V'}\,\mathrm{d}\tau \end{aligned}$$

and moreover

$$\begin{aligned} &\|v_\varepsilon(t)-\overline{v_\varepsilon}(t)\|_{V'}^2+\widetilde{\rho}\int_0^t\int_\Omega\frac{(v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau))^2}{m_\varepsilon}\mathrm{d}\tau\mathrm{d}x\le \\ \le\ &\|v_0-\overline{v_0}\|_{V'}^2+\left(\frac{M^2}{\widetilde{\rho}\varepsilon}+2\right)\int_0^t\|v_\varepsilon(\tau)-\overline{v_\varepsilon}(\tau)\|_{V'}^2\,\mathrm{d}\tau+ \\ &+\int_0^T\left\|f(\tau)-\overline{f}(\tau)\right\|_{V'}^2\mathrm{d}\tau+c_{tr}^2\int_0^T\left\|f_\alpha(\tau)-\overline{f_\alpha}(\tau)\right\|_{L^2(\Gamma_\alpha)}^2\mathrm{d}\tau. \end{aligned}$$

We obtain the estimate (3.46), via Gronwall lemma with α_0 depending on ε.

b) Now, we assume that $f\in L^2(0,T;V')$ and $\frac{v_0}{m}\in L^2(\Omega)$, $\frac{v_0}{m}\le w_s$.

Due to some obvious densities we can take $\{f_n\}_{n\ge1}\subset W^{1,1}(0,T;V')$, $\{f_\alpha^n\}_{n\ge1}\subset W^{1,1}(0,T;L^2(\Gamma_\alpha))$ and $\{v_0^n\}_{n\ge1}\subset D(B_\varepsilon)=V$, such that

$$\begin{aligned} f_n &\to f \text{ strongly in } L^2(0,T;V'), \\ f_\alpha^n &\to f_\alpha \text{ strongly in } L^2(0,T;L^2(\Gamma_\alpha)) \\ v_0^n &\to v_0 \text{ strongly in } L^2(\Omega). \end{aligned} \tag{3.61}$$

Then, for each $\varepsilon > 0$, the problem

$$\begin{aligned} \frac{\mathrm{d}v_\varepsilon^n}{\mathrm{d}t} + B_\varepsilon v_\varepsilon^n &= f_n + f_{\Gamma_\alpha}^n, \text{ a.e. } t \in (0,T), \\ v_\varepsilon^n(0) &= v_{0\varepsilon}^n \end{aligned} \tag{3.62}$$

has a unique solution v_ε^n according to a), satisfying the estimate (3.60) with the right-hand side independent of n, namely,

$$\begin{aligned} &\int_\Omega m_\varepsilon(x) j_\varepsilon\left(\frac{v_\varepsilon^n}{m_\varepsilon}(t) + S_s\right)\mathrm{d}x + \int_0^t \left\|\frac{\mathrm{d}v_\varepsilon^n}{\mathrm{d}\tau}(\tau)\right\|_{V'}^2 \mathrm{d}\tau + \\ &+ \int_0^t \left\|\beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}(\tau)\right)\right\|_V^2 \mathrm{d}\tau \le \\ \le\ & \beta_0\left(\int_0^T \|f_n(t)\|_{V'}^2\,\mathrm{d}t + \int_0^T \|f_\alpha^n(t)\|_{L^2(\Gamma_\alpha)}^2\,\mathrm{d}t + 1\right). \end{aligned} \tag{3.63}$$

We stress that ε is fixed and the second term in the previous relation is uniformly bounded due to (3.61). By this estimate we deduce that $\left\{\beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right)\right\}_n$ is in a bounded subset of $L^2(0,T;V)$ and $\left\{\frac{\mathrm{d}v_\varepsilon^n}{\mathrm{d}t}\right\}_n$ is in a bounded subset of $L^2(0,T;V')$, so we can select a subsequence such that

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon^n}{m_\varepsilon}\right) \to \zeta_\varepsilon \text{ weakly in } L^2(0,T;V) \text{ as } n \to \infty,$$

and

$$\frac{\mathrm{d}v_\varepsilon^n}{\mathrm{d}t} \to \frac{\mathrm{d}v_\varepsilon}{\mathrm{d}t} \text{ weakly in } L^2(0,T;V') \text{ as } n \to \infty.$$

We get immediately that

$$\frac{v_\varepsilon^n}{m_\varepsilon} \to w_\varepsilon \text{ weakly in } L^2(0,T;V) \text{ as } n \to \infty.$$

But $m_\varepsilon \in C^1(\overline{\Omega})$ and so the sequence $\{v_\varepsilon\}_n = \left\{m_\varepsilon \frac{v_\varepsilon^n}{m_\varepsilon}\right\}_n$ is bounded in $L^2(0,T;V)$, whence

$$v_\varepsilon^n \to v_\varepsilon \text{ weakly in } L^2(0,T;V) \text{ as } n \to \infty.$$

Since V is compact in $L^2(\Omega)$ it follows by Lions-Aubin's theorem that

$$v_\varepsilon^n \to v_\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } n \to \infty. \tag{3.64}$$

By (3.37) we have that $\{B_\varepsilon v_\varepsilon^n\}_n$ is bounded in $L^2(0,T;V')$ so that

$$B_\varepsilon v_\varepsilon^n \to \chi \text{ weakly in } L^2(0,T;V') \text{ as } n \to \infty. \tag{3.65}$$

But B_ε is quasi m-accretive so its realization on $L^2(0,T;V')$ is quasi m-accretive too, hence it is demiclosed and by (3.64) and (3.65) we get that $\chi = Bv_\varepsilon$ a.e. on Q.

Now we can pass to the limit in (3.62) as $n \to \infty$ and get (3.37), proving thus that this problem has the solution $v_\varepsilon \in C([0,T], L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V)$.

Finally, passing to the limit in (3.63), as $n \to \infty$, and using the lower semicontinuity property we get (3.43) as claimed. Estimates (3.44)–(3.45) have been proved in (3.54)–(3.55).

The uniqueness of the approximating solution follows by (3.46). ∎

3.3. Existence for the original problem

As we specified before the domains

$$\Omega_m = \{x \in \Omega;\ m(x) > 0\} \text{ and } \Omega_0 = \mathrm{int}\{x \in \Omega;\ m(x) = 0\}$$

have the common C^1-boundary, $\partial\Omega_0$, see again Fig. 1. Here, the notation "int" represents the interior of the subset.

THEOREM **3.1** *Let*

$$\begin{aligned} m &\in C^1(\overline{\Omega}),\ 0 \le m \le 1,\ f \in L^2(0,T;V'),\ f_\alpha \in L^2(0,T;L^2(\Gamma_\alpha)), \\ v_0 &\in L^2(\Omega),\ \frac{v_0}{m} \in L^2(\Omega),\ \frac{v_0}{m} \le w_s \textit{ a.e. on } \Omega. \end{aligned}$$

Then, the Cauchy problem (3.27) *has a solution*

$$w \in L^2(0,T;V), \tag{3.66}$$

such that

$$\zeta \in L^2(0,T;V),\ \zeta \in \beta^*(w(x,t)) \textit{ a.e. on } Q, \tag{3.67}$$

$$mw \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V'), \tag{3.68}$$

$$w \le w_s \textit{ a.e. } (x,t) \in Q. \tag{3.69}$$

Proof. By the hypotheses it follows that the approximating problem (3.37) (and consequently (3.33)) has, for each ε, a unique solution according to

Proposition 3.1, including the estimates (3.43)–(3.45). These do not depend on ε and imply that we can select a subsequence such that

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \to \zeta \text{ weakly in } L^2(0,T;V), \tag{3.70}$$

$$\widetilde{\beta}_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}+S_s\right) \to \zeta + \widetilde{K}_s^* \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{3.71}$$

$$\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}t} \to \mu \text{ weakly in } L^2(0,T;V'), \tag{3.72}$$

$$w_\varepsilon = \frac{v_\varepsilon}{m_\varepsilon} \to w \text{ weakly in } L^2(0,T;V). \tag{3.73}$$

We also get that the trace of $\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right)$ on Σ_u is well defined and since $\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^2(0,T;V)$ it follows that $\zeta = 0$ on Σ_u. Now

$$v_\varepsilon = m_\varepsilon \frac{v_\varepsilon}{m_\varepsilon} \tag{3.74}$$

and since $m_\varepsilon \to m$ uniformly on Ω and $m \in C(\overline{\Omega})$ it follows that

$$v_\varepsilon \to v \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{3.75}$$

By (3.73) and (3.75) we get

$$v = mw \tag{3.76}$$

and obviously

$$v = 0, \text{ a.e. on } Q_0 = \Omega_0 \times (0,T). \tag{3.77}$$

Using (3.73), (3.74) and (3.75) we still obtain that

$$\sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon} \to \sqrt{m}w \text{ weak-star in } L^\infty(0,T;L^2(\Omega)),$$

$$v_\varepsilon = \sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}\sqrt{m_\varepsilon} \to v \text{ weak-star in } L^\infty(0,T;L^2(\Omega)).$$

Again by (3.74) and $m \in C^1(\overline{\Omega})$ we deduce that

$$\|v_\varepsilon\|_{L^2(0,T;V)} \leq \text{ constant independent of } \varepsilon. \tag{3.78}$$

By Lions-Aubin compactness theorem we conclude then that $\{v_\varepsilon\}_\varepsilon$ is compact in $L^2(0,T;L^2(\Omega))$, i.e.,

$$v_\varepsilon \to v \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0, \tag{3.79}$$

and $\mu = \frac{\mathrm{d}v}{\mathrm{d}t}$. Also, by Ascoli-Arzelà theorem we can prove that $v_\varepsilon(t) \to v(t)$ strongly in V' (using (3.72) and (3.78)). Using (3.76) we can deduce by letting $\varepsilon \to 0$ in the second equation in (3.37) that

$$mw(0) = v_0. \tag{3.80}$$

We set now

$$\Omega_\delta = \{x \in \Omega;\ m(x) > \delta\} \text{ for arbitrary } \delta > 0,$$

$$Q_\delta = \Omega_\delta \times (0,T),\ Q_m = \Omega_m \times (0,T),$$

and notice that Ω_δ and Ω_m are open because $m \in C^1(\overline{\Omega})$. We have

$$\frac{1}{m_\varepsilon} = \frac{1}{m+\varepsilon} < \frac{1}{m} < \frac{1}{\delta} \text{ on } \Omega_\delta$$

and by (3.79)

$$w_\varepsilon = \frac{1}{m_\varepsilon} v_\varepsilon \to \frac{v}{m} = w \text{ strongly in } L^2(0,T;L^2(\Omega_\delta)),\ \forall \delta > 0.$$

Recall that $\beta_\varepsilon^*(r) = \widetilde{\beta}_\varepsilon^*(r + S_s) - \widetilde{K}_s^*$.
Let us fix $(x,t) \in Q_\delta$. Using the same argument like in the proof of Theorem 3.1, in Sect. 5.3 in [34], we obtain that

$$\widetilde{\beta}_\varepsilon^*(w_\varepsilon + S_s) \to \widetilde{\zeta} \in \widetilde{\beta}^*(w + S_s) \text{ weakly in } L^2(0,T;H^1(\Omega_\delta)).$$

By (3.32) and (3.71) we get that

$$\beta_\varepsilon^*(w_\varepsilon + S_s) \to \widetilde{\beta}^*(w + S_s) - \widetilde{K}_s^* \text{ weakly in } L^2(0,T;H^1(\Omega_\delta)).$$

Since δ is arbitrary we obtain

$$\zeta(x,t) \in \widetilde{\beta}^*(w(x,t) + S_s) - \widetilde{K}_s^* \text{ a.e. } (x,t) \in Q_m = \bigcup_{\delta>0} Q_\delta. \tag{3.81}$$

Proving that the subset

$$Q_m^+ = \{(x,t) \in Q_m;\ w(x,t) > w_s\}$$

has a zero measure, we deduce similarly to the proof of Corollary 3.3 in Sect. 5.3 in [34], that $w \le w_s$ a.e. $(x,t) \in Q_m$.

Finally, since $\left\{\widetilde{K}(x, w_\varepsilon + S_s)\right\}_\varepsilon$ is bounded in $L^2(Q)$, we have

$$\widetilde{K}(x, w_\varepsilon + S_s) \to \kappa \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.82}$$

and we assert that

$$\kappa(x,t) = \widetilde{K}(x, w(x,t)), \text{ a.e. } (x,t) \in Q.$$

Indeed, $\left\{\widetilde{K}_m(w_\varepsilon + S_s)\right\}_\varepsilon$ is weakly convergent to κ, on Q_m, too. On the other hand, it is strongly convergent to $\widetilde{K}_m(w + S_s)$ on each Q_δ, because $\widetilde{K}_m$ is Lipschitz. By the uniqueness of the limit the restriction of the weak limit to Q_δ should coincide with $\widetilde{K}_m(w + S_s)$. This implies that

$$\kappa = \widetilde{K}(x, w + S_s), \text{ a.e. on } Q_m. \tag{3.83}$$

On the subset Q_0 the function $\widetilde{K}$ does not depend on w, so the limit is equal to $\widetilde{K_0}(x)$.

Now we can pass to limit as $\varepsilon \to 0$ in (3.38) and obtain

$$\begin{aligned}&\int_0^T \left\langle \frac{\mathrm{d}(mw)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t + \int_Q \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}(x, w + S_s) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t = \\ &= \int_0^T \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t) \rangle_{V',V} \, \mathrm{d}t, \ \forall \phi \in L^2(0,T;V),\end{aligned} \tag{3.84}$$

where ζ is given by (3.70).

In (3.84) taking $\phi \in L^2(0,T;H_0^1(\Omega_m))$ we still deduce that w is the solution to (3.27) on Q_m too,

$$\begin{aligned}&\int_0^T \left\langle \frac{\mathrm{d}(mw)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t + \int_{Q_m} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}_m(w + S_s) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t = \\ &= \textstyle\int_0^T \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t) \rangle_{V',V} \, \mathrm{d}t, \ \forall \phi \in L^2(0,T;H_0^1(\Omega_m)),\end{aligned} \tag{3.85}$$

where $\zeta(x,t) \in \beta^*(w(x,t))$ a.e. on Q_m.

Taking now $\phi \in L^2(0,T;H_0^1(\Omega_0))$, we obtain the weak form of the equation on this subset

$$\int_{Q_0} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}_0(x) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t = 0, \ \forall \phi \in L^2(0,T;H_0^1(\Omega_0)), \tag{3.86}$$

where ζ is given by (3.70).

On the other hand, (3.84) corresponds to the problem

$$\begin{aligned} \frac{\partial(mw)}{\partial t} - \Delta\zeta + \frac{\partial\widetilde{K}(x,w+S_s)}{\partial x_3} &= f \text{ in } Q, \\ \zeta &= 0 \text{ on } \Sigma_u, \\ (\widetilde{K}(x,w+S_s)i_3 - \nabla\zeta)\cdot\nu &= f_\alpha \text{ on } \Sigma_\alpha, \end{aligned} \tag{3.87}$$

and (3.85)–(3.86) to the problem

$$\begin{aligned} \frac{\partial(mw)}{\partial t} - \Delta\zeta + \frac{\partial\widetilde{K}_m(w+S_s)}{\partial x_3} &= f \text{ in } Q_m, \\ -\Delta\zeta + \frac{\partial\widetilde{K}_0(x)}{\partial x_3} &= f \text{ in } Q_0, \\ \zeta &= 0 \text{ on } \Sigma_u, \\ (\widetilde{K}_m(w+S_s)i_3 - \nabla\zeta)\cdot\nu &= f_\alpha \text{ on } \Sigma_\alpha. \end{aligned} \tag{3.88}$$

We recall that the common boundary of the domains Ω_m and Ω_0 is regular due to the fact that $m \in C^1(\overline{\Omega})$. Since $\zeta \in L^2(0,T;V)$, we deduce that the trace of $\zeta(t) \in \beta^*(w(t))$ belongs to V a.e. t, so it is continuous across the boundary $\partial\Omega_0$ (more exactly along lines $\mathcal{L}$ that cross the boundary), a.e. $t \in (0,T)$. Thus if we take $x_0 \in \partial\Omega_0$ and denote

$$\zeta^+(t) = \lim_{x\to x_0,\ x\in\mathcal{L}\cap\Omega_m} \zeta(t),$$

then we have

$$\zeta^+(t) = \lim_{x\to x_0,\ x\in\mathcal{L}\cap\Omega_0} \zeta(t) \text{ a.e. } t \in (0,T).$$

We take into account that $\zeta^+ \in \beta^*(w(t))$ a.e. on Q_m, hence ζ turns out to be the solution to the elliptic problem

$$\begin{aligned} -\Delta\zeta(t) &= f(t) + f_{\Gamma_\alpha}(t) \text{ in } \Omega_0 \\ \zeta(t) &= \zeta^+(t) \in \beta^*(w(t)) \text{ on } \partial\Omega_0, \text{ a.e. } t \in (0,T) \end{aligned} \tag{3.89}$$

for a.e. t fixed in $(0,T)$, and w is the solution to (3.85) (equivalently to (3.24)) in Q_m.

Then, we define the function

$$w^*(x,t) = \begin{cases} w(x,t), \text{ if } (x,t) \in Q_m \\ (\beta^*)^{-1}(\zeta(x,t)), \text{ if } (x,t) \in Q_0 = \Omega_0 \times (0,T), \end{cases} \tag{3.90}$$

where ζ is the solution to (3.89) and show that it is the solution to (3.27) in the sense of Definition 3.1. Indeed, $\zeta(x,t) \in \beta^*(w^*(x,t))$ and $\zeta \in L^2(0,T;V)$, so it follows that $w^* \in D(A)$, implying that $w^* \leq w_s$ a.e. on Q. Then, mw^* belongs to the spaces specified in (3.23) (we take into account that $mw^* = 0$ on Q_0). Finally, we have to check that w^* satisfies the equation (3.26) and this follows by a straightforward computation using (3.84)–(3.86). Indeed, if we replace w^* in (3.26) we obtain

$$\begin{aligned}
&\int_0^T \left\langle \frac{\mathrm{d}(mw^*)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t + \\
&+ \int_0^T \int_{\Omega_m} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}(x, w + S_s) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t + \\
&+ \int_0^T \int_{\Omega_0} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}(x, w^*) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t = \\
= \ &\int_0^T \left\langle \frac{\mathrm{d}(mw)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t + \\
&+ \int_Q \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}(x, w + S_s) \frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t = \\
= \ &\int_0^T \langle f(t) + f_{\Gamma_\alpha}, \phi(t) \rangle_{V',V} \mathrm{d}t, \ \forall \phi \in L^2(0, T; V).
\end{aligned}$$

We took into account the expressions assigned to w^* and $\widetilde{K}(x, w + S_s)$ on each subset, (3.81) and (3.84). □

COROLLARY **3.1** *Under the assumptions of Theorem* 3.1 *the solution to (3.27) is unique if in addition*

$$\widetilde{\rho} > c_\Omega M. \tag{3.91}$$

Proof. Let us denote by w_1^* and w_2^* two solutions to (3.27) corresponding to the same data. We multiply the difference of equations (3.27) written for w_1^* and w_2^* by $(w_1^* - w_2^*)$ scalarly in V', integrate on $(0, T)$ and use the Lipschitz property of $\widetilde{K}$. We get

$$\begin{aligned}
&\|m(w_1^*(\tau) - w_2^*(\tau))\|_{V'}^2 + \widetilde{\rho} \int_0^T \|w_1^*(\tau) - w_2^*(\tau)\|^2 \,\mathrm{d}\tau \leq \qquad (3.92) \\
\leq \ &\frac{M^2}{\widetilde{\rho}} \int_0^T \|w_1^*(\tau) - w_2^*(\tau)\| \, \|w_1^*(\tau) - w_2^*(\tau)\|_{V'} \,\mathrm{d}\tau \leq \\
\leq \ &\frac{M^2}{\widetilde{\rho}} c_\Omega^2 \int_0^T \|w_1^*(\tau) - w_2^*(\tau)\|^2 \,\mathrm{d}\tau
\end{aligned}$$

where c_Ω is the constant in Poincaré's inequality. Here we took into account that for $w \in L^2(\Omega)$ we have $\|w\|_{V'} \leq c_\Omega \|w\|$.

It follows by (3.91) that $mw_1^* = mw_2^*$ a.e. on Q and $w_1^* = w_2^*$ a.e. on Q_m where $m(x) > 0$. Now we subtract the equations (3.88) corresponding to w_1^* and w_2^* and get

$$\begin{aligned} -\Delta(\zeta_1 - \zeta_2) &= 0 \text{ in } Q, \\ \zeta_1 - \zeta_2 &= 0 \text{ on } \Sigma_u, \\ -\nabla(\zeta_1 - \zeta_2) \cdot \nu &= 0 \text{ on } \Sigma_\alpha, \end{aligned}$$

where $\zeta_1 \in \beta^*(w_1^*)$, $\zeta_2 \in \beta^*(w_2^*)$ a.e. on Q. Hence $\zeta_1 = \zeta_2$ and since $(\beta^*)^{-1}$ is single valued then $w_1^* = w_2^*$ a.e. on Q. □

Remark 3.1 We observe that in the degenerate case the uniqueness of the solution can be obtained only if the transport is dominated in a sense (see (3.91)) by the diffusivity. In particular, this is true when $\widetilde{K} = 0$, i.e., when we deal with a horizontal infiltration, also called sorption.

Remark 3.2 By the proof of the solution existence we also ascertain a consequence that can be inferred at an intuitive level, i.e., the boundary value problem is separated into two problems corresponding to the domains Q_m and Q_0, connected by the flux continuity.

Indeed, if we test the first two equations in (3.88) at $\phi \in L^2(0,T;V)$ and integrate the sum over $(0,T)$ we obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{\mathrm{d}(mw)}{\mathrm{d}t}(t), \phi(t) \right\rangle_{V',V} \mathrm{d}t + \\ &+ \int_0^T \int_{\Omega_m} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}_m(w + S_s)\frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t - \\ &- \int_0^T \int_{\partial\Omega_m} \left(\widetilde{K}_m(w + S_s)i_3 - \nabla\zeta \right) \cdot \nu^+ \phi \mathrm{d}\sigma\mathrm{d}t + \\ &+ \int_0^T \int_{\Omega_0} \left(\nabla\zeta \cdot \nabla\phi - \widetilde{K}_0(x)\frac{\partial \phi}{\partial x_3} \right) \mathrm{d}x\mathrm{d}t - \\ &- \int_0^T \int_{\partial\Omega_0} \left(\widetilde{K}_0(x)i_3 - \nabla\zeta \right) \cdot \nu^- \phi \mathrm{d}\sigma\mathrm{d}t = \\ = \;&\int_0^T \int_\Omega \langle f(t) + f_{\Gamma_\alpha}(t), \phi(t) \rangle_{V',V} \,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

for any $\phi \in L^2(0,T;V)$, where ν^+ is the outer normal to $\partial\Omega_m$, ν^- is the outer normal to $\partial\Omega_0$ and $\zeta \in \beta^*(w)$ a.e. on Q_m. Taking into account (3.84) we obtain the flux continuity on the common boundary $\partial\Omega_0 \times (0,T)$

$$\left(\widetilde{K}_m(w+S_s)i_3 - \nabla\zeta\right)\cdot\nu^+ = \left(\widetilde{K}_0(x)i_3 - \nabla\zeta\right)\cdot\nu^+ \text{ on } \partial\Omega_0 \times (0,T). \quad (3.93)$$

The previous integrals on $\partial\Omega_m$ and $\partial\Omega_0$ are considered in the sense of distributions, e.g., as the value of $\left(\widetilde{K}(x,w+S_s)i_3 - \nabla\zeta\right)\cdot\nu$ at ϕ. By the trace theorem we see that, generally, the flux $\left(\widetilde{K}(x,w+S_s)i_3 - \nabla\zeta\right)\cdot\nu$ is well defined as an element of the space $L^2(0,T;H^{-1/2}(\partial\Omega_0))$.

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