Topics in Applied Mathematics & Mathematical Physics

Diffusion Processes. Physical Models and Numerical Approximation

by Stelian Ion [1]

Contents

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1. Introduction

We report some mathematical results on the numerical approximation of a class of nonlinear diffusion problems. We are concerned with the convection-diffusion-reaction equation (CDRE)

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}\left(\kappa(u)\nabla u + \boldsymbol{f}(u)\right) = g(t,x,u), \tag{1.1}$$

and generalized porous medium equation (GPME),

$$\frac{\partial u}{\partial t} - \triangle\phi(u) = r(u), \tag{1.2}$$

where div and ∇ are taken with respect to $x \in \mathbb{R}^n$; $\triangle = \operatorname{div}\nabla$ is the Laplace operator and $u(t,x)$ is the scalar unknown function.

There are some reasons to work with two different equations. The both equations quantify diffusion phenomena but in different manner. The diffusion flux is modeled by $\kappa(u)\nabla u$ in the CDRE and by $\operatorname{grad}\phi(u)$ in the GPME. In some cases the two forms can be interchanged but in other cases is not possible. For example, if $\kappa(\cdot)$ is an integrable function one can put $\phi(u) = \int^u \kappa(s)\mathrm{d}s$. Although in almost any physicaly interesting cases this transformation can be done the calculation of the function ϕ, especially when one deals with numerical approximation, can be a hard problem. In such a case is recomandable to use the CDRE form. On the other hand if $\phi(\cdot)$ is a differentiable function one has $\kappa(u) = \phi'(u)$. If $\phi(\cdot)$ is onlya continuous function it is not posible to evaluate the diffusion coefficient.

The outline of the paper follows.

In Section 2 we delineate some mechanical problems and we will make comments on the constitutive functions.

In Section 3 we present the essential facts relative to solvability of the Cauchy problem. We revise the concepts of weak solution and weak entropy solution and we will present a comparison criterion.

Section 4 is devoted to the numerical approximation.

The numerical solution of the Cauchy problem is obtained in two steps. In the first step a system of ordinary differential equation is set up and in the second step this ODE system is numericaly integrated.

The mathematical properties of the ODE model are strongly determined by the numerical diffusion flux and the numerical convective flux. We will define a numerical approximation of the diffusion flux and a numerical approximation of the convective flux that lead to a quasimonotone ODE system. Using this property we will show that there exists a comparison principle and we will provide the bounds for the solutions of the discrete model that are independent of the mesh size of triangulation.

In Section 5 we give two numerical algorithms to solve GPME equation and Richards' equation respectively. To integrate the ODE system which approximate the GPME equation we will use implicit Euler method and we we will setup an iterative algorithm to solve the system of nonlinear algebraic equation that results.

To solve Richards' equation we use an adaptive time marching scheme and an inexact Newton type method to solve nonlinear equation.

2. Physical Models

The mathematical models (1.1) and (1.2) cover a wide range of physical phenomena such that: heat transfer, infiltration of water through porous media, transport of contaminant in porous media, the flow of the gas through porous media, plasma radiation, to remaind a few.

The simplest example of the model problem (1.1) is the linear caloric equation:

$$\frac{\partial u}{\partial t} = \operatorname{div}(\kappa \nabla u), \tag{2.1}$$

where u models the temperature and $\kappa > 0$ represents the thermal conductivity. Here it is supposed that the caloric flux obeys the Fourier law $q = -\kappa \nabla T$ and that the thermal conductivity is independent of temperature. The condition $\kappa > 0$ reflects the fact that heat propagates from high to lower temperature.

If the temperature of the body is high enough one must consider the radiation effects and the temperature dependence of thermal conductivity. For example, if the power radiated by a body to environment follows the Stefan-Boltzmann law of the forth powers, for both the body and the medium, the heat equation becomes [8]

$$\frac{\partial u}{\partial t} = \operatorname{div}(\kappa(u) \nabla u) - k_r(u^4 - u_e^4). \tag{2.2}$$

The unsaturated water flow through porous media is described by the well known Richards' equations [7]

$$\frac{\partial \theta(h)}{\partial t} - \operatorname{div}(K(h)\nabla h + \boldsymbol{e}_3 K(h)) = 0, \tag{2.3}$$

where θ represents the relative volumetric water content, h represents the pressure head, K is the hydraulic conductivity and $\boldsymbol{e}_3$ is the upward vertical versor. The function $\theta(h)$ is a continuous positive function and it is strictly increasing function on the interval $(-\infty, 0]$ and a constant function on $h > 0$. Also the hydraulic conductivity is a continuous positive function strictly increasing on $(-\infty, 0]$ and a constant function on the set $h > 0$. The hydraulic conductivity becomes zero as h approaches $-\infty$.

The transport of contaminant in porous media is governed by an equation of the form [9], [10]

$$\frac{\partial\left(C+\lambda C^p\right)}{\partial t}+\boldsymbol{v}\cdot\nabla C=\operatorname{div}(D\nabla C)+g(x,C), \tag{2.4}$$

where C represents the mass concentration of the contaminant, $\boldsymbol{v}$ denotes the velocity of the fluid flow, supposed to be constant. The term λC^p, $\lambda \geq 0$ takes into account the adsorption reaction by means of Freundlich isotherm. The absorption reaction is described by the term $g(x, C)$ that usually is given by

$$g=-\alpha C^q \tag{2.5}$$

with $\alpha > 0$, $q > 0$ (the order of the reaction).

An extremely used form of the GPME is given by the

$$\frac{\partial u}{\partial t}=\triangle u^m+\lambda u^r. \tag{2.6}$$

For $m > 1$ (slow diffusion) the equation models the flow of the gas through porous medium for $m < 1$ (fast diffusion) the model is encountered in plasma physics, kinetic theory and solid state.

The Stefan problem can be written as a GPME equation with

$$\phi(u)=\lambda\begin{cases}\max\{0,(u-1)\}, & \text{if } u\geq 0,\\ u, & \text{if } u<0.\end{cases}$$

3. Mathematical Settings

In this section we review some results concerning the solution of the nonlinear diffusion equations.

The constitutive functions are supposed to satisfy:

A1
$b : \mathbb{R} \to \mathbb{R}$,	is a continuous and nondecreasing function,
$\kappa : \mathbb{R} \to \mathbb{R}_+$,	is a continuous and nondecreasing function,
$f : \mathbb{R} \to \mathbb{R}^n$,	is a local Lipschitz vector function,
$g : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$,	is a Caratheodory function.

A2 $\phi \in C([0,\infty)) \cup C^1((0,\infty))$, $\phi(0) = 0$, nodecreasing function,
$r \in C([0,\infty))$, $r(0) = 0$.

We consider the Cauchy problem for both equations. The domain Ω on which the problem is considered satisfies:

A3 $\Omega \in \mathbb{R}^n$, is an open, bounded and connected set.

The initial conditions and boundary data are written as

$$\begin{cases} u(0,x) = u_0(x), & x \in \Omega. \\ u = u_D, & t > 0, x \in \partial\Omega. \end{cases} \tag{3.1}$$

We assume that

A4 $u_0 \in L^\infty(\Omega)$,
$u_D \in L^2((0,T) : W^{1,2}(\Omega)) \cap L^\infty((0,T) \times \Omega)$.

Cauchy problem for CDRE. The Cauchy problem is defined by the equation (1.1) in a domain Ω in $\mathbb{R}^n$, the initial condition and boundary data (3.1).

Due to the nonlinear parabolic term $b(u)$ and nonlinear diffusion coefficient $\kappa(u)$ the problem (1.1) can be a degenerate problem and consequently there exists no classical solutions.

The notion of *weak solution* for the problem of the type (1.1) was introduced by Alt and Luckhaus in [1]. By imposing some proper conditions on the constitutive functions, boundary data and initial conditions, the authors were able to prove the existence of the weak solution in the case of the parabolic-elliptic degeneration, $b(u)$ is a constant function on some interval of positive measure and the diffusion coefficient is a strict positive function.

DEFINITION **3.1** (Weak Solution (H. W. Alt and S. Luckhaus)) *A measurable function u is a weak solution of the Cauchy problem* (1.1) *and* (3.1) *if it satisfies:*

1) $u - u_D \in L^2((0,T) : W_0^{1,2}(\Omega))$,

2) $b(u) \in L^\infty((0,T) : L^1(\Omega))$ *and* $\dfrac{\partial b(u)}{\partial t} \in L^2((0,T) : W^{-1,2}(\Omega))$ *with initial*

values $b(u_0)$, that is,

$$\int_0^T \left\langle \frac{\partial b(u)}{\partial t}, v \right\rangle \mathrm{d}t + \int_0^T \int_\Omega (b(u) - b(u_0)) \frac{\partial v}{\partial t} \mathrm{d}x\mathrm{d}t = 0 \tag{3.2}$$

for every $v \in L^2((0,T) : W_0^{1,2}(\Omega)) \cap W^{1,1}((0,T) : L^1(\Omega))$, $v(T,\cdot) \equiv 0$
3) $\kappa(u)\nabla u, g(\cdot,\cdot,u(\cdot,\cdot)) \in L^2((0,T)\times\Omega), f(u) \in \left(L^2((0,T)\times\Omega)\right)^n$ *and u satisfies the differential equation, that is,*

$$\int_0^T \left\langle \frac{\partial b(u)}{\partial t}, v \right\rangle \mathrm{d}t + \int_0^T \int_\Omega (\kappa(u)\nabla u + f(u)) \cdot \nabla v \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega g(t,x,u) v \mathrm{d}x\mathrm{d}t \tag{3.3}$$

for every test function $v \in L^2(0,T : W_0^{1,2}(\Omega))$.

In the paper [7] Carrillo extrapolates the concept of entropy solution introduced by Kruzhkov in theory of hyperbolic PDE [14]. He showed that there exists a unique *weak entropy solution* of the Cauchy problem with homogeneous boundary data, $u_D = 0$, even in the case of parabolic-hyperbolic degeneration. Such kind of degeneration appears when the diffusion coefficient is a null function on some interval with the positive measure.

The weak entropy solution is a weak solution that in addition satisfies an integral entropy inequality.

Let us introduce the function

$$K(u) = \int_0^u \kappa(s)\mathrm{d}s,$$

DEFINITION **3.2** (Weak entropy solution. Homogeneous case (Carrillo)) *An weak entropy solution of the Cauchy problem* (1.1) *and* (3.1) *with $u_D = 0$, is* *a weak solution which in addition satisfies the entropy inequality*

$$\int_0^T\int_\Omega H_0(u-s)\Big((\nabla K(u)+f(u)-f(s))\cdot\nabla v- \\ -(b(u)-b(s))\frac{\partial v}{\partial t}-gv\Big)\mathrm{d}x\mathrm{d}t-\int_\Omega (b(u_0)-b(s))^+v(0)\mathrm{d}x\le 0,$$

$$\int_0^T\int_\Omega H_0(-s-u)\Big((\nabla K(u)+f(u)-f(-s))\cdot\nabla v- \\ -(b(u)-b(-s))\frac{\partial v}{\partial t}-gv\Big)\mathrm{d}x\mathrm{d}t-\int_\Omega (b(u_0)-b(-s))^-v(0)\mathrm{d}x\ge 0, \tag{3.4}$$

for any $(s,v)\in\mathbb{R}\times\big(L^2((0,T):W^{1,2}(\Omega))\cap W^{1,1}((0,t):L^\infty(\Omega))\big)$ *such that* $s\ge 0, v\ge 0$ *and* $v(T)=0$.

In the entropy conditions the following notations:

$$H_0(s)=\begin{cases}1, & \text{if } s>0\\ 0, & \text{if } s\le 0\end{cases}\qquad s^+=\begin{cases}s, & \text{if } s>0\\ 0, & \text{if } s\le 0\end{cases}$$

were used. If $\kappa>0$ then the two definitions of the weak solution coincide and any weak solution is an entropy solution [7].

To deal with nonhomogeneous Dirichlet conditions for degenerate problem one supplementary difficulty is to give a sense to boundary conditions. In the paper [18] C. Mascia, A. Porreta and A. Terracina proved the existence of the weak entropy solution of the Cauchy problem with nonhomogeneous Dirichlet data. Their definition is as follows. Denote by Q_T the direct product $Q_T=(0,T)\times\Omega$. Also we use the notations:

$$\mathcal{E}(u,v)=\nabla|K(u)-K(v)|+\operatorname{sgn}(u-v)(f(u)-f(v)),$$
$$\mathcal{B}(u,v,w)=\mathcal{E}(u,v)+\mathcal{E}(u,w)-\mathcal{E}(v,w).$$

The domain Ω is such that there exists a C^2–covering of $\partial\Omega$, $\mathcal{A}=\{U_i\}_{i=1,m}$, of open sets such that $\partial\Omega\subset\cup\overline{U}_i$ and, in some local coordinates $x=(x',x_n)$, there exists a C^2 function $x_n=\alpha_i(x')$ such that $U_i\cap\partial\Omega=\{x_n=\alpha_i(x')\}$, $U_i\cap\Omega=\{x_n<\alpha_i(x'\}$.

A sequence $\{\vartheta_\delta\}$ of $C^2(\Omega)\cap C^0(\overline{\Omega})$ functions is named a boundary layer sequence if

$$\lim_{\delta\to 0^+}\vartheta_\delta=1,\ \text{pointwise in }\Omega,\ 0\le\vartheta_\delta\le 1,\ \vartheta_\delta=0\text{ on }\partial\Omega.$$

DEFINITION **3.3** (Weak Entropy Solution. Nonhomogeneous case (Mascia et al.)) *A function $u \in L^\infty((0,T)\times\Omega)$ is an entropy solution of Cauchy problem* (1.1) *and* (3.1) *if*

1) (regularity)

$$K(u) \in L^2((0,T) : W^{1,2}(\Omega))$$

and for any $U \in \mathcal{A}$, and any positive $\psi \in C_0^\infty(U)$ we have

$$\Big(-|u-u_D|\psi, \mathcal{E}(u,u_D)\psi \Big) \in \mathcal{DM}(Q)_2,$$

where $\mathcal{DM}(Q)_2$ is the set of divergence-measure vector fields in Q.

2) (entropy condition in interior of Q_T)

$$\int_{Q_T} \left\{ |b(u)-b(s)| \frac{\partial v}{\partial t} - \mathcal{E}(u,s)\nabla v + gv \right\} \mathrm{d}x\mathrm{d}t \geq 0$$

for any $v \in W_0^{1,2}(Q_T)$ and $v \geq 0$ and $s \in \mathbb{R}$.

3) (initial condition)

$$\lim_{t\to 0^+} \int_\Omega |u(t,x) - u_0(x)| \,\mathrm{d}x = 0$$

4) (boundary conditions) *in sense of trace in $L^2((0,T) : W^{1,2}(\Omega))$ we have*

$$K(u) = K(u_D), \quad t > 0,\ x \in \partial\Omega,$$

and for any boundary layer sequence ϑ_δ, and for any $U \in \mathcal{A}$, and any positive $\psi \in C_0^\infty(U)$ we have

$$\liminf_{\delta\to 0} \int_{Q_T} \mathcal{B}(u,s,u_D)\nabla\vartheta_\delta \xi\psi \mathrm{d}x\mathrm{d}t \geq 0,\ \forall s \in \mathbb{R},$$

for any $\xi \in L^2((0,T) : W^{1,2}(\Omega)), \xi \geq 0$.

Cauchy problem for GPME. The Cauchy problem consists in the equation (1.2) and the data (3.1).

The existence of the weak solution was proved by many authors see for example, [4], [25].

DEFINITION **3.4** (M. Borelli and M. Ughi) *A nonnegative function u defined on the* $\overline{\Omega} \times [0,T]$ *is said to be a* weak solution *of the Cauchy problem* (1.2) *and* (3.1) *if*

1) $u \in C\left([0,T]; L^1(\Omega)\right) \cap L^\infty([0,T] \times \Omega)$,

2) *for any test function* $\eta \in C^{1,0}([0,T] \times \overline{\Omega}) \cap C^{2,1}((0,T] \times \Omega)$ *such that* $\eta \geq 0$ *on* $(0,T] \times \Omega)$ *and* $\eta = 0$ *on* $(0,T] \times \partial\Omega$ *u satisfies the integral identity:*

$$\int\limits_{\Omega} u(t,x)\eta(t,x)\mathrm{d}x = \int\limits_{\Omega} u_0(x)\eta(0,x)\mathrm{d}x - \int\limits_0^t \int\limits_{\partial\Omega} \phi(u_D)\frac{\partial\eta}{\partial n} + \\ + \int\limits_0^t \int\limits_{\Omega} \left[u\partial_t\eta + \phi(u)\triangle\eta + r(u)\eta\right] \mathrm{d}t\mathrm{d}x \tag{3.5}$$

for any $0 \leq t \leq T$.

The presence of the reaction term and nonlinearity in the equation (1.2) generate interesting phenomena namely, extinction time or blow up of the solution and the finite speed of propagation of disturbance [25].

Such problems have been studied by several authors: Borelli-Ughi [4], Ferreira-Vasquez [13], Leoni [16], Levin-Sacks [17], Peletier and Z. Junninig [23]. In the case $r(u) = 0$ and $\phi(s) = s^m, 0 < m < 1, u_D = 0$ there exists an extinction time T_e such that the problem (1.2 has a unique classical solution, positive on $\Omega \times [0, T_e]$ and null for $t \geq T_e$, see [17].

For generalized fast diffusion with strong absorption and $\Omega = \mathbb{R}^2$ there also exists an extinction time and the support of the solution is bounded for any time $t > 0$, [4].

In the power case, $\phi(s) = s^m, r(s) = \lambda p^s, \lambda > 0$, the numerical methods to compute the solution of the similar problem (1.2) have been proposed by M.-N. Le Roux, [21] the case $m > 1$, M.-N. Le Roux and P.-E. Mainge, [22].

Pointwise comparison principle. For both Cauchy problems CDRE and GPME there exists several comparison criteria [1], [10], [25]. We will give here a result that allows one to compare two solutions with respect to their boundary and initial conditions.

For any two real functions $f(x)$ and $g(x)$ we write $f \leq g$ if $f(x) \leq g(x), \forall x \in \Omega$. In addition to assumptions **A1** the constitutive functions in CDRE problem satisfy

$$\mathbf{A1'}\ \left\|\ \begin{array}{l} (1)\, \kappa : \mathbb{R} \to \mathbb{R}_+,\ \kappa(u) \geq \eta, \\ (2)\, |\kappa(u_1) - \kappa(u_2)| < C|u_1 - u_2|^{\gamma_1},\ \gamma_1 > \dfrac{1}{2},\ \forall u_1, u_2 \in \mathbb{R}, \\ (3)\, |\boldsymbol{f}(u_1) - \boldsymbol{f}(u_2)| < C|u_1 - u_2|^{\gamma_2},\ \gamma_2 > \dfrac{1}{2},\ \forall u_1, u_2 \in \mathbb{R}, \\ (4)\, g(u_1) - g(u_2) < C(b(u_1) - b(u_2)),\ \text{for } u_1 > u_2. \end{array} \right.$$

THEOREM **3.1** (Comparison Theorem) *Let* (u_D, u_0)*,* $(\widehat{u}_D, \widehat{u}_0)$ *be such that* $u_D \leq \widehat{u}_D, u_0 \leq \widehat{u}_0$*. Let* u *and* $\widehat{u}$*) be two bounded weak solutions of the Cauchy problem* (1.1)*,* (3.1) *associated to* (u_D, u_0) *and* $(\widehat{u}_D, \widehat{u}_0)$ *respectively. Assume, in addition, that*

$$b(u)_t, b(\widehat{u})_t \in L^1((0,T) \times \Omega).$$

Then

$$u \leq \widehat{u}$$

on $(0,T) \times \Omega$*.*

Proof. We follow the main ideas from [1]. As in [1] for any $\delta > 0$ let $\Psi_\delta(\alpha) = \min(1, \max(0, \alpha/\delta))$. The function $w = \Psi_\delta(u - \widehat{u})$ belongs to $L^2(0, T : W_0^{1,2}(\Omega))$ and its gradient is given by

$$\nabla w = \begin{cases} \dfrac{1}{\delta}\left(\nabla u - \nabla \widehat{u}\right), & \text{if } 0 < u - \widehat{u} < \delta \\ 0, & \text{otherwise} \end{cases}$$

Set w as test function in (3.3). Then

$$\int_0^t \int_\Omega \left(b(u)_t - b(\widehat{u})_t\right) w \mathrm{d}x\mathrm{d}t + \underbrace{\frac{1}{\delta} \int_0^t \int_{\Omega_\delta} \left(\kappa(u)\nabla u - \kappa(\widehat{u})\nabla \widehat{u}\right) \nabla(u - \widehat{u}) \mathrm{d}x\mathrm{d}t}_{I_1} +$$
$$+ \underbrace{\frac{1}{\delta} \int_0^t \int_{\Omega_\delta} \left(\boldsymbol{f}(u) - \boldsymbol{f}(\widehat{u})\right) \cdot \nabla(u - \widehat{u}) \mathrm{d}x\mathrm{d}t}_{I_2} = \int_0^t \int_\Omega \left(g(u) - g(\widehat{u})\right) w \mathrm{d}x\mathrm{d}t, \tag{3.6}$$

where $\Omega_\delta := \{x | 0 < h - \widehat{h} < \delta\}$. The integral I_1 can be rewritten as

$$I_1 = \int_0^t \int_{\Omega_\delta} \kappa(u) ||\nabla(u - \widehat{u})||^2 \mathrm{d}x\mathrm{d}t + \int_0^t \int_{\Omega_\delta} (\kappa(u) - \kappa(\widehat{u})) \nabla \widetilde{u} \cdot \nabla(u - \widehat{u}) \mathrm{d}x\mathrm{d}t.$$

Using Young inequality, $ab \leq C(\epsilon)p^{-1}a^p + \epsilon q^{-1}b^q$, and $\mathbf{A1'}$-(1) we obtain

$$I_1 \geq \left(\eta - \frac{\epsilon}{2}\right) \int\limits_0^t \int\limits_{\Omega_\delta} ||\nabla(u-\widehat{u})||^2 \mathrm{d}x\mathrm{d}t - \frac{C(\epsilon)}{2} \int\limits_0^t \int\limits_{\Omega_\delta} (\kappa(u) - \kappa(\widehat{u}))^2 ||\nabla \widetilde{u}||^2 \mathrm{d}x\mathrm{d}t$$

and

$$I_2 \geq -\frac{\epsilon}{2} \int\limits_0^t \int\limits_{\Omega_\delta} ||\nabla(u-\widehat{u})||^2 \mathrm{d}x\mathrm{d}t - \frac{C(\epsilon)}{2} \int\limits_0^t \int\limits_{\Omega_\delta} ||\boldsymbol{f}(u) - \boldsymbol{f}(\widehat{u})||^2 \mathrm{d}x\mathrm{d}t.$$

Then

$$I_1 + I_2 \geq (\eta - \epsilon) \int\limits_0^t \int\limits_{\Omega_\delta} ||\nabla(u-\widehat{u})||^2 \mathrm{d}x\mathrm{d}t - C\delta^{2\gamma} \int\limits_0^T \int\limits_{\Omega_\delta} (||\nabla \widetilde{u}||^2 + 1) \mathrm{d}x\mathrm{d}t.$$

From $\mathbf{A1'}(4)$ the production can be estimate as

$$\int\limits_0^t \int\limits_{\Omega} (g(u) - g(\widehat{u}))\, w \mathrm{d}x\mathrm{d}t \leq \int\limits_0^t \int\limits_{\Omega} 1_{\{u-\widehat{u}>0\}} \max\{0, g(u) - g(\widehat{u}\} \mathrm{d}x\mathrm{d}t \leq$$

$$\leq C \int\limits_0^t \int\limits_{\Omega} \max\{0, b(u) - b(\widehat{u})\} \mathrm{d}x\mathrm{d}t.$$

Taking $\epsilon < \eta$ we obtain

$$\begin{aligned} \int\limits_0^t \int\limits_{\Omega} \quad & (b(u)_t - b(\widehat{u})_t)\, w \mathrm{d}x\mathrm{d}t + \frac{c}{\delta} \int\limits_0^t \int\limits_{\Omega_\delta} ||\nabla(u-\widehat{u})||^2 \mathrm{d}x\mathrm{d}t \leq \\ & \leq C\delta^{2\gamma-1} \int\limits_0^T \int\limits_{\Omega_\delta} (||\nabla \widetilde{u}||^2 + 1) \mathrm{d}x\mathrm{d}t + \int\limits_0^t \int\limits_{\Omega} \max\{0, b(u) - b(\widehat{u})\} \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.7}$$

For $\delta \to 0$ the first term on the right converge to 0 and the first term on left becomes

$$\lim_{\delta \to 0} \int\limits_0^t \int\limits_{\Omega} (b(u)_t - b(\widehat{u})_t)\, w \mathrm{d}x\mathrm{d}t = \int\limits_0^t \int\limits_{\Omega} 1_{\{u-\widehat{u}>0\}} (b(u)_t - b(\widehat{u})_t) \mathrm{d}x\mathrm{d}t =$$

$$= \int_0^t \int_\Omega \partial_t \max\{b(u) - b(\widehat{u}), 0\} \mathrm{d}x\mathrm{d}t = \int_\Omega \max\{b(u) - b(\widehat{u}), 0\}(t,x)\mathrm{d}x.$$

One obtains

$$\int_\Omega \max\{0, b(u) - b(\widehat{u})\} \mathrm{d}x\mathrm{d}t \le \int_0^t \int_\Omega \max\{0, b(u) - b(\widehat{u})\} \mathrm{d}x\mathrm{d}t,$$

and using Gronwall's inequality we get

$$b(u) \le b(\widehat{u}),$$

and using this inequality in (3.7) we have $\nabla(u - \widehat{u}) = 0$ on the set $\{0 < u - \widehat{u}\}$. So, we have $u - \widehat{u}$=const. which implies $u - \widehat{u} \le 0$ since on boundary $u \le \widehat{u}$. As a corollary of the comparison principle one can obtain an upper bound for the solution of Cauchy problems in the both case CDRE and GPME equations.

COROLLARY **3.1** *Assume that* **A1** *and* **A1′** *are fullfiled and* $g(t,x,u) = g(u)$, $g(0) = 0$. *Let* u *be the solution of the problem* (1.1), (3.1) *on some interval* $[0,T]$. *Then*

1) *if* $u_D \ge 0$ *and* $u_0 \ge 0$ *so is* $u \ge 0$,

2) *Let* $\alpha = ||u_D||_{L^\infty([0,T]\times\partial\Omega)}$, $\beta = \max\{||u_0||_\infty, \alpha\}$. *If* $\alpha > 0$ *we assume that* $g(w) \ge 0$. *Let* $w(t)$ *be the solution of the differential equation*

$$\begin{aligned} &\partial_t b(w) = g(w) \\ &w(0) = \beta. \end{aligned}$$

on the same interval $t \in [0,T]$. *Then the solution* u *satisfies*

$$u < w \text{ on } [0,T].$$

Proof. 1). One compares the solution u with the trivial solution $v = 0$.

2). Define the function $v(t,x) = w(t), \forall x \in \overline{\Omega}$. The function $v(t,x)$ verifies the equation (1.1), at the time $t = 0$ $v(0,x) = \beta > u_0$ and on boundary $v(t,x)|_{x\in\partial\Omega} = w(t) \ge \beta > u_D$ that implies $u < v$.

COROLLARY **3.2** *In the GPME the diffusion function and production function are given by* $\phi(u) = u^m$, $r(u) = -\lambda u^s$ *respectively* $\lambda > 0, m > 0, s > 0$. *The initial conditions satisfy* **A4**, $u_0 > 0$ *and* $u_D = 0$. *Let* $\beta = ||u_0||_\infty$.
1) *If* $s > 1$ *then the solution* u *of the problem* 1.2, 3.1 *satisfies*

$$||u||_\infty < \beta \left(1 - \lambda(1-s)\beta^{s-1} t\right)^{\frac{1}{1-s}}.$$

2) *If* $s < 1$ *then there exists a time* T^*, *extinction time, given by*

$$T^* = \frac{1}{\lambda}\frac{\beta^{1-s}}{1-s}$$

such that the solution exists on the interval $[0, T^*]$ *and it satisfies*

$$||u||_\infty < \beta\left(1 - \frac{t}{T^*}\right)^{\frac{1}{1-s}}.$$

Proof. In the generalized porous medium equation

$$\partial_t u = \triangle u^m - \lambda u^s$$

we make the substitution $u^m = v$ and we obtain

$$\partial_t v^p = \triangle v - \lambda v^r,$$

$$v_{t=0} = u_0^m, v|_{x\in\partial\Omega} = 0,$$

where $p = 1/m, r = s/m$. By using the corollary 1 one obtain that the function v is bounded from above by the solution of differential equation

$$\begin{array}{l} pw^{p-1}w' = -\lambda w^r, \\ w(0) = \beta^m, \end{array}$$

which has the solution

$$w = \beta^m(1 - \lambda(1-s)\beta^{s-1}t)^{\frac{m}{1-s}}.$$

4. Quasimonotone ODE Approximation

4.1. Discrete Approximation

By the method of lines (MOL), one can associate an ordinary differential system of equations (ODE) to a parabolic partial differential equation. The MOL consists in the discretization of the space variable using one of the standard methods as finite element, finite differences or finite-volume method (FVM). The FVM fits very well to conservative equations and there exists a large literature devoted to the method, we recall here the papers that deal with Dirichlet problem, [6] for hyperbolic PDE, [11], [12], [19] for nonlinear parabolic PDE.

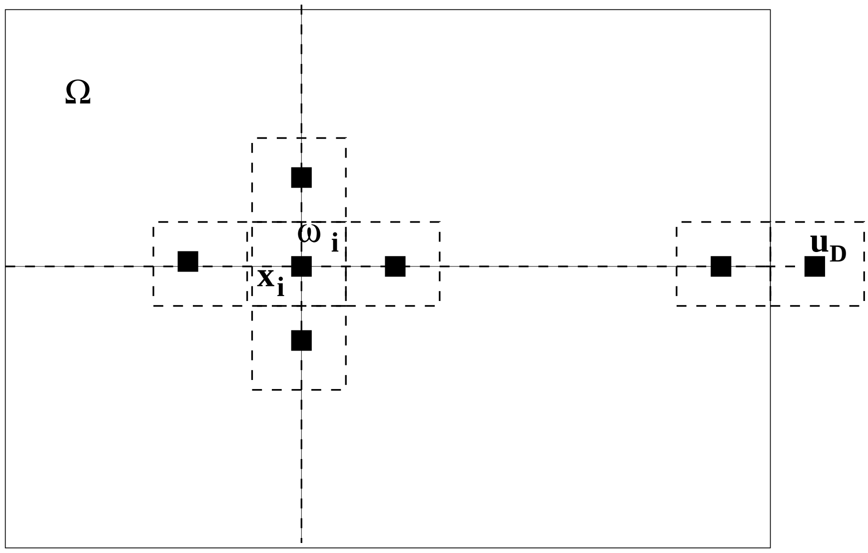


Fig. 1: Triangulation of polygonal domain in $\mathbb{R}^2$.

The FVM deals with a decomposition of the domain Ω into small polygonal domains ω_i and a net of the inner knots x_i. The assembly $\{\omega_i, x_i\}$ defines a triangulation of the domain and it is an admissible mesh if it satisfies the following conditions, [12].

DEFINITION **4.1** (Admissible mesh) *The triangulation $\mathcal{T} = \{(\omega_i, x_i)\}_{i\in I}$ is called an admissible mesh if it satisfies:*

$$
\textbf{A5}\ \left\|\ \begin{array}{l}
\omega_i \textit{ is open poligonal set} \subseteq \Omega,\ x_i \in \overline{\omega}_i \\
(1)\ \overline{\cup_{i\in I}\omega_i} = \overline{\Omega} \\
(2)\ \forall i \neq j \in I \textit{ and } \overline{\omega}_i \cap \overline{\omega}_j \neq \Phi, \textit{either } \mathcal{H}_{n-1}(\overline{\omega}_i \cap \overline{\omega}_j) = 0 \textit{ or} \\
\quad \sigma_{ij} := \overline{\omega}_i \cap \overline{\omega}_j \textit{ is a common } (n-1)\text{-face of } \omega_i \text{ and } \omega_j \\
(3)\ \textit{for all } \sigma_{ij}, \quad [x_i, x_j] \perp \sigma_{ij}
\end{array}\right.
$$

Here $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. For each volume ω_i that has a common $(n-1)$-face with the boundary $\partial\Omega$ one defines an external volume $\omega_{i_b} \in C\Omega$ by the reflection of the ω_i with respect to the face $\sigma_{i_b} = \omega_i \cap \partial\Omega$. Denotes by $\mathcal{T}^b$ the collection of all external volumes $\{(\omega_{i_b}, x_{i_b})\}$ and by I^b the set of their indices. Let $\mathcal{T}^e = \mathcal{T} \cup \mathcal{T}^b$ and $I^E = I \cup I^b$. We say that the volumes $\omega_i,\ \omega_j \in \mathcal{T}^e$ are neighbours if they share a common $n-1$–face and we denote by $\boldsymbol{n}_{i,j}$ the unit normal vector to the face σ_{ij} that point to ω_j.

Discrete form of CDRE. The space discretized equations are derived from the integral form of (1.1) for each control volume ω_i

$$\int_{\omega_i} \frac{\partial b(u)}{\partial t} \mathrm{d}x - \int_{\partial\omega_i} (\kappa(u)\nabla u + \boldsymbol{f}(u)) \cdot \boldsymbol{n} \mathrm{d}a = \int_{\omega_i} g(t,x,u)\mathrm{d}x, \ \forall i \in I. \quad (4.1)$$

By a proper approximation of the volume integrals and surface integrals one obtains discrete form of CDRE.

We define the numerical diffusion coefficient $\widetilde{\kappa} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\widetilde{\kappa}(u,v) = \max(\kappa(u), \kappa(v)). \quad (4.2)$$

It is easy to show that numerical diffusion coefficient satisfies

A6
$$\begin{array}{ll} \widetilde{\kappa}(u,v) = \widetilde{\kappa}(v,u), & \text{symmetry,} \\ (\widetilde{\kappa}(u_1,v) - \widetilde{\kappa}(u_2,v))(u_1-u_2) > 0, & \text{monotonicity,} \\ \widetilde{\kappa}(u,u) = \kappa(u), & \text{consistency.} \end{array}$$

Corresponding to each face σ_{ij} we admit that there exists a numerical flux function $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following properties:

A7
$$\begin{array}{ll} \widetilde{f}_{i,j}(u,v) = -\widetilde{f}_{j,i}(v,u), & \text{conservation,} \\ (\widetilde{f}_{i,j}(u_1,v) - \widetilde{f}_{i,j}(u_2,v))(u_1-u_2) \le 0, & \text{monotonicity,} \\ (\widetilde{f}_{i,j}(u,v_1) - \widetilde{f}_{i,j}(u,v_2))(v_1-v_2) \ge 0, & \\ \widetilde{f}_{i,j}(u,u) = \boldsymbol{f}(u) \cdot \boldsymbol{n}_{i,j}, & \text{consistency.} \end{array}$$

A numerical convective flux which satisfies **A7** is systematicaly used in the approximation of hyperbolic equation see, for example [6]. The integrals in (4.1) will be approximated as follows:

$$\begin{aligned} &\int_{\omega_i} \frac{\partial b(u)}{\partial t} \mathrm{d}x \approx \mathrm{m}(\omega_i) \frac{\partial b(u_i)}{\partial t}, \\ &\int_{\partial\omega_i} \kappa(u)\nabla u \cdot \boldsymbol{n} \mathrm{d}a \approx \sum_{j \in \mathcal{N}(i)} \mathrm{m}(\sigma_{ij}) \widetilde{\kappa}(u_i,u_j) \frac{u_j - u_i}{d_{ij}}, \\ &\int_{\partial\omega_i} \boldsymbol{f}(u) \cdot \boldsymbol{n} \mathrm{d}a \approx \sum_{j \in \mathcal{N}(i)} \widetilde{f}_{i,j}(u_i,u_j), \\ &\int_{\omega_i} g(t,x,u)\mathrm{d}x \approx \int_{\omega_i} g(t,x,u_i)\mathrm{d}x := g_i(t,u_i). \end{aligned}$$

$\mathcal{N}(i)$ denotes all neighbours in $\mathcal{T}^e$ of ω_i, $\mathrm{m}(\omega_i)$ represents the volume of polyhedron ω_i and $\mathrm{m}(\sigma_{ij})$ represents the $n-1$-dimensional measure of the face σ_{ij} and $d_{i,j} = |x_i - x_j|$.

The initial data and boundary conditions are approximated by:

$$u_{0i} = \frac{1}{\mathrm{m}(\omega_i)} \int\limits_{\omega_i} u_0(x)\mathrm{d}x, \tag{4.3}$$

$$u_{i_b} = \frac{1}{\mathrm{m}(\sigma_{i_b})} \int\limits_{\sigma_{i_b}} u_D \mathrm{d}a, \tag{4.4}$$

respectively.

As a result one can define a Cauchy problem for a system of ordinary differentila equations whose solution gives an approximation of the Cauchy problem (1.1), (3.1).

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \sum\limits_{j\in\mathcal{N}(i)} \dfrac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u_i, u_j)\dfrac{u_j - u_i}{d_{ij}} + \widetilde{f}_{i,j}(u_i, u_j)\right] + g_i(t, u_i) \\ u_i|_{t=0} = u_{0i}, \end{cases} \tag{4.5}$$

for $t > 0$ and for any $i \in I$.

Let us introduce the numerical diffusion-convection flux functions

$$\mathcal{F}_i(\boldsymbol{u}; \boldsymbol{u}_D) = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u_i, u_j)\frac{u_j - u_i}{d_{ij}} + \widetilde{f}_{i,j}(u_i, u_j)\right] \tag{4.6}$$

then the ODE approximation reads as

$$\frac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \mathcal{F}_i(\boldsymbol{u}; \boldsymbol{u}_D) + g_i(t, u_i). \tag{4.7}$$

The boundary conditions are taken into account by the volume elements next to boundary $\partial\Omega$. For such element the contribution of the boundary values to the $\mathcal{F}_i$ is given by

$$\frac{\mathrm{m}(\sigma_{i_b})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u_{i_b}, u_j)\frac{u_{i_b} - u_i}{d_{i_b}} + \widetilde{f}_{i,i_b}(u_i, u_{i_b})\right]. \tag{4.8}$$

Infiltration model. Here is an example of a numerical convective flux that satisfies **A7** with $\boldsymbol{f}(u) = \boldsymbol{e}_3 K(u)$ that appears in the Richards' equation (2.3).

$$\widetilde{f}_{i,j}(u, v) = \frac{1}{2}\left(\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j} + |\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j}|\right) K(v) + \frac{1}{2}\left(\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j} - |\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j}|\right) K(u). \tag{4.9}$$

Discrete form of GPME. For each control volume ω_i we write

$$\int_{\omega_i} \frac{\partial u}{\partial t}\mathrm{d}x - \int_{\partial\omega_i} \frac{\partial \phi(u)}{\partial n}\mathrm{d}a = \int_{\omega_i} r(u)\mathrm{d}x,\ \forall i \in I. \tag{4.10}$$

To approximate (4.10) we use the same schemes as in previous paragraph. The new integral that contains the diffusion function ϕ will be approximated by

$$\int_{\partial\omega_i} \frac{\partial \phi(u)}{\partial n}\mathrm{d}a \approx \sum_{j\in\mathcal{N}(i)} \mathrm{m}(\sigma_{ij})\frac{\phi(u_j)-\phi(u_i)}{d_{ij}}. \tag{4.11}$$

The ODE approximation of (4.10) is given by

$$\frac{\partial u_i}{\partial t} = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)}\frac{\phi(u_j)-\phi(u_i)}{d_{ij}} + r(u_i). \tag{4.12}$$

The boundary conditions are taken into account by the volume elements next to boundary $\partial\Omega$. For such an element the boundary values enters into the play by a term of the form

$$\frac{\mathrm{m}(\sigma_{i^e})}{\mathrm{m}(\omega_i)}\frac{\phi(u_D^{i_e})-\phi(u_i)}{d_{ij}^e}. \tag{4.13}$$

For shortness we introduce the notation

$$\mathcal{G}_i = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)}\frac{\phi(u_j)-\phi(u_i)}{d_{ij}}.$$

4.2. ODE Model

As in the continuum case we want to prove that the solutions of ODE (4.5) and (4.10) obey a comparison criterion.

For that, we firstly prove that $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$ satisfy Kamke conditions.

Lemma **4.1** *Assume* **A2**, **A6** *and* **A7**. *Then*

$$\mathcal{F}_i(\boldsymbol{u}^e) = 0,\ \mathcal{G}_i(\boldsymbol{u}^e) = 0 \tag{4.14}$$

for any constant state $u_i = u, \forall i \in I^e$.

$\boldsymbol{\mathcal{F}}$ *and* $\boldsymbol{\mathcal{G}}$ *satisfy Kamke conditions, that is*

$$\mathcal{F}_i(\boldsymbol{v}^e) \ge \mathcal{F}_i(\boldsymbol{w}^e),\ \mathcal{G}_i(\boldsymbol{v}^e) \ge \mathcal{G}_i(\boldsymbol{w}^e),\ \forall i \in I, \tag{4.15}$$

for any two vectors that satisfy $v_k \ge w_k,\ \forall k \in I^e$, *and* $v_i = w_i$.

Proof. To prove (4.14) we have

$$\mathcal{F}_i(\boldsymbol{u}^e) = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \boldsymbol{f}(u)\cdot \boldsymbol{n}_{ij} = 0.$$

We only prouve the counterpart relativ to $\boldsymbol{\mathcal{F}}$. To prove the Kamke conditions we have

$$\mathcal{F}_i(\boldsymbol{v}^e) - \mathcal{F}_i(\boldsymbol{w}^e) =$$

$$\sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u,v_j)\frac{v_j-u}{d_{ij}} + \widetilde{f}_{i,j}(u,v_j) - \widetilde{\kappa}(u,w_j)\frac{w_j-u}{d_{ij}} - \widetilde{f}_{i,j}(u,w_j)\right]$$

and from (4.2) and the monotonicity property of **A7** the affirmation results.

As $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}$ are both quasimonotone and nondecreasing with respect to boundary data vectorial functions the next two results are equaly true for discrete ODE (4.12).

Assumptions on source term

A1″
- There exists the real numbers $\underline{\alpha} < \alpha < \beta < \overline{\beta}$ such that
- (1) $b \in C^1((\underline{\alpha}, \overline{\beta}))$ and $b' > 0$ on $(\underline{\alpha}, \overline{\beta})$.
- There exists two Lipschitz functions $\overline{g}, \underline{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that
- (2) $\underline{g}(t,u) \le g(t,x,u) \le \overline{g}(t,u),\ \forall u \in (\underline{\alpha}, \overline{\beta})$,
- (3) $\underline{g}(t,\alpha) \le 0,\ \overline{g}(t,\beta \ge 0$.

THEOREM **4.1 (**Boundedness of discrete solutions) *Consider the Cauchy problem* (4.5). *Assume* **A1**, **A1″**, **A4**, **A6**, **A7**. *We suppose also that initial conditions and boundary data satisfy*

$$\alpha \le u_0(x) \le \beta, \text{a.e } x \in \Omega, \alpha \le u_D(t,x) \le \beta, \text{ a.e } (t,x) \in (0,T)\times\Omega. \quad (4.16)$$

Let $\underline{u}(t)$ *be the solution of the problem*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} = \underline{g}(t,u) \\ \\ |u|_{t=0} = \alpha, \end{cases} \quad (4.17)$$

$\overline{u}(t)$ *be the solution of the problem*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} = \overline{g}(t,u) \\ \\ |u|_{t=0} = \beta \end{cases} \quad (4.18)$$

and $T_{\sup} = \inf(\sup\{t|\underline{u}(t) > \underline{\alpha}, \overline{u}(t) < \overline{\beta}\}, T)$ *Then the solution* $\boldsymbol{u}(t)$ *of the Cauchy problem is bounded by* $\underline{u}$ *and* $\overline{u}$ *on the interval* $[0, T_{\sup}]$ *i.e.,*

$$\underline{u}(t) \le u_i(t) \le \overline{u}(t)\, \forall i \in I, \forall t \in [0, T_{\sup}] \tag{4.19}$$

Proof. The essential tool in the proof is the Nickel's theorem that provide the monotony of the solution of the quasimonotone ODE. The Kamke conditions ensure us that we deal with quasimonotone system.

Observe that the conditions $\mathbf{A1}''$-3 guaranties that

$$\underline{\alpha} \le \underline{u}(t) \le \alpha,\ \beta \le \overline{u}(t) \le \overline{\beta}. \tag{4.20}$$

Define

$$\underline{\mathcal{F}}_i(\boldsymbol{u}) = \mathcal{F}_i(\boldsymbol{u}; \underline{\boldsymbol{u}}), \overline{\mathcal{F}}_i(\boldsymbol{u}) = \mathcal{F}_i(\boldsymbol{u}; \overline{\boldsymbol{u}}).$$

From (4.4), (4.8), (4.15), (4.20) and the conditions $\mathbf{A1}'$-2 one obtains

$$\underline{\mathcal{F}}_i(\boldsymbol{u}) + \underline{g}(t, u) \le \mathcal{F}_i(\boldsymbol{u}; \boldsymbol{u}_D) + g_i(t, u) \le \overline{\mathcal{F}}_i(\boldsymbol{u}) + \overline{g}(t, u).$$

Since $u_i^{\sup}(t) = \overline{u}(t), \forall i \in I$ is a solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \overline{\mathcal{F}}_i(\boldsymbol{u}) + \overline{g}(t, u_i) \\ u_i|_{t=0} = \beta, \end{cases} \tag{4.21}$$

$u_i^{\inf}(t) = \underline{u}(t), \forall i \in I$ is a solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \underline{\mathcal{F}}_i(\boldsymbol{u}) + \underline{g}(t, u_i) \\ u_i|_{t=0} = \alpha, \end{cases} \tag{4.22}$$

and $\alpha \le u_{0i} < \beta$ one can apply the Nickel's theorem and one obtains

$$u_i^{\inf}(t) \le u_i(t) \le u_i^{\sup}(t),$$

which is (4.19).

THEOREM **4.2** (Comparison theorem. Discrete case) *Assume we are as in the boundedness theorem. Let* $\boldsymbol{u}(t)$ *and* $\widehat{\boldsymbol{u}}(t)$, $t \in (0, T)$, *be the solutions of the problem* (4.5) *associated to* $(\boldsymbol{u}_D, \boldsymbol{u}_0)$ *and* $(\widehat{\boldsymbol{u}}_D, \widehat{\boldsymbol{u}}_0)$ *respectively. Suppose that*

$$\boldsymbol{u}_D \le \widehat{\boldsymbol{u}}_D < 0,\ \boldsymbol{u}_0 \le \widehat{\boldsymbol{u}}_0 < 0.$$

Then

$$\boldsymbol{u} \le \widehat{\boldsymbol{u}}$$

on $(0, T)$.

Proof. The same as in the boundedness theorem.

5. Numerical Algorithms and Numerical Results

In this section we give two numerical algorithms to solve GPME equation and Richards' equations respectively.

5.1. Fast Diffusion with Strong Absorption

We will present here an algorithm to solve numerically (4.12) in the case of the fast diffusion with strong absorption. In addition to assumptions **A2** the constitutive functions ϕ and r satisfy

A2′ $\left\| \begin{array}{l} \phi \text{ is increasing function and } \lim_{s\to 0}\phi(x)/x = \infty, \\ r(s) \leq 0, \text{ for } s > 0, \end{array} \right.$

The ODE can be rewritten as

$$\frac{\partial u_i}{\partial t} = A_{ij}\phi(u_j) + r(u_i). \tag{5.1}$$

We use the classical full implicit Euler time integration scheme to integrate the system. It follows

$$u^{n+1} = u^n + \triangle t \left(A\phi(\mathbf{u}^{n+1}) + r(u^{n+1})\right), \tag{5.2}$$

where $\triangle t$ represents the time step. Depending on the initial data u_0 and the type of nonlinearity of the functions ϕ and r to solve the arising system can be a very hard problem, in the vicinity of the zero the derivative of the function ϕ in the case of fast diffusion become infinite. We propose here an algorithm suggested by the Gauss-Sidel iterative method. The method uses the very special structure of the matrix A generated by finite volume method. One writes the matrix A as

$$A = \widetilde{A} + \Gamma,$$

where Γ is a diagonal matrix containing the diagonal entries of the matrix A. We point the following properties of the two matrices

$$\widetilde{A}_{ij} \geq 0,\ \Gamma_{ii} < 0,\ \sum_j \widetilde{A}_{ij} \leq -\Gamma_{ii}. \tag{5.3}$$

We rewrite also the functions ϕ and r as

$$\phi(x) = \widetilde{\phi}(x)\cdot x,\ r(x) = -\widetilde{r}(x)\cdot x. \tag{5.4}$$

The equation (5.2) can be written now as

$$\left(I + \triangle t\left(-\Gamma\widetilde{\phi}(u^{n+1}) + \widetilde{r}(u^{n+1})\right)\right)u^{n+1} = u^n + \triangle t\widetilde{A}\phi(u^{n+1}). \tag{5.5}$$

The next theorem gives the main properties of the solution of implicit Euler method.

THEOREM **5.1** *In addition of the conditions* **A2** *and* **A2′** *we assume that* $\widetilde{\phi}$ *is a nonincreasing function and* $\widetilde{r} \geq 0$. *If the initial data and boundary conditions are positive and upper bounded functions, i.e.*

$$0 \leq u_0 \leq \rho,\ 0 \leq u_D \leq \rho,$$

then for any time step $\triangle t$ *there exists a solution of the equation* (5.2) *that satisfies*

$$0 \leq u^n \leq \rho,\ \forall n. \tag{5.6}$$

Proof. Let us assume that for a time level n there exists a solution u^n that satisfies (5.6). We will use the Browder fixed point theorem to demonstrate the existence of u^{n+1} with the same properties. Define the $\mathbb{R}^N$-values function Ψ by

$$\Psi_i(y) = \frac{u_i^n + \triangle t\sum_j \widetilde{A}_{ij}\phi(y_j)}{1 + \triangle t\left(-\Gamma_{ii}\widetilde{\phi}(y_i) + \widetilde{r}(y_i)\right)}.$$

We claim that the function Ψ is a continuous function on the set $[0,\rho]^N$ and take values in the same set. So, it has a fixed point.

Since $\widetilde{\phi}$ and $\widetilde{r}$ are continuous functions on $(0,\infty)$ and let us assume that their limits in 0 are finite we can prolong by continuity the function Ψ in 0. It is obviously that $\Psi_i > 0$. For the upper bound we have

$$\Psi_i(y) - \rho \leq \frac{u_i^n + \triangle t\sum_j \widetilde{A}_{ij}\phi(y_j)}{1 - \triangle t\Gamma_{ii}\widetilde{\phi}(y_i)} - \rho =$$

$$= \frac{u_i^n - \rho + \triangle t\left(\sum_j \widetilde{A}_{ij}\phi(y_j) + \rho\Gamma_{ii}\widetilde{\phi}(y_i)\right)}{1 - \triangle t\Gamma_{ii}\widetilde{\phi}(y_i)}.$$

For any $y \in [0,\rho]^N$ we have

$$\sum_j \widetilde{A}_{ij}\phi(y_j) + \rho\Gamma_{ii}\widetilde{\phi}(y_i) \leq \phi(\rho)\sum_j \widetilde{A}_{ij} + \rho\Gamma_{ii}\widetilde{\phi}(y_i) \leq$$

$$\leq -\phi(\rho)\Gamma_{ii} + \rho\Gamma_{ii}\widetilde{\phi}(y_i) = -\rho\Gamma_{ii}(\widetilde{\phi}(\rho) - \widetilde{\phi}(y_i)) \leq 0.$$

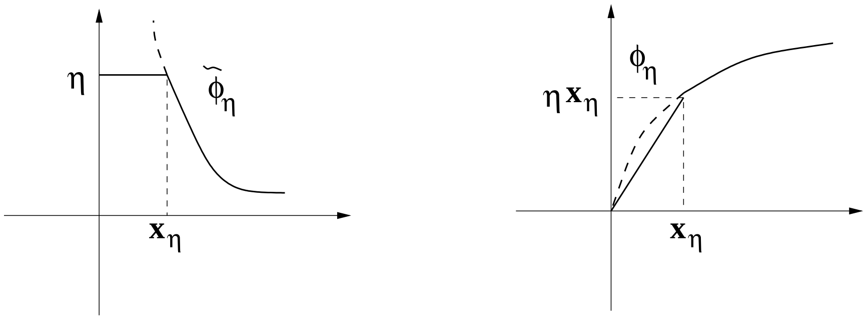


Fig. 2: The regularization of the flux function.

To obtain the first inequality one uses: assumptions **A2′** (ϕ is a nondecreasing function), boundary data is bounded from above by ρ and $\widetilde{A}_{ij} > 0$, the second inequality results from the structure of the matrix A and the last inequality from the constitutive assumption on the $\widetilde{\phi}$.

So, we have

$$0 \leq \Psi_i(y) \leq \rho$$

and for it results the existences of a fixed point, say u. Since for any i one has

$$1 + \triangle t \left(-\Gamma_{ii}\widetilde{\phi}(y_i) + \widetilde{r}(y_i) \right) < \infty, \text{ on } [0, \rho],$$

it follows that the fix point u is a solution of the of the nonlinear equation (5.6).

Let us analyse the case in which the functions $\widetilde{\phi}$ and $\widetilde{r}$ have infinite limits in 0. One regularises the function $\widetilde{\phi}$ by

$$\widetilde{\phi}_\eta(x) = \begin{cases} \eta, & \text{if } \widetilde{\phi}(x) > \eta \\ \widetilde{\phi}(x), & \text{if } \widetilde{\phi}(x) \leq \eta \end{cases} \tag{5.7}$$

and from it one has

$$\phi_\eta(x) = \begin{cases} x\eta, & \text{if } \phi(x) > x\eta \\ \phi(x), & \text{if } \phi(x) \leq x\eta. \end{cases} \tag{5.8}$$

Obviously

$$\phi_\eta(x) \leq \phi(x), \ \lim_{\eta\to\infty} \phi_\eta(x) = \phi(x).$$

In a similar manner we define r_η.

With the functions ϕ_η and r_η we are in the previous case and then results that there exists a solution $u_\eta \in [0, \rho]^N$ of the equation

$$u_\eta = u^n + \triangle t \left(A\phi_\eta(u_\eta) + r_\eta(u_\eta) \right). \tag{5.9}$$

As the sequence u_η is bounded we can extract a subsequence u_{η_n} that converges to a point $u \in [0,\rho]^N$. The problem is to demonstrate that the limit point u is a solution of the original equation, i.e.

$$u = u^n + \triangle t \left(A\phi(u) + r(u)\right).$$

Let us denote by $F_\eta(u)$ and F r.h.s., of the preceding equations, respectively. We have

$$\begin{aligned} ||u - F(u)||_\infty = \; & ||u - u_{\eta_n} + (F_{\eta_n}(u_{\eta_n}) - F_{\eta_n}(u)) + (F_{\eta_n}(u) - F(u)||_\infty \leq \\ & \leq ||u - u_{\eta_n}||_\infty + ||F_{\eta_n}(u_{\eta_n}) - F_{\eta_n}(u)||_\infty + \\ & + ||F_{\eta_n}(u) - F(u)||_\infty . \end{aligned}$$

We will show that, for any $\varepsilon > 0$,

$$||u - F(u)||_\infty \leq \varepsilon.$$

Observe that the first term and the last term can be made arbitrary small,

$$||u - u_{\eta_n}||_\infty + ||F_{\eta_n}(u) - F(u)||_\infty < \frac{\varepsilon}{2}$$

for any $n > n^\varepsilon$. The middle term can be evaluate as $||\cdot||_\infty$

$$\begin{aligned} ||F_{\eta_n}(u_{\eta_n}) - F_{\eta_n}(u)||_\infty \leq \; & \triangle t\big(||A(\phi_{\eta_n}(u_{\eta_n}) - \phi_{\eta_n}(u))||_\infty + \\ & + ||r_{\eta_n}(u_{\eta_n}) - r_{\eta_n}(u)||_\infty \big) \leq \\ \leq \; & \triangle t(||A||\,||\phi_{\eta_n}(u_{\eta_n}) - \phi_{\eta_n}(u)||_\infty + \\ & + ||r_{\eta_n}(u_{\eta_n}) - r_{\eta_n}(u)||_\infty). \end{aligned}$$

For each component i we look at

$$|\phi_{\eta_n}(u_{\eta_n i}) - \phi_{\eta_n}(u_i)|$$

and note that if u_i is not equal with zero then for a great enough number n one has

$$|\phi_{\eta_n}(u_{\eta_n i}) - \phi_{\eta_n}(u_i)| = |\phi(u_{\eta_n i}) - \phi(u_i)|,$$

if u_i equals zero then

$$|\phi_{\eta_n}(u_{\eta_n i}) - \phi_{\eta_n}(u_i)| = \phi_{\eta_n}(u_{\eta_n i}) \leq \phi(u_{\eta_n i}).$$

Using the continuity of the function ϕ we can find a number n_1^ε such that

$$||\phi_{\eta_n}(u_{\eta_n}) - \phi_{\eta_n}(u)||_\infty \leq \frac{\varepsilon}{4||A||\triangle t}$$

for any $n > n_1^{\varepsilon}$. Using the same arguments we can prove that

$$||r_{\eta_n}(u_{\eta_n}) - r_{\eta_n}(u)||_{\infty}) < \frac{\varepsilon}{4\triangle t}$$

for any $n > n_2^{\varepsilon}$. Hence, there exists a n^{ε} such that

$$||F_{\eta_n}(u_{\eta_n}) - F_{\eta_n}(u)||_{\infty} \leq \frac{\epsilon}{2}$$

for any $n > n_{\epsilon}$.

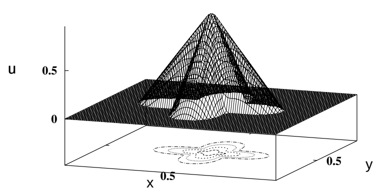
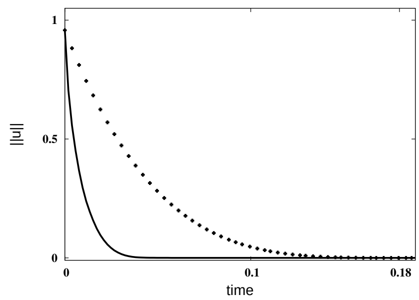
This end the proof of the theorem.

In our implementation we calculate the solution of the Euler scheme by the following iterative solver

$$\begin{gathered}\left(I + \triangle t \left(-\Gamma\widetilde{\phi}_{\eta}(u^{n+1,k}) + \widetilde{r}_{\eta}(u^{n+1,k})\right)\right) u^{n+1,k+1} = \\ u^n + \triangle t \widetilde{A}\phi_{\eta}(u^{n+1,k}).\end{gathered} \tag{5.10}$$

Numerical Simulation. For the numerical simulation we chose a very simple domain $\Omega = [0,1] \times [0,1]$. The fast diffusion with absorption is modeled by $\phi(s) = s^m, r(s) = -\lambda \cdot s^p$.

Table 1: Extinction phenomenon, extinction time $T^e = 0.18$. $\phi(s) = s^{0.75}$, $r(s) = -21. \times s^{0.5}$, $u_D = 0$

	
Initial Profile	Comparison of the numerical solution (solid line) with a theoretical estimation (points drawing).

5.2. Water Infiltration through Stratified Soil. Richard's Equation

We consider stratified soil. Hereafter the stratified soil means a block-wise homogeneous soil with horizontal parallel homogeneous strata, see figure (3).

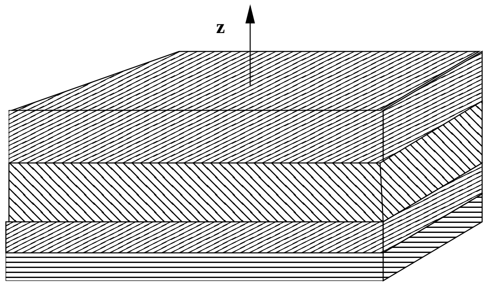


Fig. 3: Stratified porous soil. Each layer is modelled by different constitutive function.

In the case of stratified soil the different mechanical properties of the soils require different constitutive functions which in turn lead to a partial differential equation with discontinuous coefficient. On an interface of two different strata one must impose some compatible conditions to have a well defined problem. Physical considerations require the continuity of the pressure head and normal components of the velocity. So, we have

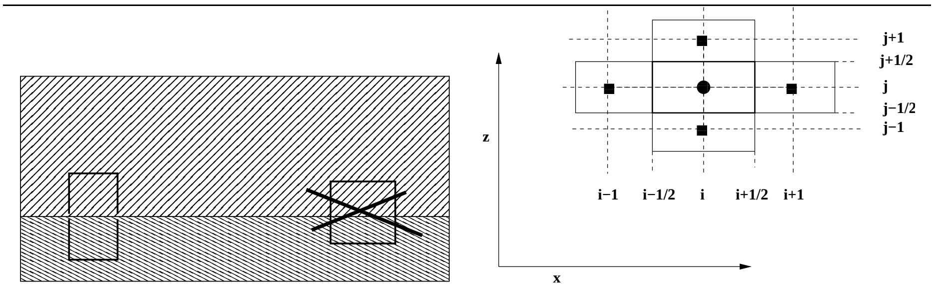
$$\begin{aligned} h|_- &= h|_+, \\ \boldsymbol{v}\cdot\mathbf{n}|_- &= \boldsymbol{v}\cdot\mathbf{n}|_+. \end{aligned} \tag{5.11}$$

Taking into account the compatibility relations (5.11) appear that it is more convenient to work with the $\theta - h$ form of Richards' equation, i.e.,

$$\begin{aligned} \partial_t \int_V \theta \mathrm{d}x &= \int_{\partial V} K(\theta)\frac{\partial(h+z)}{\partial n}\mathrm{d}s, \\ \theta &= \theta(h) \end{aligned} \tag{5.12}$$

We assume that the flow domain is the 2D rectangle $\Omega = [0,a]\times[0,b]$ which is stratified in N_s strata $[0,a]\times[Z_{i-1}, Z_i]$ with $Z_0 = 0, Z_{N_s} = b$.

Let $0 = x_{1/2} < x_{1+1/2} < \cdots < x_{N+1/2} = a$, $0 = z_{1/2} < z_{1+1/2} < \cdots < z_{M+1/2} = b$ be two partitions of the intervals $[0,a]$ and $[0,b]$ respectively. We define the control volumes $\omega_{i,j} = \left[x_{i-1/2}, x_{i+1/2}\right]\times\left[z_{j-1/2}, z_{j+1/2}\right]$, $i = \overline{1,N}$, $j = \overline{1,M}$ and the net inner knots $\boldsymbol{r}_{i,j} = (x_i, z_j)$, $x_i = \dfrac{x_{i-1/2}+x_{i+1/2}}{2}$, $z_j = \dfrac{y_{j-1/2}+y_{j+1/2}}{2}$, $i = \overline{1,N}$, $j = \overline{1,M}$. We assume that the partition $\{\omega_{i,j}\}$ is a *conform partition* with respect to stratification of the domain Ω,



Conformal Mesh with respect to strata

Volumes of the control.

Fig. 4: 2D mesh.

i.e for any j the line $z = Z_j$ does not intersect the interior of any control volume $\omega_{i,j}$.

On each volume $\omega_{i,j}$ one approximates the pressure by a constant value $h_{i,j}$ and water content by a constant value $\theta_{i,j}$. On the common boundary $\sigma_{i+1/2,j} = \omega_{i,j} \cup \omega_{i+1,j}$ of two neighbors we approximate the flux by

$$\int\limits_{\sigma_{i+1/2,j}} K(\theta)\frac{\partial(h+z)}{\partial n}\mathrm{ds} \approx K_{i+1/2,j}\frac{h_{i+1,j}-h_{i,j}}{\triangle x_{i+1}} \tag{5.13}$$

where the numerical hydraulic conductivity $K_{i+1/2,j}$ is an approximation of the hydraulic conductivity $K(\theta)$,

$$K_{i+1/2,j} = \widetilde{K}(\theta_{i,j}, \theta_{i+1,j}). \tag{5.14}$$

We assume that the function $\widetilde{K}(\cdot,\cdot)$ is a symmetric and continuous function with respect to its arguments. As result, we obtain a differential algebraic system of equation (DAE), $\theta - h$ form of Richards' equation,

$$\left\{\begin{array}{l} m_{i,j}\dfrac{\mathrm{d}\theta_{i,j}}{\mathrm{d}t} = K_{i+1/2,j}\dfrac{h_{i+1,j}-h_{i,j}}{\triangle x_{i+1}} - K_{i-1/2,j}\dfrac{h_{i,j}-h_{i-1,j}}{\triangle x_i} + \\ \qquad +K_{i,j+1/2}\left(\dfrac{h_{i,j+1}-h_{i,j}}{\triangle z_{j+1}}+1\right) - K_{i,j-1/2}\left(\dfrac{h_{i,j}-h_{i,j-1}}{\triangle z_j}+1\right), \\ \theta_{i,j} = \theta(h_{i,j}). \end{array}\right. \tag{5.15}$$

To integrate the DAE (5.15) we use an implicit multi-step method, [5]. Let $\{t_{n-k}, t_{n-k+1}, ..., t_n\}$ be a sequence of moments of time and denotes by $\theta^m = \theta(t_m) \in \mathbb{R}^{NM}$, $NM = N \times M$. Supposing that one knows the values $\{\theta^{n-k}, \theta^{n-k+1}, ..., \theta^n\}$, the values θ^{n+1} and h^{n+1} at the next moment of time

t_{n+1} are calculated as follows. Define a predictor polynomial $\omega^P(t)$ and a corrector polynomial $\omega^C(t)$. The predictor polynomial interpolates the values $\{\theta^{n-k}, \theta^{n-k+1}, ..., \theta^n\}$ at moments of time $\{t_{n-k}, t_{n-k+1}, ..., t_n\}$, Lagrange interpolation,

$$\omega^P(t) = \sum_{j=0}^{k} q_j(t)\theta^{n-j}. \tag{5.16}$$

For each $j = \overline{0,k}$ the polynomial $q_j(t)$ is given by

$$q_j(t) = \prod_{i=0, i\neq j}^{k} \frac{t - t_{n-i}}{t_{n-j} - t_{n-i}}.$$

The corrector polynomial $\omega^C(t)$ interpolates the unknowns θ^{n+1} and the values of $\omega^P(t)$ at the moments of time t_{n+1} and $\{t_{n+1} - j\triangle t_n; j = \overline{1,k}\}$, respectively. The unknowns θ^{n+1} and h^{n+1} are determined by imposing to the corrector polynomial $\omega^C(t)$ and to h^{n+1} to satisfies the DAE. Then a system of nonlinear equation results. By denoting

$$\begin{aligned}
&\mathcal{F}_{i,j}(\boldsymbol{\theta}^{n+1}, \boldsymbol{h}^{n+1}) := \\
&K_{i+1/2,j}(\theta^{n+1})\frac{h_{i+1,j}^{n+1} - h_{i,j}^{n+1}}{\triangle x_{i+1}} - K_{i-1/2,j}(\theta^{n+1})\frac{h_{i,j}^{n+1} - h_{i-1,j}^{n+1}}{\triangle x_i} + \\
&K_{i,j+1/2}(\theta^{n+1})\left(\frac{h_{i,j+1}^{n+1} - h_{i,j}^{n+1}}{\triangle z_{j+1}} + 1\right) - K_{i,j-1/2}(\theta^{n+1})\left(\frac{h_{i,j} - h_{i,j-1}}{\triangle z_j} + 1\right)
\end{aligned} \tag{5.17}$$

one obtains

$$\begin{cases} m_{i,j}\left(\dfrac{a}{\triangle t^n}\theta_{i,j}^{n+1} - w_{i,j}^{P,n}\right) = \mathcal{F}_{i,j}\left(\boldsymbol{\theta}^{n+1}, \boldsymbol{h}^{n+1}\right), \\ \theta_{i,j}^{n+1} = \theta(h_{i,j}^{n+1}), \end{cases} \tag{5.18}$$

where $w_{i,j}^{P,n}$ are known quantities as functions of the preceding values of θ.

The nonlinear system (5.18) is solved iteratively using an inexact Newton step followed by a Broyden step until a desired accuracy is obtained. Let $\mathcal{R}$ be given by

$$\boldsymbol{\mathcal{R}}(\boldsymbol{\theta}, \boldsymbol{h}) = \boldsymbol{m}\left(\frac{a}{\triangle t^n}\boldsymbol{\theta} - \boldsymbol{w}^{P,n}\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{\theta}, \boldsymbol{h}\right). \tag{5.19}$$

The matrix $\mathcal{J}(\boldsymbol{\theta}, \boldsymbol{h})$ of the iterative process in INS is an approximation of the full Jacobian of the function $\mathcal{R}$, the product of it with a vector $\boldsymbol{w}$ read as

$$\mathcal{J}(\boldsymbol{\theta},\boldsymbol{h})\boldsymbol{w}=\boldsymbol{m}\frac{a}{\triangle t^n}C(\boldsymbol{h})\boldsymbol{w}-\widetilde{\boldsymbol{\mathcal{F}}}\left(\boldsymbol{\theta},\boldsymbol{w}\right), \tag{5.20}$$

where

$$\widetilde{\boldsymbol{\mathcal{F}}}\left(\boldsymbol{\theta},\boldsymbol{w}\right)=\partial_h\boldsymbol{\mathcal{F}}\left(\boldsymbol{\theta},\boldsymbol{w}\right) \tag{5.21}$$

and

$$C(\cdot)=\frac{\mathrm{d}\theta(\cdot)}{\mathrm{d}h}.$$

The nonlinear solver is:

$$\left\|\begin{array}{ll}
\text{Inexact Newton step} & \\
\mathcal{J}(\boldsymbol{\theta}^{n+1,k},\boldsymbol{h}^{n+1,k})\boldsymbol{\delta}_h^{NS}=-\boldsymbol{\mathcal{R}}\left(\boldsymbol{\theta}^{n+1,k},\boldsymbol{h}^{n+1,k}\right), & (\text{s}\,1)\\
\overline{\boldsymbol{h}}^{n+1,k+1}=\boldsymbol{h}^{n+1,k}+\boldsymbol{\delta}_h^{NS}, & (\text{s}\,2)\\
\overline{\boldsymbol{\theta}}^{n+1,k+1}=\theta(\overline{\boldsymbol{h}}^{n+1,k+1}), & (\text{s}\,2)\\
\text{Broyden step} & \\
\mathcal{J}(\boldsymbol{\theta}^{n+1,k},\boldsymbol{h}^{n+1,k})\boldsymbol{\delta}_h^{BS}=-\boldsymbol{\mathcal{R}}\left(\overline{\boldsymbol{\theta}}^{n+1,k},\overline{\boldsymbol{h}}^{n+1,k}\right), & (\text{s}\,3)\\
\boldsymbol{\delta}_h^{k+1}=\boldsymbol{\delta}_h^{BS}\dfrac{\langle\boldsymbol{\delta}_h^{NS},\boldsymbol{\delta}_h^{NS}\rangle}{\langle\boldsymbol{\delta}_h^{NS},\boldsymbol{\delta}_h^{NS}\rangle-\langle\boldsymbol{\delta}_h^{NS},\boldsymbol{\delta}_h^{BS}\rangle}, & (\text{s}\,4)\\
\boldsymbol{h}^{n+1,k+1}=\overline{\boldsymbol{h}}^{n+1,k}+\boldsymbol{\delta}_h^{k+1}, & (\text{s}\,5)\\
\boldsymbol{\theta}^{n+1,k+1}=\theta(\boldsymbol{h}^{n+1,k+1}). & (\text{s}\,5)
\end{array}\right. \tag{5.22}$$

The linear equations in the steps s1 and s3 are solved by Conjugate Gradient Method for linear system with symmetric and positive definite matrix. We present some numerical tests obtained using the above algorithm. As empirical models for water content $\theta(h)$ and hydraulic conductivity $K(\theta)$ we use the van Genuchten model,

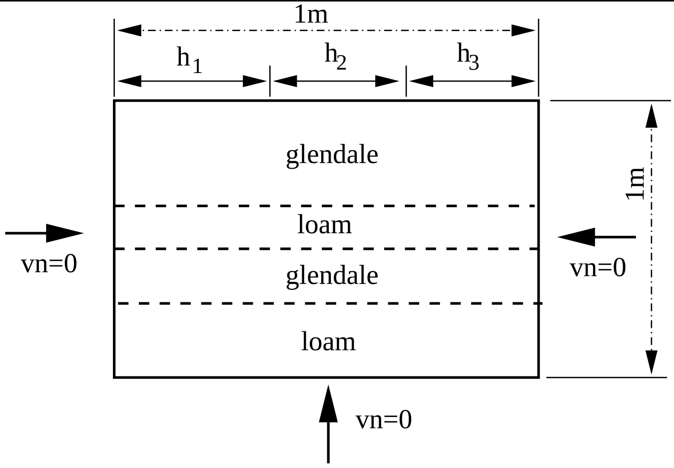
$$S(h)=\begin{cases}(1+(\alpha h)^n)^{-m}, & h<0,\\ 1, & h\geq 0,\end{cases} \tag{5.23}$$

$$K(S)=\begin{cases}K_sS^l\left(1-\left(1-S^{1/m}\right)^m\right)^2, & 0<S<1,\\ K_s, & S\geq 1,\end{cases} \tag{5.24}$$

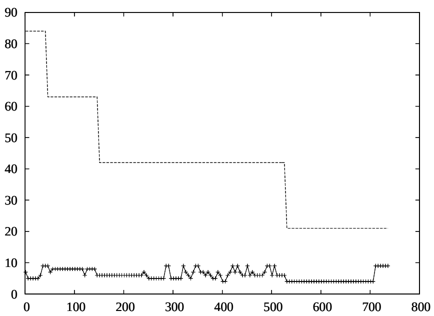
where S represents the relative water content

$$S = \frac{\theta - \theta_r}{\theta_s - \theta_r}.$$

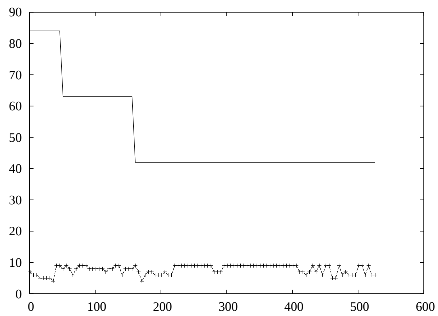
The soil in the test is a layered soil with two alternate strata.



Physical configuration. The parameters for the *loam soil* in the van Genuthen model are: $n = 2$, $\alpha = 3.35\,\mathrm{m}^{-1}$, $l = 0.5$, $K_s = 0.3318\,\mathrm{mh}^{-1}$, $\theta_r = 0.012$, $\theta_s = 0.368$ and for the *Glendale soil* are: $n = 1.3954$, $\alpha = 1.04\,\mathrm{m}^{-1}$, $l = 0.5$, $K_s = 0.545 \times 10^{-2}\,\mathrm{mh}^{-1}$, $\theta_r = 0.106$, $\theta_s = 0.4686$. The initial datum is $h^0 = -1.0\,\mathrm{m}$ in the whole domain. The boundary conditions are of the mixt type.

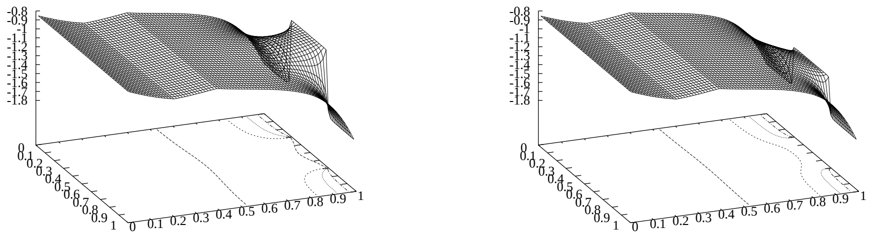


$h_1 = -0.75\,\mathrm{m}$, $h_2 = -0.0\,\mathrm{m}$, $h_3 = -0.75\,\mathrm{m}$.

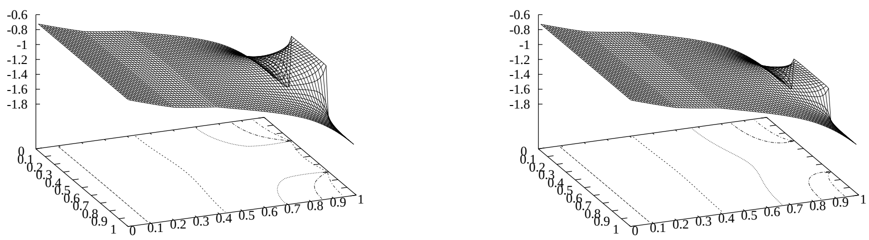


$h_1 = -0.75\,\mathrm{m}$, $h_2 = -0.3\,\mathrm{m}$, $h_3 = -0.75\,\mathrm{m}$.

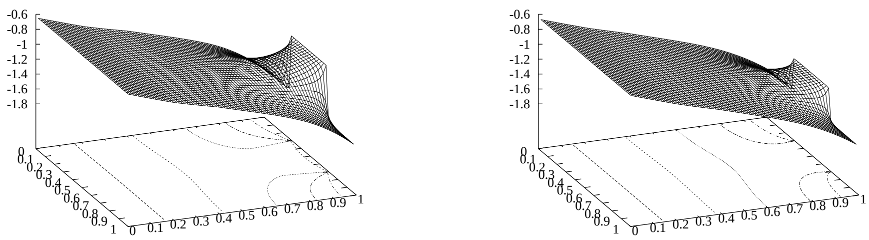
The number of the iterations versus time step. Numbers of iteration in nonlinear solver (line-point) and the total numbers of iteration time step in CGM method The time simulation was 48h.



$t = 6\text{h}$



$t = 24\text{h}$



$t = 48\text{h}$

The comparative profiles of the pressure head for two different Dirichlet datum on the top boundary at different moments of time. $h_1 = -0.75\,\text{m},\ h_2 = -0.\,\text{m},\ h_3 = -0.75\,\text{m}$.(left), $h_1 = -0.75\,\text{m},\ h_2 = -0.3\,\text{m},\ h_3 = -0.75\,\text{m}$.(right).

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