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Approximate inertial manifolds, induced trajectories, and approximate solutions for semilinear parabolic equations, based upon these; applications to flow and diffusion problems

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1. Introduction

In the study of dissipative semi-dynamical systems generated by semilinear parabolic equations, the theory of qualitative behavior of the system at large times plays an important role. By parabolic semilinear equations we mean partial differential equations that can be written as abstract equations in a Hilbert space, of the form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + R(u) = f, \tag{1.1}$$

where u is a function of time with values in a Hilbert space H (whose definition comprises the boundary value conditions imposed to equation (1.1)). We attach to the above equation an initial condition

$$u(0) = u_0, \tag{1.2}$$

with u_0 in H. We assume that A is a linear operator, defined on a dense subspace $D(A)$ of H, self-adjoint, positive definite, with compact inverse, while R is a nonlinear operator defined on $D(R) \subset D(A)$. We do not insist here on the hypotheses on R, but we assume that it is such that the Cauchy problem (1)-(2) has an unique solution on $[0,T]$, for every $u_0 \in H$ and every $T > 0$. Hence a semi-dynamical system is generated by the above problem, by setting $S(t)u_0 = u(t,u_0)$, where $u(t,u_0)$ is the solution of (1.1)–(1.2).

For this presentation we assume that f is in H. We also assume that the semi-dynamical system generated by (1.1) is dissipative in the sense that there is a bounded absorbing set for it. An absorbing set is a set B having the property that, for every bounded set $M \subset H$, there is a value of t, depending on M, let us denote it by t_M, with the property that, $S(t)M \subset B$ for $t \geq t_M$. For the particular problems we consider here, there also are absorbing balls in some subspace V of H, with $D(A) \subset V \subset H$.

In the theory of qualitative behavior at large times of solutions of equations of the form (1.1), the notion of global attractor plays an important role. A global attractor [3] is a compact set of the phase space H, invariant to the semigroup $S(t)_{t\geq 0}$, that attracts the bounded sets of the phase space, when time tends to infinity. This means that the global attractor bears in its structure the properties of the behavior of the semi-dynamical system at large times. For many problems of interest the existence of an attractor was proved [37].

The study of the geometrical and topological properties of the global attractors flourished since the last two decades of the XXth century and the major

hope was that a connection between the structure of the attractor and very complex phenomena like turbulence in the flow of the fluids will be found.

In this context, another interesting notion appeared, that of *inertial manifold* (i.m.) [11]. It is a finite dimensional, invariant and at least Lipschitz manifold having the property that it exponentially attracts all the trajectories of the problem. More than that, an i.m. has the property of *asymptotic completeness* meaning that for every u_0 in H there is a v_0 on the i.m. such that the distance between the trajectories passing through the two points decreases exponentially with time.

The invariance of the i.m. implies the fact that we can construct a restriction of the problem to this manifold. The restricted problem is named *inertial form* [11], [37] and, since the i.m. is finite dimensional, is equivalent with a system of ODEs. The above defined asymptotic completeness of the i.m. implies that the asymptotic behavior at large times of the dynamical system is described by the asymptotic behavior of the inertial form. Hence the large times study of the initial semi-dynamical system (infinite dimensional since its phase space is H) can be reduced to that of a finite-dimensional one.

Another important consequence of the properties of the i.m.s is that, when a global attractor exists, it is contained in the i.m. These considerations explain the large interest shown by the scientific community in inertial manifolds. From among the great number of papers devoted to the inertial manifolds we remind: [11] (with the extended version [12]), [8], [9], [5], [36]. The important monograph [37] had a second edition in 1997.

From a theoretical point of view, the i.m.s looked very promising, but major obstacles appeared in trying to use their properties in the study of concrete problems. One is due to the fact that existence of i.m.s is in most papers proved by a fixed point theorem, and is not constructive. There is a constructive proof in [2] but it uses some integral manifolds whose construction is equivalent with solving the equation. Another problem is a restrictive hypothesis among the hypothesis of the existence theorems- the hypothesis of a spectral gap that imposes the existence of two successive eigenvalues of A situated at a "large enough" distance [1], [12], [37]. This hypothesis is not fulfilled by many problems, (e.g. is not fulfilled for the two-dimensional Navier-Stokes equations).

In this situation the approximate inertial manifolds were defined as approximations of i.m.s or as substitutes of these, when the i.m.s could not be proved to exist. An approximate inertial manifold (a.i.m.) is a finite dimensional, at least Lipschitz manifold in the space H, with the property that all the trajectories of the dynamical system enter a narrow neighborhood of the manifold

at a certain moment and never leave the neighborhood after. Even if it has not the invariance property, an a.i.m. is important because, if the problem has a global attractor, it is contained in the narrow neighborhood mentioned above.

The localization of the attractors in the space of phases was a first interesting application field of the a.i.m.s. Besides this, a.i.m.s found very interesting applications in the construction of some approximate solutions (the numerical integration) of the nonlinear evolution problems. Examples of papers devoted to a.i.m.s are: [10], [13], [23], [26], [27], [28], [33], [35], [37], [38], [39].

In Section 2 we present some methods, that use a.i.m.s, for the construction of approximate solutions for problems of the type (1.1)–(1.2), the so-called *non-linear Galerkin method* and *post-processed Galerkin method.*

We include a method conceived by us, that we named *repeatedly adjusted and post-processed Galerkin method*, that is connected to the preceding methods but brings some simplifications to these. In Section 3 we present the way these method work for the two-dimensional Navier-Stokes equations with periodic boundary conditions, and in Section 4, for a two-dimensional reaction-diffusion equation, with Von Neumann boundary conditions.

In order to settle the notations and the functional framework of our presentation, we shortly remind below the Galerkin spectral method for the abstract equation (1.1).

1.1. The Galerkin method

In the hypotheses we assumed on the operator A of equation (1.1), it follows that A has positive eigenvalues that form a tending to infinity sequence:

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ..., \ \lambda_n \underset{n\to\infty}{\to} \infty.$$

The eigenfunctions of A form a total (orthonormal) system for H. We consider the set, denoted Γ_m, of the first distinct m eigenvalues (in increasing order) and the eigenfunctions corresponding to these. We denote by P the orthogonal projection operator on the subspace spanned by these eigenfunctions and we set $Q = I - P$ (where I is the identity application on H). The solution u of (1.1)–(1.2) is projected by the two projectors and we set

$$p = Pu,$$
$$q = Qu.$$

It follows that the functions p and q are solutions of

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p+q) = Pf, \tag{1.3}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} + \nu Aq + QR(p+q) = Qf, \tag{1.4}$$

$$p(0) = Pu_0, \tag{1.5}$$

$$q(0) = Qu_0. \tag{1.6}$$

Usually, the component q of the solution is proved to be, at large times, "little" in the norm of H compared to the p component. That is, an inequality of the form

$$|q\,(t)| \leq C_0 \delta^a, \tag{1.7}$$

where

$$\delta = \frac{\lambda_1}{\lambda_{m+1}}, \tag{1.8}$$

and a is some positive number, is true. For the Navier-Stokes equations it is proved in [38] that a inequality of the type (1.7) holds, with $a = 1$ and C_0 depending on m. We proved in [19] that the inequality can be improved in the sense that it is true with a C_0 that does not depend on m. For the reaction-diffusion equation, $|q(t)|$ is of the order of δ for large enough times [4].

If in the equation (1.4) q is neglected in the presence of p, we find the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p) = Pf. \tag{1.9}$$

This is the Galerkin approximation of the equation (1.1). The solution of the problem (1.9) with the initial condition (1.5), that we denote by $p_G(.)$, is the Galerkin approximation of the solution of (1.1)-(1.2). For several problems it is proved in the literature that inequalities of the type

$$|u(t) - p_G(t)| \leq C\delta^\alpha,$$

where $u(t)$ is the solution of the problem (1.1)-(1.2), $\delta > 0$ is defined by (1.8), and $\alpha > 0$.

As example, for a reaction-diffusion equation with Neumann boundary values and for the two-dimensional Navier-Stokes equations, $\alpha = 1$ (in the hypothesis $f \in H$). The problem (1.9), (1.5) is equivalent to a system of ordinary differential equations for the coordinates of $p(t)$ along the eigenfunctions that span PH. The definition of δ shows that the greater will be m, (hence the dimension of PH), the smaller will be the error.

In the construction of the Galerkin equation, the q component of the solution (that is proved to be small for large times) is approximated with 0. The nonlinear Galerkin (and/or post-processed) methods of approximation are based upon the idea of approximating $q(t)$ by using a a.i.m instead of the manifold $\mathbf{q}_0$.

2. Modified Galerkin methods

The nonlinear Galerkin (and/or post-processed) methods of approximation are based upon the idea of approximating $q(t)$ by using an a.i.m instead of taking $q \simeq 0$.

2.1. Families of a.i.m.s used in the modified Galerkin methods

There are several types of a.i.m.s defined in the literature. Among them, those defined in [10], [38], [39] (for the Navier-Stokes equations – NSE) generated new numerical integration methods, based on the Galerkin method. They form a family $\{\mathcal{M}_j\}_{j\geq 0}$ and are the graphs of some functions $\Phi_j : P\mathcal{H} \to Q\mathcal{H}$. The definitions of these a.i.m.s for the NSE are presented in Section 3 while those for the RDE are given in Section 4. A.i.m.s of the type of those cited above may be (and were) defined for many particular problems of the form (1.1)–(1.2). The main property of these a.i.m.s, on which their use in the construction of the numerical methods is based, is the following: the distance (in the norm of H) between *q(t)* and the image of *p(t)* on the a.i.m. $\mathcal{M}_n$ is of the order of $\delta^{a(n)}$ that is

$$|q(t) - \Phi_n(p(t))| \leq C\delta^{a(n)}, \tag{2.1}$$

where $a(n)$ is increasing with n.

For example, for the two-dimensional NSE it is proved [38], [39] that $a(n) = (n+3)/2$. Since, for NSE, about the H norm of $q(t)$ only the fact of being of the order of δ is known, it is clear that any of the above a.i.m.s provides a better approximation of $q(t)$ than the so-called plane manifold $q = 0$, for the mentioned problem.

2.2. The nonlinear Galerkin methods

The nonlinear Galerkin method (NL Galerkin method) was first defined in [29]. The method relies on the idea that $\Phi_0(p(t))$ is a better approximation of $q(t)$ than 0, and considers, instead of the Galerkin equation (3.25), the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p + \Phi_0(p)) = Pf, \tag{2.2}$$

with initial condition (1.5). By denoting with $\widetilde{p}_0(.)$ the solution of this problem, the approximate solution of (1.1)–(1.2) is taken as

$$v_0(t) = \widetilde{p}_0(t) + \Phi_0(\widetilde{p}_0(t)).$$

As it is natural, since $\Phi_n(p(t))$ approximates $q(t)$ better and better with the increase of n, the next idea, appeared in [6], was to consider the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p + \Phi_n(p)) = Pf, \tag{2.3}$$

with the initial condition (1.5). Let $\widetilde{p}_n(.)$ the solution of this problem. The approximate solution is then defined as

$$v_n(t) = \widetilde{p}_n(t) + \Phi_n(\widetilde{p}_n(t)).$$

For the problems considered in the context of nonlinear Galerkin problems, it is proved that the error is of the order of $\delta^{b(n)}$, where $b(n)$ is increasing with n.

E.g., for the Navier-Stokes equations it is proved in [7] that $b(n) = (n+3)/2$, while for the reaction-diffusion equation it is asserted in [32] that $b(n) = n+2$ provided $f \in H$.

2.3. Post-processed Galerkin methods

In [14] the following modified Galerkin method is proposed, that also uses a.i.m.s. Let again $p_G(.)$ be the solution of (1.9), (1.5). Then the value of $\Phi_0(p_G(t))$ is computed at the right end side of the time interval $[0, T]$, that is in T. The approximate solution in T is defined as

$$w(T) = p_G(T) + \Phi_0(p_G(T)).$$

This method is named *the post-processed Galerkin method (PP Galerkin method)* because the solution of the Galerkin problem is corrected only in the final phase, after finishing the numerical integration of the Galerkin problem, by using the first a.i.m. of the family described in 2.1 (hence post-processed). The error of this approximate solution is less than that of the Galerkin method. Thus, for the two-dimensional Navier-Stokes equations, it is shown in [14] to be of the order of $\delta^{5/4}$. Another estimate is proved in [15], i.e. the error is proved to be of the order of $L^2\delta^{3/2}$, where $L = 1 + \ln(2m^2)$. This latter estimate of the error is not necessary better than the former because of the coefficient L^2.

The next idea appeared in the literature [32] was to postprocess the NL Galerkin method of the preceding section. More precisely, the equation (2.3) is considered, it is integrated on all the time interval $[0, T]$, then $\Phi_{n+1}(\widetilde{p}_n(T))$, is computed, and the approximate solution in T is defined as

$$w_n(T) = \widetilde{p}_n(T) + \Phi_{n+1}(\widetilde{p}_n(T)).$$

This method is called the nonlinear post-processed Galerkin method (NL PP Galerkin method). In [32] the use of the method is exemplified on the reaction-diffusion equation and it is proved that, if $f \in H$, then the error is of the order of $\ln m\, \delta^{n+3}$.

2.4. A new modified Galerkin method

In [38], in the context of the study of the NSE, a family of functions, $\{q_j\}_{j\geq 0}$, $q_j : \mathbb{R}^+ \to Q\mathcal{H}$, having the property

$$|q_j(t) - q(t)| \leq k_j L^{1+j/2}\delta^{(3+j)/2} \tag{2.4}$$

for large enough times is constructed. Here the coefficients k_j depend on the data of the problem (ν, $|f|$, λ_1), and $L = 1 + \ln\frac{\lambda_{m+1}}{\lambda_1}$. Actually, the function q_0 is of the form

$$q_0 = \Phi_0(p),$$

while, for $j \geq 1$, q_j are recursively defined by relations of the type

$$q_j = F_j(Qf, p, q_0, ..., q_{j-1}). \tag{2.5}$$

The functions $u_j = p + q_j$, $j \geq 0$ define the so-called *induced trajectories*, $\{u_j(t); t \geq 0\}$, associated to the trajectory $\{u(t); t \geq 0\}$ of the dynamical system. Relation (2.4) shows that the functions u_j, $j \geq 0$, are approximations

of the exact solution, of increasing with j accuracy. The definition of the a.i.m.s $\mathcal{M}_j$ used in the nonlinear Galerkin methods for the NSE are based upon the definitions of the functions q_j.

In [20], for the two-dimensional NSE with periodic boundary conditions, we defined a new type of modified Galerkin method, that uses some approximations of the induced trajectories and not the a.i.m.s. We describe here the method in the general context of equation (1.1). The purpose of the method is that of working with a very low-dimensional projection space PH, and the idea from which we started is that, however small is the dimension of PH, if we have a very good approximation for q, let us denote it by $\widetilde{q}$, then a very good approximation for p will be obtained by solving the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + g(p + \widetilde{q}) = Pf.$$

In consequence, a good approximation of u may be obtained. The method is structured on several levels. One of the ideas we followed in developing this method is that of having to integrate only differential equations of the same level of difficulty as the Galerkin equation. This was possible by using approximations of induced trajectories instead of a.i.m.s.

Level 0. This level has two stages. The first is the classical Galerkin method, i.e. we solve the problem (1.9), (1.5) and we consider its solution, $p_G(.)$.

The second stage consists in defining the function of time, with values in $Q\mathcal{H}$:

$$\widetilde{q}_0(t) = \Phi_0(p_G(t)), \tag{2.6}$$

the function Φ_0 being the one that defines the first a.i.m. of the family cited in 2.1.

Then we define the approximate solution at this first level as

$$\widetilde{u}_0 = p_G + \widetilde{q}_0.$$

Since the function $\widetilde{q}_0(t)$ will be used at the second level of our method, in the numerical implementation of this method, the function $\widetilde{q}_0$ should be computed in each point of the time mesh, unlike in the post-processing defined in [14], where it is computed only at the final point of the integration interval $[0, T]$. Besides this, Level 0 of our method is essentially the Galerkin post-processed method.

Level 1. We consider the problem

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p + \widetilde{q}_0) = Pf, \tag{2.7}$$

$$p(0) = Pu_0$$

and we denote by $\widetilde{p}_0$ its solution. This is an "adjusted" Galerkin problem.

This equation is essentially different from the corresponding one of the NL Galerkin method (see equation (2.3)) since $\widetilde{q}_0$ is known from Level 1.

Then we define

$$\widetilde{q}_1(t) = F_1(Qf, \widetilde{p}_0(t), \widetilde{q}_0(t)).$$

The approximate solution is

$$\widetilde{u}_1 = \widetilde{p}_0 + \widetilde{q}_1.$$

Level $j > 1$.

We assume that $\widetilde{q}_0,\ \widetilde{q}_1,\ \widetilde{q}_2, ..., \widetilde{q}_{j-1}$ were constructed. The problem

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \nu Ap + PR(p + \widetilde{q}_{j-1}) = Pf, \qquad (2.8)$$
$$p(0) = Pu_0,$$

is considered and its solution is denoted by $\widetilde{p}_{j-1}$. Then we denote

$$\widetilde{q}_j = F_j\left(Qf,\ \widetilde{p}_{j-1},\ \widetilde{q}_0,\ \widetilde{q}_1, ..., \widetilde{q}_{j-1}\right)$$

and the approximate solution is

$$\widetilde{u}_j = \widetilde{p}_{j-1} + \widetilde{q}_j.$$

At first sight, the idea of performing several time integrations seems a bad idea, since every such integration involves a large amount of computations. However, a careful analysis shows that the amount of computations involved in the NL Galerkin method (based upon the a.i.m. $\mathcal{M}_j$) is greater than that involved in solving the problems from Level 1 to the eq. (2.8) of Level j, inclusive. Such an analysis is performed for the Navier-Stokes equations in 3.8. Hence our method, that we call *the repeatedly adjusted and post-processed Galerkin method (R-APP Galerkin method)* is an alternative to the NL Galerkin method. The final post-processing, by adding $\widetilde{q}_j$ to $\widetilde{p}_{j-1}$ is equivalent to the post-processing of NL Galerkin method and does not imply a large amount of calculi since it will be performed only in some selected moments of time (eventually only at the last moment, T). In what concerns the error, for the problems discussed below we can prove that the error of R-APP Galerkin method is of the same order of magnitude as that for NL PP Galerkin method, for the two particular problems in Sections 3 and 4.

3. Modified Galerkin methods for the Navier-Stokes equation

We present here the modified Galerkin methods for the Navier-Stokes equations: the NL, NL PP Galerkin methods already defined in the literature and our R-APP Galerkin method.

3.1. The setting of the problem

We consider the problem of the two-dimensional flow of a incompressible Newtonian fluid, modeled by the Navier-Stokes equations. We impose periodic boundary conditions and choose the periodicity cell to be a square, $\Omega = (0,l) \times (0,l)$. Thus the problem is

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p = \mathbf{f}, \tag{3.1}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{3.2}$$

where $\mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^2$ is the velocity of the fluid, $t \geq 0$, $\mathbf{x} \in \Omega$, $p(t,\mathbf{x}) \in \mathbb{R}$ is the pressure of the fluid, ν is the kinematic viscosity, and $\mathbf{f}$ is the volume force. We add the initial condition

$$\mathbf{u}(0,\cdot) = u_0(\cdot). \tag{3.3}$$

We assume that $\mathbf{f}$ is independent of time and is an element of $\left[L^2_{per}(\Omega)\right]^2$. As is usual in the study of the Navier-Stokes equations with periodic boundary conditions, we assume that [40], [34]

$$\overline{\mathbf{f}} = \frac{1}{l^2} \int_\Omega \mathbf{f}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathbf{0}, \tag{3.4}$$

and that the pressure is a periodic function on Ω. For simplicity we will assume also that the average $\overline{\mathbf{u}}$ of the velocity over the periodicity cell is zero.

The velocity $\mathbf{u}$ is thus looked for in the space $\mathcal{H} = \left\{\mathbf{v};\ \mathbf{v} \in \left[L^2_{per}(\Omega)\right]^2,\ \operatorname{div} \mathbf{v} = \mathbf{0}, \overline{\mathbf{v}} = 0\right\}$ with the scalar product $(\mathbf{u},\mathbf{v}) = \int_\Omega (u_1 v_1 + u_2 v_2)\, \mathrm{d}\mathbf{x}$, (where $\mathbf{u} = (u_1,u_2)$, $\mathbf{v} = (v_1,v_2)$) and the induced norm is denoted by $|\cdot|$. Let us also consider the space $\mathcal{V} = \left\{\mathbf{u} \in \left[H^1_{per}(\Omega)\right]^2,\ \operatorname{div} \mathbf{u} = \mathbf{0}, \overline{\mathbf{u}} = 0\right\}$, with the scalar product $((\mathbf{u},\mathbf{v})) = \sum_{i,j=1}^2 \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)$, and the induced norm, denoted by $\|\cdot\|$.

The variational formulation of the Navier-Stokes equations [40] leads, for the periodic boundary conditions, to the Cauchy problem

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \mathbf{f} \quad \text{in } \mathcal{V}', \tag{3.5}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_0 \in \mathcal{H}. \tag{3.6}$$

The notations

$$\mathbf{B}(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v}, \tag{3.7}$$

$$\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u},\mathbf{u}), \tag{3.8}$$

will be used below.

We remind here the classical existence and uniqueness results for the Navier-Stokes equations in $\mathbb{R}^2$, with periodic boundary conditions.

We denote $\mathbf{A} = -\boldsymbol{\Delta}$. The definition domain of the linear operator $\mathbf{A}$ is $D(\mathbf{A}) = \mathcal{V} \cap H^2_{per}(\Omega)$.

THEOREM **3.1** [40]. *a) If* $\mathbf{u}_0 \in \mathcal{H}$, $\mathbf{f} \in \mathcal{H}$, *then the problem* (3.5), (3.6) *has an unique solution* $\mathbf{u} \in C^0([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$. *b) If, in addition,* $\mathbf{u}_0 \in \mathcal{V}$, *then* $\mathbf{u} \in C^0([0,T];\mathcal{V}) \cap L^2(0,T;D(\mathbf{A}))$. *The solution is, in this latter case, analytic in time on the positive real axis.*

The semi-dynamical system $\{S(t)\}_{t\geq 0}$ generated by problem (3.5) is dissipative [37]. More precisely, there is a $\rho_0 > 0$ such that for every $R > 0$, there is a $t_0(R) > 0$ with the property that for every $\mathbf{u}_0 \in \mathcal{H}$ with $|\mathbf{u}_0| \leq R$, we have $|S(t)\,\mathbf{u}_0| \leq \rho_0$ for $t > t_0(R)$. In addition, there are absorbing balls in $\mathcal{V}$ and $\mathbf{D}(\mathbf{A})$ for $\{S(t)\}_{t\geq 0}$, [34] i.e. there are $\rho_1 > 0$, $\rho_2 > 0$ and, for every $R > 0$, there are $t_1(R)$, $t_2(R)$ with $t_2(R) \geq t_1(R) \geq t_0(R)$ such that $|\mathbf{u}_0| \leq R$ implies $\|S(t)\,\mathbf{u}_0\| \leq \rho_1$ for $t > t_1(R)$ and $|\mathbf{A}S(t)\,\mathbf{u}_0| \leq \rho_2$ for $t > t_2(R)$.

3.2. The decomposition of the space, the projected equations

The eigenvalues of $\mathbf{A}$ are $\lambda_{j_1,j_2} = \frac{4\pi^2}{l^2}\left(j_1^2 + j_2^2\right)$, $(j_1,\ j_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$, and the corresponding eigenfunctions are

$$\mathbf{w}^{s\pm}_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2, \mp j_1)}{|\mathbf{j}|}\sin\left(2\pi\frac{j_1x_1 \pm j_2x_2}{l}\right),$$

$$\mathbf{w}^{c\pm}_{j_1,j_2} = \frac{\sqrt{2}}{l}\frac{(j_2, \mp j_1)}{|\mathbf{j}|}\cos\left(2\pi\frac{j_1x_1 \pm j_2x_2}{l}\right),$$

where $|\mathbf{j}| = \left(j_1^2 + j_2^2\right)^{\frac{1}{2}}$ [38]. These eigenfunctions form a total system for $\mathcal{H}$. For a fixed $m \in \mathbb{N}$ we consider the set Γ_m of eigenvalues λ_{j_1,j_2} having $0 \leq j_1, j_2 \leq m$. We define

$$\lambda := \lambda_{1,0} = \lambda_{0,1} = \frac{4\pi^2}{l^2},$$
$$\Lambda := \lambda_{m+1,0} = \lambda_{0,m+1} = \frac{4\pi^2}{l^2}\left(m+1\right)^2,$$
$$\delta = \delta\left(m\right) := \frac{\lambda}{\Lambda} = \frac{1}{(m+1)^2}.$$

Λ is the least eigenvalue not belonging to Γ_m. The eigenfunctions corresponding to the eigenvalues of Γ_m span a finite-dimensional subspace of $\mathcal{H}$. We denote by $\mathbf{P}$ the orthogonal projection operator on this subspace and by $\mathbf{Q}$ the orthogonal projection operator on the complementary subspace. We write for the solution $\mathbf{u}$ of (3.5), (3.6), $\mathbf{u} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p} = \mathbf{P}\mathbf{u}$, $\mathbf{q} = \mathbf{Q}\mathbf{u}$.

By projecting equation (3.5) on the above constructed spaces, we obtain

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{P}\mathbf{B}(\mathbf{p}+\mathbf{q}) = \mathbf{P}\mathbf{f}, \tag{3.9}$$
$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} - \nu\Delta\mathbf{q} + \mathbf{Q}\mathbf{B}(\mathbf{p}+\mathbf{q}) = \mathbf{Q}\mathbf{f}. \tag{3.10}$$

In [10] is proved that for every $R > 0$, there is a moment $t_3\left(R\right) \geq t_2(R)$ such that for every $|\mathbf{u}_0| \leq R$,

$$\begin{aligned} &|\mathbf{q}\left(t\right)| \leq K_0 L^{\frac{1}{2}}\delta, \quad \|\mathbf{q}\left(t\right)\| \leq K_1 L^{\frac{1}{2}}\delta^{\frac{1}{2}}, \\ &\left|\mathbf{q}'\left(t\right)\right| \leq K_0' L^{\frac{1}{2}}\delta, \quad |\Delta\mathbf{q}\left(t\right)| \leq K_2 L^{\frac{1}{2}}, \qquad t \geq t_3\left(R\right), \end{aligned} \tag{3.11}$$

where, for our choice of the set of eigenvalues Γ_m, $L = 1 + \ln(2m^2)$. In [19] we proved that estimates of the same order are true for the various norms of $\mathbf{q}\left(t\right)$ above, but with coefficients of the powers of δ not depending on m.

3.3. Induced trajectories for the Navier-Stokes problem

In [38] the notion of induced trajectory is defined and a family of induced trajectories is constructed for this problem. The asymptotic expansions that rely behind this construction are not made explicit there.

A family of functions, $\{\mathbf{q}_j;\ j \in \mathbb{N}\}$, that satisfy the equations

$$-\nu\Delta\mathbf{q}_0 + \mathbf{QB}(\mathbf{p}) = \mathbf{Qf}, \tag{3.12}$$

$$-\nu\boldsymbol{\Delta}\mathbf{q}_1 + \mathbf{QB}(\mathbf{p}) + \mathbf{QB}(\mathbf{p},\mathbf{q}_0) + \mathbf{QB}(\mathbf{q}_0,\mathbf{p}) = \mathbf{Qf}, \tag{3.13}$$

$$-\nu\Delta\mathbf{q}_2 + \mathbf{QB}(\mathbf{p}) + \mathbf{QB}(\mathbf{p},\mathbf{q}_1) + \mathbf{QB}(\mathbf{q}_1,\mathbf{p}) + \mathbf{QB}(\mathbf{q}_0,\mathbf{q}_0) + \mathbf{q}_0' = \mathbf{Qf}, \tag{3.14}$$

$$-\nu\boldsymbol{\Delta}\mathbf{q}_j + \mathbf{q}_{j-2}' + \mathbf{QB}(\mathbf{p}) + \mathbf{QB}(\mathbf{p},\mathbf{q}_{j-1}) + \tag{3.15}$$

$$+\mathbf{QB}(\mathbf{q}_{j-1},\mathbf{p}) + \mathbf{QB}(\mathbf{q}_{j-2},\mathbf{q}_{j-2}) = \mathbf{Qf},\ \ j \geq 2,$$

is defined.

If $\mathbf{p}(t)$ is, as above, the $\mathbf{P}$ projection of the solution $\mathbf{u}(t)$ of the NSE, the sets $\{\mathbf{u}_j(t) = \mathbf{p}(t) + \mathbf{q}_j(t);t \geq 0\}$ are called *induced trajectories* associated to the trajectory $\{\mathbf{u}(t) = \mathbf{p}(t) + \mathbf{q}(t);t \geq 0\}$. The inequalities

$$|\mathbf{q}_j| \leq \kappa_j\delta L^{1/2},\ \ \|\mathbf{q}_j\| \leq \kappa_j\delta^{1/2}L^{1/2},\ |\mathbf{q}_j'| \leq \kappa_j\delta L^{1/2},$$

are proved in [38], as well as the following

$$|\mathbf{q}(t) - \mathbf{q}_j(t)| \leq \bar{\kappa}_j L^{(1+j)/2}\delta^{(3+j)/2}. \tag{3.16}$$

3.4. A family of approximate inertial manifolds for the Navier-Stokes equations

The family of induced trajectories above, more precisely the functions $\mathbf{q}_j$, $j \geq 0$, form the starting point for the construction of a family of approximate inertial manifolds defined in the literature, the first one in [10] and the following in [38] and [39]. The first a.i.m. of this family is the graph $\mathcal{M}_0$ of the function $\Phi_0 : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$, that satisfies the relation

$$-\nu\boldsymbol{\Delta}\Phi_0(\mathbf{X}) + \mathbf{QB}(\mathbf{X}) = \mathbf{Qf},$$

where $\mathbf{X} \in \mathbf{P}\mathcal{H}$. Thus $\Phi_0(\mathbf{X})$ is explicitly given by

$$\Phi_0(\mathbf{X}) = (-\nu\boldsymbol{\Delta})^{-1}(\mathbf{Qf} - \mathbf{QB}(\mathbf{X})). \tag{3.17}$$

The connection between this definition and the definition (3.12) of $\mathbf{q}_0$ is obvious: the set of points $\{\mathbf{p}(t)+\mathbf{q}_0(t);\ t\geq 0\}$ lies on $\mathcal{M}_0$. The next a.i.m. defined in [38] is $\mathcal{M}_1$, the graph of the function $\Phi_1 : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$, given by the solution of the problem

$$-\nu\mathbf{\Delta}\Phi_1(\mathbf{X}) + \mathbf{QB}(\mathbf{X}) + \mathbf{QB}(\mathbf{X},\Phi_0(\mathbf{X})) + \mathbf{QB}(\Phi_0(\mathbf{X}),\mathbf{X}) = \mathbf{Qf},$$

that is

$$\Phi_1(\mathbf{X}) = -(\nu\mathbf{\Delta})^{-1}\left[\mathbf{Qf} - \mathbf{QB}(\mathbf{X}) - \mathbf{QB}(\mathbf{X},\Phi_0(\mathbf{X})) - \mathbf{QB}(\Phi_0(\mathbf{X}),\mathbf{X})\right]. \tag{3.18}$$

The relation with the definition (3.13) of the corresponding function $\mathbf{q}_1$ is clear.

For $j\geq 2$, inspired by the definition (3.15) of $\mathbf{q}_j$, the a.i.m. $\mathcal{M}_j$ is defined as the graph of $\Phi_j : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$, with $\Phi_j(\mathbf{X})$ the solution of

$$\begin{aligned}-\nu\mathbf{\Delta}\Phi_j(\mathbf{X}) + \mathbf{QB}(\mathbf{X}) + \mathbf{QB}(\mathbf{X},\Phi_{j-1}(\mathbf{X})) + \mathbf{QB}(\Phi_{j-1}(\mathbf{X}),\mathbf{X}) + \\ + \mathbf{QB}(\mathbf{\Phi}_{j-2}(\mathbf{X})) + \mathbf{D\Phi}_{j-2}(\mathbf{X})\,\Gamma_{j-2}(\mathbf{X}) = \mathbf{Qf},\end{aligned}$$

where $\mathbf{D\Phi}_{j-2}(\mathbf{X})\,\Gamma_{j-2}(\mathbf{X})$ is the Fréchet differential of $\mathbf{\Phi}_{j-2}(\mathbf{X})$, applied to

$$\Gamma_{j-2}(\mathbf{X}) = \nu\mathbf{\Delta X} - \mathbf{PB}(\mathbf{X} + \mathbf{\Phi}_{j-2}(\mathbf{X})) + \mathbf{Pf}. \tag{3.19}$$

Hence

$$\begin{aligned}\mathbf{\Phi}_j(\mathbf{X}) = -(\nu\mathbf{\Delta})^{-1}\left[\mathbf{Qf} - \mathbf{QB}(\mathbf{X}) - \mathbf{QB}(\mathbf{X},\mathbf{\Phi}_{j-1}(\mathbf{X})) - \right. \\ \left. - \mathbf{QB}(\mathbf{\Phi}_{j-1}(\mathbf{X}),\mathbf{X}) - \mathbf{QB}(\mathbf{\Phi}_{j-2}(\mathbf{X})) - \mathbf{D\Phi}_{j-2}(\mathbf{X})\,\Gamma_{j-2}(\mathbf{X})\right].\end{aligned} \tag{3.20}$$

The inequalities (3.16) allow us to estimate the distance between the trajectories of the problem and the a.i.m.s. This is immediate for the first two a.i.m.s, since for $j=0,1$, we have $\mathbf{u}_j(t)\in\mathcal{M}_j$, and thus

$$\mathrm{dist}_{\mathcal{H}}(\mathbf{u}(t),\ \mathcal{M}_j) \leq \mathrm{dist}(\mathbf{u}(t),\mathbf{u}_j(t)) = |\mathbf{q}(t)-\mathbf{q}_j(t)|.$$

For the a.i.m.s $\mathcal{M}_j$ with $j>1$, some extra work is necessary, since $\mathbf{D\Phi}_{j-2}(\mathbf{p}(t))\,\Gamma_{j-2}(\mathbf{p}(t))$ is only an approximation of $[\mathbf{q}_{j-2}(\mathbf{p}(t))]'$. However, in [38] and [39] it is proved that

$$\mathrm{dist}_{\mathcal{H}}(\mathbf{u}(t),\mathcal{M}_j) \leq \bar{\kappa}_j L^{(1+j)/2}\delta^{(3+j)/2}.$$

3.5. Nonlinear Galerkin method for the Navier Stokes equations

The nonlinear Galerkin method was first presented in [29]. It is defined for a class of equations that contains the Navier-Stokes equations as a particular case, i.e. an equation of the type (1.1) with

$$R(u) = B(u) + Cu,$$

where $B(u) = B(u,u)$, $B(.,.)$ is a bilinear operator having essentially the properties of $\mathbf{B}$ and C is a linear operator. It is assumed that $A + C$ is positive in H and C is bounded from $V = D(A^{1/2})$ to H.

We write the method for the Navier-Stokes problem we considered here (that is we take $A = -\nu\boldsymbol{\Delta}$, $B = \mathbf{B}$, $C = 0$). It consists in approximating in the $\mathbf{P}$ projection of the equation, the function $\mathbf{q}$ with help of the first a.i.m. of the family described above. That is, instead of the Galerkin equation, the equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{P}\left[\mathbf{B}(\mathbf{p}) + \mathbf{B}(\mathbf{p}, \boldsymbol{\Phi}_0(\mathbf{p})) + \mathbf{B}(\boldsymbol{\Phi}_0(\mathbf{p}), \mathbf{p})\right] = \mathbf{P}\mathbf{f}, \tag{3.21}$$

with the initial condition

$$\mathbf{p}(0) = \mathbf{P}\mathbf{u}_0,$$

is considered, where $\boldsymbol{\Phi}_0$ is given by (3.17) (the notations are adapted to ours). We see from the term $\mathbf{PB}(\mathbf{p} + \boldsymbol{\Phi}_0(\mathbf{p}))$ the term $\mathbf{PB}(\boldsymbol{\Phi}_0(\mathbf{p}), \boldsymbol{\Phi}_0(\mathbf{p}))$ is missing. This is because it is of lower order than the preceding terms.

As for the equation of $\boldsymbol{\Phi}_0$, this is taken in [29] as

$$\Phi_0(\mathbf{p}) = (-\nu\boldsymbol{\Delta})^{-1}\mathbf{Q}_{2m}\left[\mathbf{f} - \mathbf{B}(\mathbf{p})\right], \tag{3.22}$$

where $\mathbf{Q}_{2m}$ is the projection operator defined as $\mathbf{Q}_{2m} = \mathbf{Q}\mathbf{P}_{2m}$, where $\mathbf{P}_{2m}$ is the projector on the space spanned by the eigenfunctions corresponding to the eigenvalues in Γ_{2m} (of λ_{j_1,j_2} having $0 \leq j_1, j_2 \leq 2m$). This is because the space $\mathbf{Q}\mathcal{H}$ is infinite dimensional and a truncation must be made (at least for $\mathbf{f}$, since for periodic boundary conditions, if $\mathbf{X} \in \mathbf{P}\mathcal{H}$ then $\mathbf{B}(\mathbf{X})$ is anyway in $\mathbf{P}_{2m}\mathcal{H}$).

Let us denote, together with the authors of [29], the solution of (3.21) by $\mathbf{u}_m$. It is proved in the paper we refer at, that, if $\mathbf{u}_0 \in \mathcal{H}$ then $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^2(0,T;\mathcal{V})$, $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^p(0,T;\mathcal{H})$, strongly (for any $T > 0$, $p \geq 1$) and $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^\infty(\mathbb{R}^+;\mathcal{H})$ weak-star.

If $\mathbf{u}_0 \in \mathcal{V}$ then $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^2(0,T;D(A))$, $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^p(0,T;\mathcal{V})$, strongly (for any $T>0$, $p\geq 1$) and $\mathbf{u}_m \underset{m\to\infty}{\to} \mathbf{u}$ in $L^\infty(\mathbb{R}^+;\mathcal{V})$ weak-star.

As an alternative nonlinear Galerkin method, that starting from the equation (similar to (2.2))

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{PB}(\mathbf{p}+\mathbf{\Phi}_0(\mathbf{p})) = \mathbf{Pf}, \tag{3.23}$$

with Φ_0 defined by (3.22) is also given in [29]. Convergence results similar to those asserted above are proved.

In [7] an estimate of the error of the method is given

$$|\mathbf{u}(t) - [\mathbf{u}_m(t) + \mathbf{\Phi}_0(\mathbf{u}_m(t))]| \leq C(t)\delta^{3/2}.$$

In [6] the NL Galerkin method is improved by using more accurate a.i.m.s. The equation that provides the approximate solution is (we write it here also for the N-S equations)

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{PB}(\mathbf{p}+\mathbf{\Phi}_j(\mathbf{p}))] = \mathbf{Pf}, \tag{3.24}$$

where $\mathbf{\Phi}_j$ is the the function whose graph is the corresponding a.i.m. (similar to that defined in (3.20), but slightly different). Let us denote by $\mathbf{u}_{m,j}$ the solution of (3.24) and by $\mathbf{v}_{m,j} = \mathbf{u}_{m,j} + \mathbf{\Phi}_j(\mathbf{u}_{m,j})$. It is proved in [6] that if $\mathbf{u}_0 \in \mathcal{V}$, both $\mathbf{u}_{m,j}$ and $\mathbf{v}_{m,j}$ converge to $\mathbf{u}$ (when $m\to\infty$) in $L^2(0,T;D(A))$ and in $L^p(0,T;\mathcal{V})$, strongly (for all $p\geq 1$ and all $T>0$), and in $L^\infty(\mathbb{R}^+;\mathcal{V})$ weak-star. It is also proved that, for a fix j, $\mathbf{z}_{m,j} = \mathbf{\Phi}_j(\mathbf{u}_m)$ converges (when $m\to\infty$) to 0 in $L^\infty(\mathbb{R}^+;\mathcal{V})$ and $L^2(0,T;D(A))$ strongly for any $T>0$. In [7] some estimates for the error are obtained. More precisely, for the NSE, it is shown that (with our numbering of the a.i.m.s)

$$|\mathbf{u}(t) - \mathbf{v}_{m,j}(t)| \leq K_j L_m^{(j+3)/2}\delta^{(j+3)/2}.$$

3.6. Post-processed Galerkin method for the Navier-Stokes equations

The ideas on which the post-processed Galerkin method relies are exposed in 2.3. In [14] a general equation is considered and the Navier-Stokes equation

is treated as a particular case. The solution $\mathbf{p}_G$ of the Galerkin equation,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{PB}(\mathbf{p}) = \mathbf{Pf}, \tag{3.25}$$
$$\mathbf{p}(\mathbf{0}) = \mathbf{Pu}_0,$$

is post-processed. This means, at a certain moment T (the end of the time interval on which the integration of (3.25) was performed) the image of $\mathbf{p}_G$ on the first a.i.m. $\mathcal{M}_0$, that is $\Phi_0(\mathbf{p}_G(T))$, is computed and is added to $\mathbf{p}_G(T)$. It is proved that, if $\mathbf{f} \in \mathcal{H}$, then

$$|\mathbf{u}(T) - (\mathbf{p}_G(T) + \Phi_0(\mathbf{p}_G(T)))| \leq C\delta^{5/4}. \tag{3.26}$$

In a subsequent paper, [15], the same authors prove another estimate for the Navier-Stokes problem. More exactly, they prove that, for $\mathbf{f} \in \mathcal{H}$,

$$|\mathbf{u}(T) - (\mathbf{p}_G(T) + \Phi_0(\mathbf{p}_G(T)))| \leq CL^2\delta^{3/2}. \tag{3.27}$$

Estimate (3.27) is not necessarily better than (3.26), since the coefficient L^2 appears (as before, $L = 1 + \ln(2m^2)$). In [32] the method is improved. That paper considers a reaction-diffusion equation, but the algorithm works for the Navier-Stokes equations as well. Instead of the Galerkin equations, the NL Galerkin equations (3.24) are considered. The solution $\mathbf{u}_{m,j}(t)$ of these equations is post-processed, i.e. the sum

$$\mathbf{u}_{m,j}(T) + \Phi_{j+1}(\mathbf{u}_{m,j}(T))$$

is considered and proposed as an approximate solution. The estimate of the error is made in [32] for the reaction-diffusion equation, hence is not relevant for the Navier-Stokes equation.

3.7. The repeatedly adjusted and post-processed Galerkin method for the Navier-Stokes equation

We adapt the general method presented in 2.4 to the Navier-Stokes equations.

Level 0. We define the first step of this level as the classical Galerkin method. Let us consider the Cauchy problem

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\boldsymbol{\Delta}\mathbf{p} + \mathbf{PB}(\mathbf{p}) = \mathbf{Pf}, \tag{3.28}$$
$$\mathbf{p}(0) = \mathbf{Pu}_0.$$

We denote by $\mathbf{p}_G(t)$ its solution and define

$$\widetilde{\mathbf{q}}_0(t) = \mathbf{\Phi}_0(\mathbf{p}_G(t)).$$

In the implementation of the method, the equation (3.28) must be numerically integrated. We remark that the values of $\widetilde{\mathbf{q}}_0(t)$ must be computed in every point of the time mesh used in the course of the numerical integration, since they will be used at the next level of the method.

We define the function

$$\widetilde{\mathbf{u}}_0(t) = \mathbf{p}_G(t) + \widetilde{\mathbf{q}}_0(t). \tag{3.29}$$

This preliminary level differs from the PP Galerkin method only in the post-processing part, in the fact that we compute $\widetilde{\mathbf{q}}_0(t)$ at any moment of time and not only at the end of the time interval on which (3.28) is integrated.

Level 1. Now we consider the problem

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\mathbf{\Delta p} + \mathbf{PB}(\mathbf{p}+\widetilde{\mathbf{q}}_0) &= \mathbf{Pf}, \\ \mathbf{p}(0) &= \mathbf{Pu}_0, \end{aligned} \tag{3.30}$$

with $\widetilde{\mathbf{q}}_0(t)$ computed at the preceding step. Since $\widetilde{\mathbf{q}}_0(t)$ is already known, this equation is not more difficult to integrate than the simple Galerkin equation attached to the Navier-Stokes equation. It is an adjusted Galerkin equation since the nonlinear term is adjusted by adding to $\mathbf{p}(t)$ the term $\widetilde{\mathbf{q}}_0(t)$ that approximates $\mathbf{q}(t)$ better than $\mathbf{0}$ does. We denote by $\widetilde{\mathbf{p}}_0(t)$ the solution of problem (3.30). The computation of the error showed that $\widetilde{\mathbf{p}}_0$ is a better approximation of $\mathbf{p}$ than $\mathbf{p}_G$ (see the comments in 3.8).

Then we define

$$\begin{aligned} \widetilde{\mathbf{q}}_1(t) = \; & -(\nu\triangle)^{-1}[\mathbf{Qf} - \mathbf{QB}(\widetilde{\mathbf{p}}_0(t)) - \mathbf{QB}(\widetilde{\mathbf{p}}_0(t), \widetilde{\mathbf{q}}_0(t)) - \\ & -\mathbf{QB}(\widetilde{\mathbf{q}}_0(t), \widetilde{\mathbf{p}}_0(t))] \end{aligned}$$

The approximate solution will be defined at this level as

$$\widetilde{\mathbf{u}}_1(t) = \widetilde{\mathbf{p}}_0(t) + \widetilde{\mathbf{q}}_1(t). \tag{3.31}$$

This function is an approximation of $\mathbf{u}_1$ that defines the second induced trajectories.

Level j ($j \geq 2$). We assume that we constructed the functions $\widetilde{\mathbf{q}}_{j-2},\ \widetilde{\mathbf{q}}_{j-1}(t)$. We consider the adjusted Galerkin problem

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\boldsymbol{\Delta}\mathbf{p} + \mathbf{PB}\left(\mathbf{p}+\widetilde{\mathbf{q}}_{j-1}\right) = \mathbf{Pf}, \tag{3.32}$$

$$\mathbf{p}(0) = \mathbf{Pu}_0,$$

and denote by $\widetilde{\mathbf{p}}_{j-1}\left(t\right)$ its solution. Then we set

$$\widetilde{\mathbf{q}}_j\left(t\right) = \left(-\nu\boldsymbol{\Delta}\right)^{-1}\Big[\mathbf{Qf} - \mathbf{QB}(\widetilde{\mathbf{p}}_{j-1}\left(t\right)) - \mathbf{QB}(\widetilde{\mathbf{p}}_{j-1}\left(t\right), \widetilde{\mathbf{q}}_{j-1}\left(t\right)) - \tag{3.33}$$

$$-\mathbf{QB}(\widetilde{\mathbf{q}}_{j-1}\left(t\right), \widetilde{\mathbf{p}}_{j-1}\left(t\right)) - \mathbf{QB}(\widetilde{\mathbf{q}}_{j-2}\left(t\right), \widetilde{\mathbf{q}}_{j-2}\left(t\right)) - \widetilde{\mathbf{q}}'_{j-2}\left(t\right)\Big].$$

We define the approximate solution at this level as

$$\widetilde{\mathbf{u}}_j(t) = \widetilde{\mathbf{p}}_{j-1}\left(t\right) + \widetilde{\mathbf{q}}_j(t).$$

We remark that $\widetilde{\mathbf{u}}_j\left(t\right)$ is an approximation of $\mathbf{u}_j\left(t\right)$ (that defines a induced trajectory of the family constructed in [38]).

We must say that, at the last level, as in the NL PP Galerkin method, we may correct $\widetilde{\mathbf{p}}_{j-1}$ by adding $\widetilde{\mathbf{q}}_j$ only at some moments of interest (the final postprocessing step).

We also must remark that, when the method is numerically implemented, the projector $\mathbf{Q}$ must be replaced by a finite dimensional projector as, e.g. $\mathbf{Q}_{2m}$ defined in Section 3.5.

3.8. The error of the R-APP Galerkin method

It is not the purpose of this work to present the explicit calculus of the error of the methods presented. We proved in [20] that

$$|\mathbf{p}(t) - \widetilde{\mathbf{p}}_j(t)| \leq C\delta^{5/4+j/2}$$

and

$$|\mathbf{q}(t) - \widetilde{\mathbf{q}}_j(t)| \leq C\delta^{3/2+j/2},$$

where C depends on the data of the problem: $\Omega,\ \mathbf{f},\ \nu,\ \lambda_1$, and on t but not on m.

With other methods, other estimates may be obtained. If we start from estimates of [15] of $\left|\mathbf{p}\left(t\right) - \mathbf{p}_G\left(t\right)\right|$, where $\mathbf{p}_G\left(t\right)$ is, as before, the classical Galerkin approximation of the solution, that is

$$\left|\mathbf{p}\left(t\right) - \mathbf{p}_G\left(t\right)\right| \leq C'L^2\delta^{3/2},$$

an improvement of the estimate of the error of the successive solutions by a factor of $\delta^{1/4}$ seems to be obtained. However, the appearance of the factor L^2 ($L = 1 + \ln(2m^2)$) diminishes this success. A very careful analysis of the constants C, C' should be performed in order to see what approach is better.

Anyway, the R-APP Galerkin provides approximates solutions as accurate as those provided by the NL PP Galerkin method.

3.9. R-APP Galerkin method compared to the high-order accuracy NLPP Galerkin method

The R-APP Galerkin method is intended to bring some simplifications to the NL Galerkin methods that use high accuracy approximate inertial manifolds. Hence this method makes sense only if more of its levels are passed through.

The simplifications come from the following facts:

a) the use of some already known functions (the $\widetilde{\mathbf{q}}_j$s) for the adjustment of the Galerkin equation, makes the equations for the approximations of $\mathbf{p}$ to have essentially the same structure as the Galerkin equation; this imply simplifications of the algorithms for the numerical integration of these equations, compared to the corresponding equations of the NL Galerkin equations;

b) the use of the "approximate induced trajectories" instead of the approximate inertial manifolds makes some computations easier, because, in the function $\widetilde{\mathbf{q}}_j$ the term $\widetilde{\mathbf{q}}'_{j-2}$ appears instead of the corresponding term $\mathbf{D}\mathbf{\Phi}_{j-2}(\mathbf{X})\,\Gamma_{j-2}(\mathbf{X})$ of the a.i.m. Φ_j; the term $\widetilde{\mathbf{q}}'_{j-2}$ can be approximated by the numerical derivative (since we know its values in the points of the time mesh);

c) when we proceed to Level j of the method, all we need are the values of $\widetilde{\mathbf{q}}_{j-2}$ and $\widetilde{\mathbf{q}}_{j-1}$, while all values of $\widetilde{\mathbf{p}}_k$, $k < j-1$ and $\widetilde{\mathbf{q}}_k$, $k < j-2$ may be erased from the memory of the computer; this must be compared to the NL Galerkin method that uses $\mathcal{M}_j$, where in the course of a single numerical integration one must handle the values of all functions Φ_k, $k \leq j$, and all these must be stored in the memory of the computer.

In order to compare the R-APP Galerkin method with the NL PP Galerkin method, we must look at the levels j with $j \geq 2$.

Let us analyze in parallel the first stage of Level 3 (that delivers us the function $\widetilde{\mathbf{p}}_2$) of our method and the corresponding NL Galerkin method (that uses the a.i.m. $\mathcal{M}_2$). It is easier to follow our reasoning on this particular case than than on the general one.

In order to make the following as clear as possible, we describe the computations necessary for the simple Euler integration method. Of course, more elaborated algorithms must be used, but the difficulties added by these should be evaluated for each specific algorithm individually.

In order to proceed, we consider a time-mesh $0 = t_0 < t_1 < t_2 < ... < t_k < ... < t_N = T$ on the time integration interval $[0, T]$.

Let us make the notations

$$\Gamma_G(\mathbf{p}) = \mathbf{Pf} + \nu\boldsymbol{\Delta}\mathbf{p} - \mathbf{PB}(\mathbf{p})$$

$$\widetilde{\Gamma}_j(\mathbf{p}) = \mathbf{Pf} + \nu\boldsymbol{\Delta}\mathbf{p} - \mathbf{PB}(\mathbf{p}+\widetilde{\mathbf{q}}_j).$$

R-APP Galerkin method, at the third level, requires the following computations for the determination of $\widetilde{\mathbf{p}}_2(t)$:

at Level 0 – computation of $\mathbf{p}_G(t_k)$, $k = 1, ..., N$, by numerical integration of eq. (3.28) (this is equivalent with the computation of $\Gamma_G(\mathbf{p_G}(t_{k-1}))$); then computation of $\widetilde{\mathbf{q}}_0(t_k)$;

at Level 1 – computation of $\widetilde{\mathbf{p}}_0(t_k)$, $k = 1, ..., N$, by numerical integration of eq. (3.30) (this is equivalent with the computation of $\widetilde{\Gamma}_0(\widetilde{\mathbf{p}}(t_{k-1}))$); then computation of $\widetilde{\mathbf{q}}_1(t_k)$;

at Level 2 – computation of $\widetilde{\mathbf{p}}_1(t_k)$, $k = 1, ..., N$, by numerical integration of eq. (3.32) with $j = 2$, (this is equivalent with the computation of $\widetilde{\Gamma}_1(\widetilde{\mathbf{p}}_1(t_{k-1}))$), then computation of $\widetilde{\mathbf{q}}_2(t_k)$;

at Level 3 – computation of $\widetilde{\mathbf{p}}_2(t_k)$, by numerical integration of eq. (3.32) with $j = 3$ (this is equivalent with the computation of $\widetilde{\Gamma}_2(\widetilde{\mathbf{p}}_2(t_{k-1}))$).

NL Galerkin method that uses $\mathcal{M}_2$, presented in [29], consists in the integration of the system of ODEs

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \nu\Delta\mathbf{p} + \mathbf{P}[\mathbf{B}(\mathbf{p} + \boldsymbol{\Phi}_2(\mathbf{p}))] = \mathbf{Pf}, \qquad (3.34)$$

$$\mathbf{p}(0) = \mathbf{Pu}_0,$$

where the function Φ_2 is given by

$$-\nu\Delta\boldsymbol{\Phi}_2(\mathbf{p}) + \mathbf{Q}_{2m}\mathbf{B}(\mathbf{p} + \boldsymbol{\Phi}_1(\mathbf{p})) + \mathbf{q}_1^1 = \mathbf{Q}_{2m}\mathbf{f}, \quad (3.35)$$

$$-\nu\Delta\mathbf{q}_1^1 + \mathbf{Q}_{2m}\left[\mathbf{B}(\left(\mathbf{p}_0^1,\ \mathbf{p} + \boldsymbol{\Phi}_1(\mathbf{p})\right) + \mathbf{B}\left(\mathbf{p} + \boldsymbol{\Phi}_1(\mathbf{p}), \mathbf{p}_0^1\right)\right] = \mathbf{0},$$

$$\mathbf{p}_0^1 - \nu\boldsymbol{\Delta}\mathbf{p} + \mathbf{P}[\mathbf{B}(\mathbf{p} + \boldsymbol{\Phi}_1(\mathbf{p}))] = \mathbf{Pf},$$

$$-\nu\Delta\boldsymbol{\Phi}_1(\mathbf{p}) + \mathbf{Q}_{2m}\mathbf{B}(\mathbf{p} + \boldsymbol{\Phi}_0(\mathbf{p})) = \mathbf{Q}_{2m}\mathbf{f},$$

$$-\nu\Delta\boldsymbol{\Phi}_0(\mathbf{p}) + \mathbf{Q}_{2m}\mathbf{B}(\mathbf{p}) = \mathbf{Q}_{2m}\mathbf{f}.$$

We reproduced here the definition of $\mathcal{M}_2$ from [29], but we adapted the notations from [29] to our notations and we started counting a.i.m.s with 0, as in [38], while in [29] this count begins with 1.

In the course of the numerical integration, with $\mathbf{p}(t_{k-1})$, $k = 1, ..., N$, already determined, in order to find $\mathbf{p}(t_k)$, we have to compute:

$\mathbf{\Phi}_0(\mathbf{p}(t_{k-1}))$, $\mathbf{\Phi}_1(\mathbf{p}(t_{k-1}))$, $\Gamma_1(\mathbf{p}(t_{k-1}))$ (for the calculation of $\mathbf{p_0^1}(t_k)$, with Γ_1 given by (3.19), $j = 3$), $\mathbf{q}_1^1(t_{k-1})$, $\mathbf{\Phi}_2(\mathbf{p}(t_{k-1}))$, and finally $\Gamma_2(\mathbf{p}(t_{k-1}))$. This will yield $\mathbf{p}(t_k)$.

Now we can compare the two methods from the point of view of the computations involved. We have the following:

– computation of $\widetilde{\mathbf{q}}_0(t_j)$ is equivalent to that of $\mathbf{\Phi}_0(\mathbf{p}(t_j))$;

– computation of $\widetilde{\mathbf{q}}_1(t_j)$ is equivalent to that of $\mathbf{\Phi}_1(\mathbf{p}(t_j))$;

– computation of $\widetilde{\Gamma}_1(\widetilde{\mathbf{p}}_1(t_j))$ is equivalent to that of $\Gamma_1(\mathbf{p}(t_j))$;

– computation of $\widetilde{\mathbf{q}}_2(t_j)$ is equivalent to that of $\mathbf{\Phi}_2(\mathbf{p}(t_j))$, assuming that $\mathbf{q}_1^1(t_j)$ is already computed;

– finally we observe that the computation of $\mathbf{p}_G(t_j)$ and $\widetilde{\mathbf{p}}_0(t_j)$ (from R-APP Galerkin method) together, involve less computations than that of $\mathbf{q}_1^1(t_j)$ (from the NL Galerkin method).

This is because in computing $\mathbf{p}_G(t_j)$ we have to compute a number of $4m^2 + 2m$ projections of the term $\Gamma_G(\mathbf{p_G}(t_{j-1}))$ and in computing $\widetilde{\mathbf{p}}_0(t_j)$ we have to compute $4m^2 + 2m$ projections of the term $\widetilde{\Gamma}_0(\widetilde{\mathbf{p}}(t_{j-1}))$, while in computing $\mathbf{q}_1^1(t_j)$ we have to compute $12m^2 + 6m$ projections.

At the following level, induced trajectories, respectively a.i.m.s, of higher order are used. The definition of these involves approximations of the derivatives similar to the above. Hence, the difference in the amounts of computations between the two methods increases with the order of the method. It follows that the R-APP Galerkin method involves a smaller amount of computations than the NL Galerkin method.

The computational effort involved in the final post-processing part is eased in the R-APP Galerkin method by the fact that, by using approximations of the induced trajectories we can approximate directly (by numerical derivative) the function $\mathbf{q}'$, while in the NL PP Galerkin method it is approximated by the differential $\mathbf{D\Phi}_{j-2}(\mathbf{X})\Gamma_{j-2}(\mathbf{X})$. In conclusion, the R-APP Galerkin method brings simplifications to the NL PP Galerkin method relying on higher accuracy a.i.m.s.

4. Modified Galerkin methods for a reaction-diffusion problem

We consider a reaction-diffusion (RD) equation of the form

$$\frac{\partial u}{\partial t} - D\left(\Delta u - u\right) + g(u) = f, \tag{4.1}$$

where u is a real-valued function, $u = u(t, \mathbf{x}),\ \mathbf{x} \in \Omega = (0,\ l) \times (0,\ l),\ l > 0$, D is the diffusion coefficient and the function g is a polynomial function of odd degree. In order to simplify the following considerations we take here a polynomial function of degree 3,

$$g(u) = b_0 + b_1 u + b_2 u^2 + b_3 u^3,\ \ b_i \in \mathbb{R},\ b_3 > 0.$$

We take $f \in L^2(\Omega)$. To the equation (4.1) we associate an initial condition

$$u(0) = u_0 \tag{4.2}$$

and the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.3}$$

The phase space is here $\mathcal{H} = L^2(\Omega)$. We consider also the space $\mathcal{V} = H^1(\Omega)$ with the usual norm.

The operator $A = -\Delta + I$ is a positive-definite, self–adjoint, with compact inverse operator with definition domain $D(A) = H^2(\Omega)$. The following existence result may be obtained by the Galerkin-Faedo method [37], [34]

THEOREM **4.1** *If $u_0 \in \mathcal{H}$, then there exists a unique solution $u \in C(\mathbb{R}^+;\ \mathcal{H})$, $u \in L^2(0,T;\ \mathcal{V}) \cap L^{2p}(0,T;L^{2p}(\Omega))$ where $p > 1,\ T > 0$. If, more than that, $u_0 \in \mathcal{V}$, then $u \in C([0,T);\ \mathcal{V}) \cap L^2(0,T;\ H^2(\Omega))$.*

The semi-dynamical system $\{S(t)\}_{t\geq 0}$, generated by (4.1) is proved to be dissipative in $\mathcal{H}$ and $\mathcal{V}$ [37], [34]. Hence there is a $\rho_0 > 0$ (respectively a $\rho_1 > 0$), such that for every $R > 0$, there is a moment $t_0(R)$ (respectively $t_1(R) > t_0(R)$) with the property that for every $u_0 \in \mathcal{H}$ with $|u_0| \leq R$, we have $|S(t)u_0| < \rho_0$, for $t \geq t_0(R)$ (respectively $\|S(t)u_0\| < \rho_1$, for $t \geq t_1(R)$).

4.1. The splitting of the space

The eigenvalues of A are

$$\lambda_{j,k} = \frac{\pi^2}{l^2}[j^2 + k^2] + 1$$

and the corresponding eigenfunctions are

$$w_{j,k} = \frac{\sqrt{\alpha_j \alpha_k}}{l} \cos\frac{j\pi x}{l} \cos\frac{k\pi y}{l},$$

where $\alpha_j = 1$ for $j = 0$ and $\alpha_j = 2$ for $j \neq 0$.

As for the Navier-Stokes equations, we consider the set Γ_m of eigenvalues λ_{j_1,j_2} with $0 \leq j_1, j_2 \leq m$. We make the notations

$$\Lambda = \lambda_{m+1,0} = \lambda_{0,m+1},$$
$$\delta = \frac{1}{\Lambda}.$$

We also consider the space spanned by the eigenfunctions corresponding to these eigenvalues and we denote by P the projector on this space. We set $Q = I - P$, where I is the identity on $\mathcal{H}$, $p = Pu,\ q = Qu$.

We project the equation (4.1) by using these projectors, to obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} - D\,(\Delta p - p) + Pg(p+q) = Pf,$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} - D\,(\Delta q - q) + Qg(p+q) = Qf.$$

It can be proved (e.g. [4]) that

$$|q| \leq C\delta$$

for t great enough, where the coefficient C depends on the data of the problem.

4.2. Induced trajectories for the reaction-diffusion problem

In constructing a family of induced trajectories for the reaction-diffusion problem, we try an asymptotic analysis of the RD equations. We develop the function q in series of powers of δ

$$q = \delta\left(k_0 + \delta k_1 + \delta^2 k_2 + \delta^3 k_3 + ...\right). \tag{4.4}$$

We have

$$\begin{aligned} g(p+q) &= g(p) + g'(p)q + \frac{1}{2}g''(p)q^2 + \frac{1}{6}g'''(p)q^3 = \\ &= g(p) + g'(p)\delta\left(k_0 + \delta k_1 + \delta^2 k_2 + \delta^3 k_3 + ...\right) + \\ &+ \frac{1}{2}g''(p)\left[\delta\left(k_0 + \delta k_1 + \delta^2 k_2 + \delta^3 k_3 + ...\right)\right]^2 + \\ &+ \frac{1}{6}g'''(p)\left[\delta\left(k_0 + \delta k_1 + \delta^2 k_2 + \delta^3 k_3 + ...\right)\right]^3, \end{aligned}$$

hence, by ordering the terms after the powers of δ,

$$\begin{aligned} g(p+q) &= g(p) + \delta g'(p)k_0 + \\ &+ \delta^2\left[g'(p)k_1 + \frac{1}{2}g''(p)k_0^2\right] + \\ &+ \delta^3\left[g'(p)k_2 + \frac{1}{2}g''(p)2k_0k_1 + \frac{1}{6}g'''(p)k_0^3\right] + \\ &+ \delta^4\left[g'(p)k_3 + \frac{1}{2}g''(p)\left(k_1^2 + 2k_0k_2\right) + \frac{1}{6}g'''(p)3k_0^2k_1\right] + ... \,. \end{aligned} \tag{4.5}$$

Then, by substituting (4.4) in the equation for q, we obtain

$$\begin{aligned} &\delta k_0' + \delta^2 k_1' + \delta^3 k_2' + \delta^4 k_3' + ... \\ &- D\left[\delta\Delta k_0 + \delta^2\Delta k_1 + \delta^3\Delta k_2 + \delta^4\Delta k_3 + \delta^5\Delta k_4 + ...\right] + \\ &+ D\left[\delta k_0 + \delta^2 k_1 + \delta^3 k_2 + \delta^4 k_3 + \delta^5 k_4 + ...\right] + \\ &+ Qg(p) + \delta Qg'(p)k_0 + \delta^2 Q\left[g'(p)k_1 + \frac{1}{2}g''(p)k_0^2\right] + \\ &+ \delta^3 Q\left[g'(p)k_2 + \frac{1}{2}g''(p)2k_0k_1 + \frac{1}{6}g'''(p)k_0^3\right] + \\ &+ \delta^4 Q\left[g'(p)k_3 + \frac{1}{2}g''(p)\left(k_1^2 + 2k_0k_2\right) + \frac{1}{6}g'''(p)3k_0^2k_1\right] + ... = Qf. \end{aligned}$$

In ordering the terms in (4.5) we simply performed an algebraic calculus, and treated the right-hand side as a polynomial in δ, but when we look for the terms of the same order of magnitude, a careful analysis should be performed. Since $k_j(t) \in Q\mathcal{H}$, we have

$$|\Delta k_j| \geq \Lambda\,|k_j| = \frac{1}{\delta}\,|k_j| \tag{4.6}$$

and it follows that the term $\delta^{j+1}\nu\Delta k_j$ is of the order of j. We also must evaluate carefully the terms containing products or powers of $k_j s$. E.g., for the term $\frac{1}{2}g''(p)k_0^2$ we have the estimates

$$\left|\frac{1}{2}g''(p)k_0^2\right| = \left(\int_\Omega \left(g''(p)\right)^2 k_0^4 \mathrm{d}x\right)^{1/2} \leq \left(\int_\Omega \left(g''(p)\right)^4 dx\right)^{1/4} \left(\int_\Omega k_0^8 \mathrm{d}x\right)^{1/4}.$$

Sobolev embedding theorem gives

$$\|u\|_{L^p(\Omega)} \leq C(p,s)\,\|u\|_s\,,$$

with $1/p = 1/2 - s/2$, $s < 1$, and, since

$$\|u\|_s \leq C\,\|u\|_1\,,$$

we obtain

$$\left(\int_\Omega k_0^8 dx\right)^{1/4} = \|k_0\|^2_{L^8(\Omega)} \leq C^2(8,\frac{3}{4})\,\|k_0\|^2_{3/4} \leq C^2(8,\frac{3}{4})\,\|k_0\|^2_1\,.$$

In a similar way we see that $\left(\int_\Omega \left(g''(p)\right)^4 \mathrm{d}x\right)^{1/4}$ is a function of ρ_0 and ρ_1. This together with inequality $\|k_0\|_1 \geq \left(\frac{1}{\delta}\right)^{1/2} |k_0|$ show that all we can say about the term $\frac{1}{2}\delta^2 g''(p)k_0^2$ is that it is of order δ and we have to consider it together with the terms of the same order. Similar reasonings will be considered implicit for the other terms containing products or powers of k_js. Thus we obtain the relations:

$$\begin{aligned}
-\delta D\Delta k_0 + Qg(p) &= Qf,\\
\delta k_0' - \delta^2 D\Delta k_1 + \delta D k_0 + \delta Q g'(p)k_0 + \frac{1}{2}\delta^2 Q g''(p)k_0^2 &= 0,\\
\delta^2 k_1' - \delta^3 D\Delta k_2 + \delta^2 D k_1 + \delta^2 Q g'(p)k_1 + &\\
+\frac{1}{2}\delta^3 Q g''(p)2k_0k_1 + \frac{1}{6}\delta^3 Q g'''(p)k_0^3 &= 0,\\
\delta^3 k_2' - \delta^4 D\Delta k_3 + \delta^3 D k_2 + \delta^3 Q g'(p)k_2 + &\\
\frac{1}{2}\delta^4 Q g''(p)\left(k_1^2 + 2k_0k_2\right) + \frac{1}{6}\delta^4 Q g'''(p)3k_0^2k_1 &= 0,\\
&\vdots
\end{aligned}$$

Now we define the functions

$$q_j = \delta k_0 + \delta^2 k_1 + \delta^3 k_2 + \delta^4 k_3 + ... + \delta^{j+1} k_j.$$

By summing the equations for k_j, we obtain equations for q_j:

$$\begin{aligned} -D\Delta q_0 + Qg(p) &= Qf, \qquad (4.7)\\ q_0' - D\Delta q_1 + Dq_0 + Qg\,(p) + Qg'(p)q_0 + \frac{1}{2}Qg''(p)q_0^2 &= Qf,\\ q_1' - D\Delta q_2 + Dq_1 + Qg\,(p) + Qg'\,(p)\,q_1 +&\\ \frac{1}{2}Qg''(p)q_0^2 + \frac{1}{2}Qg''(p)2q_0\,(q_1 - q_0) + \frac{1}{6}Qg'''(p)q_0^3 &= Qf,\\ q_2' - D\Delta q_3 + Dq_2 + Qg\,(p) + Qg'\,(p)\,q_2 +&\\ \frac{1}{2}Qg''(p)q_1^2 + \frac{1}{6}Qg'''(p)3q_0^2\,(q_1 - q_0) &= Qf,\\ \vdots& \end{aligned}$$

We see that the nonlinearity of the polynomial makes the equations neither "beautiful", nor with a clear structure. However, we consider the functions

$$u_j(t) = p(t) + q_j(t),$$

and define the *induced trajectories* of the problem as the sets $\{u_j(t);\ t \geq 0\}$. These will be used to define the R-APP method for the reaction-diffusion equations.

4.3. Approximate inertial manifolds for the reaction-diffusion equation

In the NL Galerkin method and in the NL PP Galerkin method described in literature [32], the following a.i.m.s are defined for the RD equation: for any $j \geq 0$, $\mathcal{M}_j$ is the graph of the function $\Phi_j : P\mathcal{H} \to Q\mathcal{H}$, described below

$$DA\Phi_0(p) + Qg(p) = Qf, \qquad (4.8)$$

$$q_{j-1}^1 + DA\Phi_j(p) + Qg\,(p + \Phi_{j-1}(p)) = Qf, \quad j \geq 1. \qquad (4.9)$$

Here $q_{j-1}^1 = \mathbf{D}\Phi_{j-1}(p)\Gamma_{j-1}(p)$, with $\mathbf{D}\Phi_{j-1}(p)$ the Fréchet differential of Φ_{j-1} computed in p and applied to $\Gamma_{j-1}(p) = Pf - DAp - Pg(p + \Phi_{j-1}(p))$.

If we would want to construct a family of a.i.m.s $\widetilde{\mathcal{M}}_j$ starting from the induced trajectories we defined above (as is done in [38] for the Navier-Stokes equation), the first a.i.m. of the family, $\widetilde{\mathcal{M}}_0$, would be identical with $\mathcal{M}_0$ since the function $\widetilde{\Phi}_0$ defining it would be identical to Φ_0 of (4.8), as the equation for $q_0(t)$ shows.

The second a.i.m., $\widetilde{\mathcal{M}}_1$, would be quite different from $\mathcal{M}_1$ above. That is, it would be the graph of the function $\widetilde{\Phi}_1$ defined by the equation

$$\begin{array}{rl} DQ\widetilde{\Phi}_0(p)\Gamma_0(p) - D\Delta\widetilde{\Phi}_1(p) & +D\widetilde{\Phi}_0(p) + Qg(p) + \\ +Qg'(p)\widetilde{\Phi}_0(p) + & \frac{1}{2}Qg''(p)\widetilde{\Phi}_0(p)^2 = Qf, \end{array} \tag{4.10}$$

with $\Gamma_0(p) = Pf + D(\Delta p - p) - Pg(p + \widetilde{\Phi}_0(p))$. We see that the difference between this equation and that for Φ_1, that we write explicitly below

$$\mathbf{D}\Phi_0(p)\Gamma_0(p) - D\Delta\Phi_1(p) + D\Phi_1(p) + Qg(p + \Phi_0(p)) = Qf, \tag{4.11}$$

consists essentially in the presence of the term $\frac{1}{6}g'''(p)\Phi_0(p)^3$ in this latter equation. If the polynomial g would be of higher degree, the difference between the two families of a.i.m.s, that defined starting from the induced trajectories and the one defined by the relations (4.8) and (4.9) would increase. However, for the sake of the elegance of the definitions, (4.11) may be taken as the equation for $\Phi_1(p)$ even if it does not spring from an accurate asymptotic analysis. The presence of the higher order terms does not affect the order of magnitude of the distance between the exact solution of the R-D equation and the first a.i.m. [21].

4.4. "Induced trajectories" inspired by a.i.m.s

For the sake of the simplicity of the definitions and having in mind some simplifications of the computations in the R-APP Galerkin method below, we can choose an alternate definition for the induced trajectories of the R-D problem, inspired from the definitions of the a.i.m. of [32]. That is, we define the functions $\widetilde{q}_j$ through the relations

$$\begin{array}{c} DA\widetilde{q}_0 + Qg(p) = Qf, \\ \widetilde{q}'_{j-1} + DA\widetilde{q}_j + Qg(p + \widetilde{q}_{j-1}) = Qf, \quad j \geq 1, \end{array} \tag{4.12}$$

where $p(t) = Pu(t)$. The functions

$$\widetilde{u}_j = p + \widetilde{q}_j$$

define the new "induced trajectories" $\{\widetilde{u}_j(t);\ t \geq 0\}$.

4.5. The NL Galerkin method for the RDE

The NL Galerkin method for RDE consists in integrating the differential equation:

$$\frac{\mathrm{d}p}{\mathrm{d}t} + DAu + g(p + \Phi_0(p)) = Pf, \tag{4.13}$$

with the initial condition

$$p(0) = Pu_0. \tag{4.14}$$

If we denote by y_m its solution, the approximate solution is taken as

$$y_m(t) + \Phi_0(y_m(t)).$$

In [32] it is asserted that, for large enough t,

$$|u(t) - (y_m(t) + \Phi_0(y_m(t)))| \leq C\delta^2.$$

Improved NL Galerkin methods make use of the higher accuracy a.i.m.s, $\mathcal{M}_j$, $j \geq 1$. That is an equation of the type

$$\frac{\mathrm{d}p}{\mathrm{d}t} + DAu + g(p + \Phi_j(p)) = Pf, \tag{4.15}$$

with the initial condition (4.14) is solved, let $y_{m,j}$ be its solution. The approximate solution of the RDE is taken as:

$$y_{m,j}(t) + \Phi_j(y_{m,j}(t)).$$

In [32] it is proved that the $\mathcal{H}$ norm of the error of this approximate solution is of the order of $C(t)\delta^{j+2}$.

4.6. The PP NL Galerkin method for the RDE

Also in [32] the NL Galerkin method is post-processed, i.e. to the solution $y_{m,j}$ of the NL Galerkin problem, considered in T, the quantity $\Phi_{j+1}(y_{m,j}(T))$ is added and

$$y_{m,j}(T) + \Phi_{j+1}(y_{m,j}(T))$$

is taken as the approximate solution in T. It is proved in [32] that

$$|u(t) - (y_{m,j}(t) + \Phi_{j+1}(y_{m,j}(t)))| \leq C \ln m\, \delta^{j+3}.$$

4.7. The R-APP Galerkin method for the RDE

We describe the R-APP Galerkin method for the reaction-diffusion equation. In [21] we presented a variant of our method that has as initial level a NL Galerkin method (this was meant to skip a numerical integration - that of the Galerkin problem). Let us denote generically

$$q_j = F_j(Qf, p, q_0, q_1, ..., q_{j-1}),$$

either the functions given by the set of relations (4.7) or the functions $\widetilde{q}_j$ given by (4.12). We see that in this latter case, F_j, $j \geq 1$ actually depends only on Qf, p, q_{j-1}, q'_{j-1}.

Level 0. We consider the NL Galerkin problem

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} - D\left(\Delta p - p\right) + Pg(p) &= Pf, \\ p(0) &= Pu_0 \end{aligned} \tag{4.16}$$

and denote it's solution by p_G.

Then we compute, at every moment of time

$$\widetilde{q}_0\left(t\right) = F_0(Qf, p_G\left(t\right)).$$

When the numerical implementation of the method is actually done, this is equivalent to the computation of q_1 at the nodes of the time mesh, and $q'_0\left(t_i\right)$ is approximated by $(q_0(t_i) - q_0(t_{i-1}))/(t_i - t_{i-1})$. The approximate solution is

$$u_0 = p_G + \widetilde{q}_0.$$

Level 1. We consider the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} - D\left(\Delta p - p\right) + Pg(p + \widetilde{q}_0) = Pf,$$

and denote its solution by $\widetilde{p}_0$. Then we compute

$$\widetilde{q}_1\left(t\right) = F_1(Qf, \widetilde{p}_0\left(t\right), \widetilde{q}_0\left(t\right)).$$

The approximate solution at this level is defined as

$$\widetilde{u}_j = \widetilde{p}_{j-1} + \widetilde{q}_j.$$

Level j > 1. We assume $\widetilde{q}_0, \widetilde{q}_1, ..., \widetilde{q}_{j-1}$ were successively constructed. We consider the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} - D\left(\Delta p - p\right) + Pg(p + \widetilde{q}_{j-1}) = Pf,$$

and denote its solution by $\widetilde{p}_{j-1}$. Then we compute

$$\widetilde{q}_j\left(t\right) = F_j(Qf, \widetilde{p}_{j-1}\left(t\right), \widetilde{q}_0\left(t\right), \widetilde{q}_1(t), ..., \widetilde{q}_{j-1}(t),).$$

The approximate solution at this level is defined as

$$\widetilde{u}_j = \widetilde{p}_{j-1} + \widetilde{q}_j.$$

Remarks: 1. While the equations for p_j are equivalent to a finite, constant number, of (differential) equations, the equations for q_j are equivalent to a system of equations having (if Qf admits non-null projections on an infinite number of eigenfunctions) a infinite number of equations.

Hence a truncation must be done. In [6] the truncation is made by using a projector, denoted P_{2m}, that is the analogous of P but with $2m$ instead of m. If Qf would have nonzero projections only on a finite number of eigenfunctions, then q_j would also be finite dimensional. In this situation, we could also compute the dimension of q_j, by using the consequences of the trigonometrical relation $2\cos\alpha\cos\beta = \cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)$, on the products of eigenfunctions. Then, in order to not affect the estimate of the error predicted by our method, we could take a truncation of Qf, let us denote it by $Q_j f$ such that $\left|\Delta^{-1}\left(Qf - Q_j f\right)\right|$ is less that the error of the level j.

2. Both families of $\{q_j\}_{j\geq 0}$ defined above present advantages and disadvantages one relative to the other. The first family, defined in (4.7), has the advantage of demanding a smaller amount of computations since in (4.7) fewer terms than in (4.12) are taken into account at a certain level. It presents the disadvantage of recalling all q_i with $i < j$, at a certain level j. The second family of approximations of q, given by (4.12), recalls at a certain level j, only the values of q_{j-1}. This is important from the point of view of organizing the memory of the computer in the numerical implementation of the method. However, this second family takes into account more terms in the polynomial g. This increases a lot the computations when g has a high degree.

4.8. Estimates of the error

By using the method of [32], we can prove that both families of induced trajectories defined above lead to the same orders of error, for every level of the R-APP method, as the corresponding NL PP Galerkin method. That is, we can prove [22] that at the level $j+1$ of our method

$$|p - \widetilde{p}_j| \leq C_j \, (\ln m \,) \, \delta^{j+3}$$

and

$$|q - \widetilde{q}_{j+1}| \leq K_j \delta^{j+3},$$

and thus

$$|u - \widetilde{u}_{j+1}| \leq [C_j \, (\ln m \,) + K_j] \, \delta^{j+3}.$$

4.9. Comments on the method

The comparison of the computational cost of the R-APP Galerkin method to that of the NL Galerkin method is similar to that we performed for the Navier-Stokes equations. The conclusions are the same: the R-APP Galerkin method is more economic than the NL PP Galerkin method. The difference in the computational cost between the two methods increases with their level.

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