Topics in Applied Mathematics & Mathematical Physics

Estimating the number of negative eigenvalues of a relativistic Hamiltonian with regular magnetic field

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1. Introduction

For the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$ $(d \geq 3)$, one has the well-known CLR (Cwikel-Lieb-Rosenblum) estimation for $N(V)$, *the number of negative eigenvalues*:

$$N(V) \;\leq\; c(d) \int_{\mathbb{R}^d} \mathrm{d}x \; |V_-(x)|^{d/2} \,. \tag{1.1}$$

V is the multiplication operator with the function $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_- := (|V|-V)/2 \in L^{d/2}(\mathbb{R}^d)$; the constant $c(d) > 0$ only depends on the dimension $d \geq 3$ (see [47], Th. XII.12).

There exist at least four different proofs of this inequality. Rosenblum [35] uses "piece-wise polynomial approximation in Sobolev spaces". Lieb [25] relies on the Feynman-Kac formula. Cwickel [4] uses ideas from interpolation theory. Finally, Li and Yau [31] make a heat kernel analysis.

The inequality (1.1) has been extended in [1] and [48] to the case of operators with magnetic fields $(-i\nabla - A)^2 + V$, where the components of the vector potential $A = (A_1, \ldots, A_d)$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. The basic ingredient of the proof is the Feynman-Kac-Ito formula. Melgaard and Rosenblum [41] generalizes this result (by a different method) to a class of differential operators of second order with variable coefficients. The idea for treating the relativistic Hamiltonian (without a magnetic field), by replacing Brownian motion with a Lévy process, appears in [5] and we follow it in our work giving all the technical details. Some similar results but for a different Hamiltonian and with different techniques have been obtained recently in [8].

Our aim in this paper is to obtain an estimation of the type (1.1) for an operator that is a good candidate for a relativistic Hamiltonian with magnetic field (for scalar particles); it is gauge covariant and obtained through a quantization procedure from the classical candidate. We shall make use of a "magnetic pseudodifferential calculus" that has been introduced and developed in some previous papers [34], [35], [27], [28], [36], [38], [24].

Let us denote by $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ the family of functions $f \in C^\infty(\mathbb{R}^d)$ for which all the derivatives $\partial^\alpha f$, $\alpha \in \mathbb{N}^d$ have polynomial growth.

Let B be a magnetic field (a 2-form) with components $B_{jk} \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$. It is known that it can be expressed as the differential $B = dA$ of a vector potential (a 1-form) $A = (A_1, \ldots, A_d)$ with $A_j \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $j = 1, \ldots, d$; an example is the transversal gauge:

$$A_j(x) = -\sum_{k=1}^{n} \int_0^1 \mathrm{d}s\ B_{jk}(sx)sx_k.$$

We denote by

$$\Gamma^A(x,y) := \int_0^1 \mathrm{d}s\, A((1-s)x+sy) = \int_{[x,y]} A, \quad x,y \in \mathbb{R}^d. \tag{1.2}$$

the circulation of A along the segment $[x,y]$, $x,y \in \mathbb{R}^d$. If a is a symbol on $\mathbb{R}^d$, one defines by an oscillatory integral the linear continuous operator $\mathfrak{Op}^A(a) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d)$ by

$$\left[\mathfrak{Op}^A(a)\right](x) := (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}y\,\mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} \mathrm{e}^{-\mathrm{i}\int_{[x,y]}A} a\left(\frac{x+y}{2},\xi\right) u(y), \tag{1.3}$$

The correspondence $a \mapsto \mathfrak{Op}^A(a)$ is meant to be a quantization and could be regarded as a functional calculus $\mathfrak{Op}^A(a) = a(Q,\Pi^A)$ for the family of non-commuting operators $(Q_1,\dots,Q_d;\Pi_1^A,\dots,\Pi_d^A)$, where Q is the position operator, $\Pi^A := D - A(Q)$ is the magnetic momentum, with $D := -\mathrm{i}\nabla$.

If a belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$, then $\mathfrak{Op}^A(a)$ acts continuously in the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}^*(\mathbb{R}^d)$, respectively. It enjoys the important physical property of being gauge covariant: if $\varphi \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ is a real function, A and $A' := A + d\varphi$ define the same magnetic field and one prove easily that $\mathfrak{Op}^{A'}(a) = \mathrm{e}^{\mathrm{i}\varphi}\mathfrak{Op}^A(a)\mathrm{e}^{-\mathrm{i}\varphi}$. The property is not shared by the quantization $a \mapsto \mathfrak{Op}_A(a) := \mathfrak{Op}(a \circ \nu_A)$, where $\mathfrak{Op}$ is the usual Weyl quantization and $\nu_A : \mathbb{R}^d \to \mathbb{R}^d$, $\nu_A(x,\xi) := (x, \xi - A(a))$ is an implementation of "the minimal coupling".

We mention that in the references quoted above, a symbolic calculus is developed for the magnetic pseudodifferential operators (1.3). In particular, a symbol composition $(a,b) \mapsto a\sharp^B b$ is defined and studied, verifying $\mathfrak{Op}^A(a)\mathfrak{Op}^A(b) = \mathfrak{Op}^A(a\sharp^B b)$. It depends only on the magnetic field B, no choice of a gauge being needed. The formalism has a C^*-algebraic interpretation in terms of twisted crossed products, cf. [35], [37], [39] and it has been used in [40] for the spectral theory of quantum Hamiltonians with anisotropic potentials and magnetic fields.

We shall denote by H_A the unbounded operator in $L^2(\mathbb{R}^d)$ defined on $C_0^\infty(\mathbb{R}^d)$ by $H_A u := \mathfrak{Op}^A(h)u$, with $h(x,\xi) \equiv h(\xi) := \langle\xi\rangle - 1 = (1+|\xi|^2)^{1/2} - 1$. One can express it as

$$(H_A u)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} h\left(\xi - \Gamma^A(x,y)\right) u(y). \tag{1.4}$$

H_A is a symmetric operator and, as seen below, essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. Also denoting its closure by H_A, we will have $H_A \geq 0$.

Ichinose and Tamura [19], [20], using the quantization $a \mapsto (Op)_A(a)$, study another relativistic Hamiltonian with magnetic field defined by

$$\left(H'_A u\right)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} h\left(\xi - A\left(\frac{x+y}{2}\right)\right) u(y), \tag{1.5}$$

for which they prove many interesting properties. Unfortunately, H'_A is not gauge covariant (cf. [24]). Many of the properties of H'_A also hold for H_A (by replacing $A\left(\frac{x+y}{2}\right)$ with $\Gamma^A(x,y)$ in the statements and proofs) and this will be used in the sequel.

Aside the magnetic field $B = dA$, we shall also consider an electric potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, real function expressed as $V = V_+ - V_-$, $V_\pm \geq 0$, such that $V_- \in L^{d+k}(\mathbb{R}^d) \cap L^{d/2+k}(\mathbb{R}^d)$ for some $k \geq 0$. We are interested in the operator $H(A,V) := H_A + V$; it will be shown that it is well-defined in form sense as a self-adjoint operator in $L^2(\mathbb{R}^d)$, with essential spectrum included into the positive real axis. Taking advantage of gauge covariance, we denote by $N(B,V)$ the number of strictly negative eigenvalues of $H(A,V)$ (multiplicity counted); it only depends on the potential V and the magnetic field B.

The main result of the article is

THEOREM **1.1** *Let $B = dA$ be a magnetic field with $B_{jk} \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $A_j \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and let $V = V_+ - V_- \in L^1_{\mathrm{loc}(\mathbb{R}^d)}$ be a real function with $V_\pm \geq 0$ and $V_- \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$. Then there exists a constant C_d, only depending on the dimension $d \geq 3$, such that*

$$N(B,V) \leq C_d \left(\int_{\mathbb{R}^d} \mathrm{d}x \, V_-(x)^d + \int_{\mathbb{R}^d} \mathrm{d}x \, V_-(x)^{d/2} \right). \tag{1.6}$$

A standard consequence is the next Lieb-Thirring-type estimation:

COROLLARY **1.1** *We assume that the components of B belong to $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and that $V = V_+ - V_- \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a real function with $V_\pm \geq 0$ and $V_- \in L^{d+k}(\mathbb{R}^d) \cap L^{d/2+k}(\mathbb{R}^d)$, $k > 0$. We denote by $\lambda_1 \leq \lambda_2 \leq \ldots$ the* *strictly negative eigenvalues of $H(A, V)$ (with multiplicity). For any $d \geq 2$ there exists a constant $C_d(k)$ such that*

$$\sum_j |\lambda_j|^k \leq C_d(k) \left(\int_{\mathbb{R}^d} \mathrm{d}x\, V_-(x)^{d+k} + \int_{\mathbb{R}^d} \mathrm{d}x\, V_-(x)^{d/2+k} \right). \tag{1.7}$$

Sections 2, 3, 4 will contain essentially known facts (usually presented without proofs), needed for checking Theorem 1.1. So, in Section 2 we introduce the Feller semigroup ([20], [17], [26]) associated to the operator $H_0 := \langle D \rangle - 1$. In the third section we define properly the operator $H(A, V)$ and study its basic properties. In Section 4 we recall some probabilistic results, as the Markov process associated to the semigroup defined by H_0 ([25], [6], [26]) and the Feynman-Kac-Itô formula adapted to a Lévy process ([20]).

In Section 5 we prove Theorem 1.1 for $B = 0$, using some of Lieb's ideas for the non-relativistic case (see [48]) in the setting proposed in [5]. The last section contains the proof of Theorem 1.1 with magnetic field as well as Corollary 1.1. The main ingredient is the Feynman-Kac-Itô formula.

2. The Feller semigroup

We consider the following symbol (interpreted as a classical relativistic Hamiltonian for $m = 1, c = 1$) $h : \mathbb{R}^d \to \mathbb{R}_+$ defined by $h(\xi) := \langle \xi \rangle - 1 \equiv \sqrt{1 + |\xi|^2} - 1$. Let us observe (as in [17]) that it defines a *conditional negative definite function* (see [47]) and thus has a Lévy-Khincin decomposition (see Appendix 2 to Section XIII of [47]). Computing $(\nabla h)(\xi)$ and $(\Delta h)(\xi)$ and using the general Lévy-Khincin decomposition (see for example [47]), one obtains that there exists a Lévy measure $\mathsf{n}(\mathrm{d}y)$, i.e. a non-negative, σ-finite measure on $\mathbb{R}^d$, for which $\min\{1, |y|^2\}$ is integrable on $\mathbb{R}^d$, such that

$$h(\xi) \;=\; -\int_{\mathbb{R}^d} \mathsf{n}(\mathrm{d}y) \left\{ \mathrm{e}^{\mathrm{i}y\cdot\xi} - 1 - \mathrm{i}\,(y \cdot \xi)\, I_{\{|x|<1\}}(y) \right\}, \tag{2.1}$$

where $I_{\{|x|<1\}}$ is the characteristic function of the open unit ball in $\mathbb{R}^d$. One has the following explicit formula (see [17]):

$$\mathsf{n}(\mathrm{d}y) \;=\; 2(2\pi)^{-(d+1)/2} |y|^{-(d+1)/2} K_{(d+1)/2}(|y|)\, \mathrm{d}y, \tag{2.2}$$

with K_ν the modified Bessel function of third type and order ν. We recall the following asymtotic behaviour of these functions:

$$0 \;<\; K_\nu(r) \;\leq\; C \max(r^{-\nu}, r^{-1/2}) \mathrm{e}^{-r}, \quad \forall r > 0, \quad \forall \nu > 0. \tag{2.3}$$

We shall denote by $\mathcal{H}^s(\mathbb{R}^d)$ the usual Sobolev spaces of order $s \in \mathbb{R}$ on $\mathbb{R}^d$ and by H_0 the pseudodifferential operator $h(D) \equiv \mathfrak{Op}(h)$ considered either as a continuous operator on $\mathcal{S}(\mathbb{R}^d)$ and on $\mathcal{S}^*(\mathbb{R}^d)$ or as a self-adjoint operator in $L^2(\mathbb{R}^d)$ with domain $\mathcal{H}^1(\mathbb{R}^d)$. The semigroup generated by H_0 is explicitly given by the convolution with the following function (for $t > 0$ and $x \in \mathbb{R}^d$):

$$\overset{\circ}{\wp}_t(x) \;:=\; (2\pi)^{-d}\frac{t}{\sqrt{|x|^2+t^2}}\int_{\mathbb{R}^d} d\xi\, e^{\left(t-\sqrt{(|x|^2+t^2)(|\xi|^2+1)}\right)} \;=$$

$$=\; 2^{-(d-1)/2}\,\pi^{-(d+1)/2}\, te^t(|x|^2+t^2)^{-(d+1)/4}K_{(d+1)/2}(\sqrt{|x|^2+t^2}) \qquad (2.4)$$

(see [20], [2]). We have

$$\overset{\circ}{\wp}_t(x) \;>\; 0 \quad \text{and} \quad \int_{\mathbb{R}^d} dx\, \overset{\circ}{\wp}_t(x) \;=\; 1. \qquad (2.5)$$

From (2.3) one easily can deduce the following estimation

$$\exists C > 0 \quad \text{such that} \quad \overset{\circ}{\wp}_t(0) \;\leq Ct^{-d}(1+t^{d/2}), \quad \forall t > 0. \qquad (2.6)$$

Let us set

$$C_\infty(\mathbb{R}^d) \;:=\; \left\{ f \in C(\mathbb{R}^d) \mid \lim_{|x|\to\infty} f(x) = 0 \right\} \qquad (2.7)$$

and endow it with the Banach norm $\|f\|_\infty := \sup_{x\in\mathbb{R}^d}|f(x)|$. Using the above properties of the function $\overset{\circ}{\wp}_t$ we can extend e^{-tH_0} to a well-defined bounded operator $P(t)$ acting in $C_\infty(\mathbb{R}^d)$.

Remark **2.1** *One can easily verify that* $\{P(t)\}_{t\geq 0}$ *is a Feller semigroup, i.e.:*

1. $P(t)$ *is a contraction:* $\|P(t)f\|_\infty \leq \|f\|_\infty$, $\forall f \in C_\infty(\mathbb{R}^d)$;
2. $\{P(t)\}_{t\geq 0}$ *is a semigroup:* $P(t+s) = P(t)P(s)$;
3. $P(t)$ *preserves positivity:* $P(t)f \geq 0$ *for any* $f \geq 0$ *in* $C_\infty(\mathbb{R}^d)$;
4. *We have* $\lim_{t\searrow 0}\|P(t)f - f\|_\infty = 0$, $\forall f \in C_\infty(\mathbb{R}^d)$.

3. The perturbed Hamiltonian

Suppose given a magnetic field of class $C^\infty_{\mathsf{pol}}(\mathbb{R}^d)$ and let us choose a potential vector A, such that $B = dA$, with components also of class $C^\infty_{\mathsf{pol}}(\mathbb{R}^d)$ (this is always possible, as said before). We shall denote by H_A the operator

$\mathfrak{Op}^A(h)$, considered either as a continuous operator on $\mathcal{S}(\mathbb{R}^d)$ and on $\mathcal{S}^*(\mathbb{R}^d)$ (by duality) or as an unbounded operator on $L^2(\mathbb{R}^d)$ with domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Using the Fourier transform one easily proves that for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$[H_0 u](x) \;=\; -\int_{\mathbb{R}^d} n(\mathrm{d}y)\,\left[u(x+y) - u(x) - I_{\{|z|<1\}}(y)\,(y\cdot\partial_x u)\,(x)\right]. \quad (3.1)$$

Recalling the definition of $\mathfrak{Op}^A(h)$, we remark that

$$[H_A u](x) \;=\; \left[\mathfrak{Op}^A(h)u\right](x) \;=\; \left[\mathfrak{Op}(h)\left(\mathrm{e}^{\mathrm{i}(x-.)\cdot\Gamma^A(x,.)}u\right)\right](x) \;= \quad (3.2)$$

$$= \;\left[H_0\left(\mathrm{e}^{\mathrm{i}(x-.)\cdot\Gamma^A(x,.)}u\right)\right](x).$$

Combining the above two equations one gets easily

$$[H_A u](x) \;=\; -\int_{\mathbb{R}^d} n(\mathrm{d}y)\,\left[\mathrm{e}^{-\mathrm{i}y\cdot\Gamma^A(x,x+y)}u(x+y) - u(x) - \right. \quad (3.3)$$

$$\left. -I_{\{|z|<1\}}(y)\,(y\cdot(\partial_x - \mathrm{i}A(x))u)\,(x)\right].$$

Repeating the arguments in [17] with $\Gamma^A(x,x+y)$ replacing $A((x+y)/2)$ one proves the following results similar to those in [17].

PROPOSITION **3.1** *Considered as unbounded operator in $L^2(\mathbb{R}^d)$, H_A is essential self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Its closure, also denoted by H_A, is a positive operator.*

PROPOSITION **3.2** *For any $u \in L^2(\mathbb{R}^d)$ such that $H_A u \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$*

$$\Re\left[(\mathrm{sign}u)(H_A u)\right] \;\geq\; H_0|u|.$$

Using the method in [49] we can prove the following result.

PROPOSITION **3.3** *For any $u \in L^2(\mathbb{R}^d)$ we have:*

1. *for any $\lambda > 0$ and for any $r > 0$*

$$\left|(H_A+\lambda)^{-r}\,u\right| \;\leq\; (H_0+\lambda)^{-r}\,|u|; \quad (3.4)$$

2. *for any $t \geq 0$*

$$\left|\mathrm{e}^{-tH_A}u\right| \;\leq\; \mathrm{e}^{-tH_0}|u|. \quad (3.5)$$

We associate to H_A its sesquilinear form

$$\mathcal{D}(\mathfrak{h}_A) = \mathcal{D}(H_A^{1/2}),$$

$$\mathfrak{h}_A(u,v) := (H_A^{1/2}u, H_A^{1/2}v), \quad \forall (u,v) \in \mathcal{D}(\mathfrak{h}_A)^2. \tag{3.6}$$

Consider now a function $V \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$, $V \geq 0$ and associate to it the sesquilinear form

$$\mathcal{D}(\mathfrak{q}_V) := \{u \in L^2(\mathbb{R}^d) \mid \sqrt{V}u \in L^2(\mathbb{R}^d)\},$$

$$\mathfrak{q}_V(u,v) := \int_{\mathbb{R}^d} \mathrm{d}x\, V(x)u(x)\overline{v(x)}, \quad \forall (u,v) \in \mathcal{D}(\mathfrak{q}_V)^2. \tag{3.7}$$

Both these sesquilinear forms are symmetric, closed and positive. We shall abbreviate $\mathfrak{h}_A(u) \equiv \mathfrak{h}_A(u,u)$ and $\mathfrak{q}_V(u) \equiv \mathfrak{q}_V(u,u)$.

PROPOSITION **3.4** *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that can be decomposed as $V = V_+ - V_-$ with $V_\pm \geq 0$ and $V_\pm \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$. Moreover let us suppose that the sesquilinear form $\mathfrak{q}_{V_-}$ is small with respect to $\mathfrak{h}_0$ (i.e. it is $\mathfrak{h}_0$-relatively bounded with bound strictly less then 1). Then the sesquilinear form $\mathfrak{h}_A + \mathfrak{q}_{V_+} - \mathfrak{q}_{V_-}$, that is well defined on $\mathcal{D}(\mathfrak{h}_A) \bigcap \mathcal{D}(\mathfrak{q}_{V_+})$, is symmetric, closed and bounded from below, defining thus an inferior semibounded self-adjoint operator $H(A;V) \equiv H := H_A \dot{+} V$ (sum in sense of forms).*

Proof. The sesquilinear form $\mathfrak{h}_A + \mathfrak{q}_{V_+}$ (defined on the intersection of the form domains) is clearly positive, symmetric and closed. We shall prove now that the sesquilinear form $\mathfrak{q}_{V_-}$ is $\mathfrak{h}_A + \mathfrak{q}_{V_+}$-bounded with bound strictly less then 1, so that the conclusion of the proposition follows by standard arguments.

Let us denote by $H_+ := H_A \dot{+} V_+$ the unique positive self-adjoint operator associated to the sesquilinear form $\mathfrak{h}_A + \mathfrak{q}_{V_+}$ by the representation theorem 2.6 in §VI.2 of [29]. As $V_+ \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$, we have $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathfrak{h}_A) \bigcap \mathcal{D}(\mathfrak{q}_{V_+})$ and thus we can use the form version of the Kato-Trotter formula from [30]:

$$\mathrm{e}^{-tH_+} \;=\; s-\lim_{n\to\infty} \left(\mathrm{e}^{-(t/n)H_A}\, \mathrm{e}^{-(t/n)V_+}\right)^n, \qquad \forall t \geq 0. \tag{3.8}$$

Let us recall the formula ($r > 0$ and $\lambda > 0$)

$$(H_+ + \lambda)^{-r} \;=\; \Gamma(r)^{-1} \int_0^\infty \mathrm{d}t\; t^{r-1}\, \mathrm{e}^{-t\lambda}\, \mathrm{e}^{-tH_+}. \tag{3.9}$$

Combining the above two equalities we obtain

$$\left|(H_+ + \lambda)^{-r} f\right| \;\leq\; \Gamma(r)^{-1} \int_0^\infty \mathrm{d}t\; t^{r-1}\, \mathrm{e}^{-t\lambda}\; \left|\mathrm{e}^{-tH_+} f\right| \;= \tag{3.10}$$

$$= \;\Gamma(r)^{-1} \int_0^\infty \mathrm{d}t\; t^{r-1} \left| s - \lim_{n\to\infty} \left(\mathrm{e}^{-(t/n)H_A}\, \mathrm{e}^{-(t/n)V_+}\right)^n f \right| \;\leq$$

$$\leq\; (H_0 + \lambda)^{-r} |f|,$$

by using the second point of Proposition 3.3.

Taking $u = (H_0 + \lambda)^{-1/2} g$ with $g \in L^2(\mathbb{R}^d)$ arbitrary and $\lambda > 0$ large enough and using the hypothesis on V_- we deduce that there exists $a \in [0,1)$, $b \geq 0$ and $a' \in [0,1)$ such that

$$\mathfrak{q}_{V_-}(u) \leq a\|H_0^{1/2} u\|^2 + b\|u\|^2 = a\|H_0^{1/2}(H_0+\lambda)^{-1/2} g\|^2 + b\|(H_0+\lambda)^{-1/2} g\|^2 \leq$$

$$\leq (a + b/\lambda)\|g\|^2 \leq a'\|g\|^2. \tag{3.11}$$

For any $v \in \mathcal{D}(\mathfrak{h}_A) \bigcap \mathcal{D}(\mathfrak{q}_{V_+})$ let $f := (H_+ + \lambda)^{1/2} v$ and $g := |f|$. Using now (3.10) with $r = 1/2$, (3.11) and the explicit form of $\mathfrak{q}_{V_-}$ we conclude that

$$\mathfrak{q}_{V_-}(v) = \mathfrak{q}_{V_-}\left((H_+ + \lambda)^{-1/2} f\right) \leq \mathfrak{q}_{V_-}\left((H_0 + \lambda)^{-1/2} g\right) \leq \tag{3.12}$$

$$\leq a'\|g\|^2 \;=\; a' \left\|(H_+ + \lambda)^{1/2} v\right\|^2 \;=\; a' \left[\mathfrak{h}_A(v) + \mathfrak{q}_+(v) + \lambda\|v\|^2\right].$$

□

DEFINITION **3.1** *For a potential function V satisfying the hypothesis of Proposition 3.4, we call the operator $H = H(A;V)$ introduced in the same proposition the relativistic Hamiltonian with potential V and magnetic vector potential A.*

The spectral properties of H only depend on the magnetic field B, different choices of a gauge giving unitarily equivalent Hamiltonians, due to the gauge covariance of our quantization procedure.

PROPOSITION **3.5** *Let B be a magnetic field with $\mathcal{C}^\infty_{\mathsf{pol}}(\mathbb{R}^d)$ components and A a vector potential for B also having $\mathcal{C}^\infty_{\mathsf{pol}}(\mathbb{R}^d)$ components. Assume that $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function that can be decomposed as $V = V_+ - V_-$ with $V_\pm \geq 0$, $V_+ \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$ and $V_- \in L^p(\mathbb{R}^d)$ with $p \geq d$. Then*

1. $\mathfrak{q}_{V_-}$ *is a $\mathfrak{h}_0$-bounded sesquilinear form with relative bound 0;*

2. *the Hamiltonian H defined in Definition* 3.1 *is bounded from below and we have* $\sigma_{\mathsf{ess}}(H) = \sigma_{\mathsf{ess}}(H_A \dot{+} V_+) \subset [0, \infty)$.

Proof. 1. Using Observation 3 in §2.8.1 from [37], we conclude that for $d > 1$, the Sobolev space $\mathcal{H}^{1/2}(\mathbb{R}^d)$ (that is the domain of the sesquilinear form $\mathfrak{h}_0$) is continuously embedded in $L^r(\mathbb{R}^d)$ for $2 \leq r \leq 2d/(d-1) < \infty$. Also using Hölder inequality, we deduce that for $r = 2p/(p-1) \in [2, 2d/(d-1)]$, for $p \geq d$

$$\|V_-^{1/2} u\|_2^2 \leq \|V_-\|_p \|u\|_r^2 \leq c\|V_-\|_p \|u\|_{\mathcal{H}^{1/2}(\mathbb{R}^d)}^2, \tag{3.13}$$

$\forall u \in \mathcal{H}^{1/2}(\mathbb{R}^d) = \mathcal{D}(\mathfrak{h}_0)$. Thus $V_-^{1/2} \in \mathbb{B}(\mathcal{H}^{1/2}(\mathbb{R}^d); L^2(\mathbb{R}^d))$; now let us prove that it is even compact. Let us observe that for $d \leq p < \infty$, $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. Thus, for $d \leq p < \infty$ let $\{W_\epsilon\}_{\epsilon>0} \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ be an approximating family for $V_-^{1/2}$ in $L^{2p}(\mathbb{R}^d)$, i.e. $\|V_-^{1/2} - W_\epsilon\|_{2p} \leq \epsilon$. Moreover, for any sequence $\{u_j\} \subset \mathcal{H}^{1/2}(\mathbb{R}^d)$ contained in the unit ball (i.e. $\|u_j\|_{\mathcal{H}^{1/2}} \leq 1$) we may suppose that it converges to $u \in \mathcal{H}^{1/2}(\mathbb{R}^d)$ for the weak topology on $\mathcal{H}^{1/2}(\mathbb{R}^d)$ and thus $\|u\|_{\mathcal{H}^{1/2}} \leq 1$. It follows that $W_\epsilon u_j$ converges to $W_\epsilon u$ in $L^2(\mathbb{R}^d)$ and due to (3.13) we have:

$$\|(V_-^{1/2} - W_\epsilon)(u - u_j)\| \leq C^{1/2} \|V_-^{1/2} - W_\epsilon\|_{L^{2p}} \|u - u_j\|_{\mathcal{H}^{1/2}} \leq 2c^{1/2}\epsilon, \quad \forall j \geq 1.$$

We conclude that $V_-^{1/2} u_j$ converges in $L^2(\mathbb{R}^d)$ to $V_-^{1/2} u$ and using the duality we also get that V_- is a compact operator from $\mathcal{H}^{1/2}(\mathbb{R}^d)$ to $\mathcal{H}^{-1/2}(\mathbb{R}^d)$. Using exercise 39 in ch. XIII of [47] we deduce that $\mathfrak{q}_-$ has zero relative bound with respect to $\mathfrak{h}_0$.

2. The conclusion of point 1 implies that the operator $V_-^{1/2}(H_0 + 1)^{-1/2} \in \mathbb{B}[L^2(\mathbb{R}^d)]$ is compact. Using the first point of Proposition 3.3 with $\lambda = -1$ and $r = 1/2$, and Pitt Theorem in [45], we conclude that the operator $V_-^{1/2}(H_A \dot{+} V_+ + 1)^{-1/2} \in \mathbb{B}[L^2(\mathbb{R}^d)]$ is also compact. Thus $V_- : \mathcal{D}(\mathfrak{h}_A + \mathfrak{q}_{V_+}) \to \mathcal{D}(\mathfrak{h}_A + \mathfrak{q}_{V_+})$ is compact and the conclusion (2) follows from exercise 39 in ch. XIII of [47]. □

4. The Feynman-Kac-Itô formula

In this section we gather some probabilistic notions and results needed in the proof of Theorem 1.1. The main idea is that we obtain a Feynman-Kac-Itô formula (following [20]) for the semigroup defined by $H(A, V)$ and this allows us to reduce the problem to the case $B = 0$. For this last one we repeat then the proof in [5] giving all the necessary details for the case of singular potentials V; here an essential point is an explicit formula for the integral kernel of the operator $\mathrm{e}^{-tH(0,V)}$ in terms of a Lévy process.

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a probability space, i.e. $\mathfrak{F}$ is a σ-algebra of subsets of Ω and P is a non-negative σ-aditive function on $\mathfrak{F}$ with $\mathsf{P}(\Omega) = 1$. For any integrable random variable $X : \Omega \to \mathbb{R}$ we denote its expectation value by

$$\mathsf{E}(X) := \int_\Omega X(\omega)\mathsf{P}(d\omega). \tag{4.1}$$

For any sub-σ-algebra $\mathfrak{G} \subset \mathfrak{F}$ we denote its associated conditional expectation by $\mathsf{E}(X \mid \mathfrak{G})$; this is the unique $\mathfrak{G}$-measurable random variable $Y : \Omega \to \mathbb{R}$ satisfying

$$\int_B Y(\omega)\mathsf{P}(\mathrm{d}\omega) = \int_B X(\omega)\mathsf{P}(\mathrm{d}\omega), \qquad \forall B \in \mathfrak{G}. \tag{4.2}$$

Let us recall the following properties of the conditional expectation (see for example [26]):

$$\mathsf{E}\left(\mathsf{E}(X \mid \mathfrak{G})\right) = \mathsf{E}(X), \tag{4.3}$$

$$\mathsf{E}(XZ \mid \mathfrak{G}) = Z\mathsf{E}(X \mid \mathfrak{G}), \tag{4.4}$$

for any $\mathfrak{G}$-measurable random variable $Z : \Omega \to \mathbb{R}$, such that ZX is integrable.

We also recall the Jensen inequality ([48], [26]): for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$, and for any lower bounded random variable $X : \Omega \to \mathbb{R}$ the following inequality is valid

$$\varphi(\mathsf{E}(X)) \leq \mathsf{E}(\varphi(X)). \tag{4.5}$$

Following [6], we can associate to our Feller semigroup $\{P(t)\}_{t\geq 0}$, defined in Section 2, a Markov process $\{(\Omega, \mathfrak{F}, \mathsf{P}_x), \{X_t\}_{t\geq 0}, \{\theta_t\}_{t\geq 0}\}$; that we briefly recall here:

- Ω is the set of *"cadlag"* functions on $[0,\infty)$, i.e. functions $\omega : [0,\infty) \to \mathbb{R}^d$ (paths) that are continuous to the right and have a limit to the left in any point of $[0,\infty)$.

- $\mathfrak{F}$ is the smallest σ-algebra for which the *coordinate functions* $\{X_t\}_{t\geq 0}$, with $X_t(\omega) := \omega(t)$, are measurable.

- P_x is a probability on Ω such that for any $n \in \mathbb{N}^*$, for any ordered set $\{0 < t_1 \leq \ldots \leq t_n\}$ and any family $\{B_1, \ldots, B_n\}$ of Borel subsets in $\mathbb{R}^d$, we have

$$\mathsf{P}_x \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\} \ = \tag{4.6}$$

$$= \int_{B_1} \mathrm{d}x_1 \, \overset{\circ}{\wp}_{t_1}(x - x_1) \int_{B_2} \mathrm{d}x_2 \, \overset{\circ}{\wp}_{t_2 - t_1}(x_1 - x_2) \ldots \int_{B_n} \mathrm{d}x_n \, \overset{\circ}{\wp}_{t_n - t_{n-1}}(x_{n-1} - x_n).$$

  One can deduce that, if E_x denotes the expectation value with respect to P_x, then for any $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ and for any $t \geq 0$ one has

$$\mathsf{E}_x(f \circ X_t) \ = \ [P(t)f](x). \tag{4.7}$$

  We also remark that P_x is the image of the probability $\mathsf{P}_0 \equiv \mathsf{P}$ under the map $S_x : \Omega \to \Omega$ defined by $[S_x\omega](t) := x + \omega(t)$.

- For any $t \geq 0$, the map $\theta_t : \Omega \to \Omega$ is defined by $[\theta_t\omega](s) := \omega(s + t)$. If we denote by $\mathfrak{F}_t$ the sub-σ-algebra of $\mathfrak{F}$ generated by the processes $\{X_s\}_{0\leq s\leq t}$, then for any $t \geq 0$ and any bounded random variable $Y : \Omega \to \mathbb{R}$

$$\mathsf{E}_x(Y \circ \theta_t \mid \mathfrak{F}_t)(\omega) \ = \ \mathsf{E}_{X_t(\omega)}(Y), \quad \mathsf{P}_x - a.e. \text{ on } \Omega. \tag{4.8}$$

We use the fact that (see [25], [20]) the probability P_x is concentrated on the set of paths X_t such that $X_0 = x$ and by the Lévy-Ito Theorem:

$$X_t = x + \int_0^{t_+} \int_{\mathbb{R}^d} y \, \tilde{N}_X(\mathrm{d}s \, \mathrm{d}y). \tag{4.9}$$

Here $\tilde{N}_X(\mathrm{d}s\,\mathrm{d}y) := N_X(\mathrm{d}s\,\mathrm{d}y) - \hat{N}_X(\mathrm{d}s\,\mathrm{d}y)$, $\hat{N}_X(\mathrm{d}s\,\mathrm{d}y) := \mathsf{E}_x(N_X(\mathrm{d}s\,\mathrm{d}y)) = \mathrm{d}s\,\mathsf{n}(\mathrm{d}y)$ with $\mathsf{n}(\mathrm{d}y)$ the Lévy measure appearing in (2.1) and N_X a 'counting measure' on $[0, \infty) \times \mathbb{R}^d$ that for $0 < t < t'$ and B a Borel subset of $\mathbb{R}^d$ is defined as $N_X((t, t'] \times B) \ :=$

$$:= \# \left\{ s \in (t, t'] \ \mid \ X_s \neq X_{s-}, \, X_s X_{s-} \in B \right\}. \tag{4.10}$$

Following the procedure developped in [20] by Ichinose and Tamura one obtains a Feynman-Kac-Itô formula for Hamiltonians of the type $H = H_A \dot{+} V$. In fact we have

PROPOSITION **4.1** *Under the same conditions as in Definition* 3.1, *for any function* $u \in L^2(\mathbb{R}^d)$ *we have*

$$\left(\mathrm{e}^{-tH}u\right)(x) \;=\; \mathsf{E}_x\left((u\circ X_t)\,\mathrm{e}^{-S(t,X)}\right), \quad t\geq 0, x\in\mathbb{R}^d \tag{4.11}$$

where

$$\begin{aligned} S(t,X) \;:=&\; \mathrm{i}\int_0^{t_+}\int_{\mathbb{R}^d}\tilde{N}_X(\mathrm{d}s\,\mathrm{d}y)\left\langle\int_0^1 \mathrm{d}r\,\left(A(X_{s_-}+ry)\right), y\right\rangle + \\ &+\mathrm{i}\int_0^t\int_{\mathbb{R}^d}\hat{N}_X(\mathrm{d}s\,\mathrm{d}y)\left\langle\left(\int_0^1 \mathrm{d}r\,A(X_s+ry)-A(X_s)\right), y\right\rangle + \\ &+\int_0^t \mathrm{d}s\,V(X_s). \end{aligned} \tag{4.12}$$

In the sequel we shall take $A=0$ and $V\in C_0^\infty(\mathbb{R}^d)$. As it is proved in [6], the operator $\mathrm{e}^{-t(H_0\dot{+}V)}$ has an integral kernel that can be described in the following way. Let us denote by $\mathfrak{F}_{t-}$ the sub-σ-algebra of $\mathfrak{F}$ generated by the random variables $\{X_s\}_{0\leq s<t}$. For any pair $(x,y)\in[\mathbb{R}^d]^2$ and any $t>0$ we define a measure $\mu_{0,x}^{t,y}$ on the Borel space $(\Omega,\mathfrak{F}_{t-})$ by the equality

$$\mu_{0,x}^{t,y}(M) \;:=\; \mathsf{E}_x\left[\chi_M\,\overset{\circ}{\wp}_{t-s}(X_s-y)\right], \tag{4.13}$$

for any $M\in\mathfrak{F}_s$ and $0\leq s<t$, where χ_M is the characteristic function of M. This measure is concentrated on the family of 'paths' $\{\omega\in\Omega \mid X_0(\omega)= x, X_{t-}(\omega)=y\}$ and we have $\mu_{0,x}^{t,y}(\Omega)=\overset{\circ}{\wp}_t(x-y)$.

PROPOSITION **4.2** *Let* $F:\Omega\to\mathbb{R}$ *be a non-negative* $\mathfrak{F}_{t-}$*-measurable random variable and let* $f:\mathbb{R}^d\to\mathbb{R}$ *be a positive borelian function. Then the following equality holds for any* $t>0$ *and any* $x\in\mathbb{R}^d$*:*

$$\begin{aligned} \int_{\mathbb{R}^d}\mathrm{d}y\left\{\int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega)\,F(\omega)\,\mathrm{e}^{-\int_0^t \mathrm{d}s\,V(X_s)}\right\}f(y) \;=& \\ =\; \mathsf{E}_x\left(F\,\mathrm{e}^{-\int_0^t \mathrm{d}s\,V(X_s)}\,f(X_t)\right). \end{aligned} \tag{4.14}$$

Proof. This is a direct consequence of relations (2.29) and (2.33) from [6]. □

Let us now take $A=0$ in Proposition 4.1 and $F=1$ in Proposition 4.2 in order to deduce that the operator $\mathrm{e}^{-t(H_0\dot{+}V)}$ is an integral operator with integral kernel given by the function

$$\wp_t(x,y) \;:=\; \int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega)\,\mathrm{e}^{-\int_0^t \mathrm{d}s\,V(X_s)}, \quad t>0,\ (x,y)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{4.15}$$

Proposition 3.3 from [6] implies that the function $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \ni (t,x,y) \mapsto \wp_t(x,y) \in \mathbb{R}$ is non-negative, continuous and verifies $\wp_t(x,y) = \wp_t(y,x)$. We shall also need the following result.

Proposition **4.3** *For any $t>0$, any $x\in\mathbb{R}^d$ and any function $g:\Omega\to\mathbb{R}$ that is integrable with respect to the measure $\mu^{t,x}_{0,x}$ we have the equality:*

$$\int_\Omega \mu^{t,x}_{0,x}(\mathrm{d}\omega)\, g(\omega) \;=\; \int_\Omega \mu^{t,0}_{0,0}(\mathrm{d}\omega)\, g(x+\omega). \tag{4.16}$$

Proof. It is evidently sufficient to prove that for any $s\in[0,t)$ and any $M\in\mathfrak{F}_s$ we have

$$\mu^{t,x}_{0,x}(M) \;=\; \left(\mu^{t,0}_{0,0}\circ S_x^{-1}\right)(M)$$

where the map $S_x:\Omega\to\Omega$ is defined by $(S_x(\omega)(t) := x+\omega(t)$. We noticed previously the identity $\mathsf{P}_x = \mathsf{P}_0\circ S_x^{-1}$; thus for any function $F:\Omega\to\mathbb{R}$ integrable with respect to P_x we have $\mathsf{E}_x(F) = \mathsf{E}_0(F\circ S_x)$. We remark that $X_s(\omega+x) = \omega(s)+x = X_s(\omega)+x$, and using the definition of the measure $\mu^{t,x}_{0,x}$ in (4.13), we obtain

$$\mu^{t,x}_{0,x}(M) = \mathsf{E}_x\left[\chi_M\, \mathring{\wp}_{t-s}(X_s - x)\right] = \mathsf{E}_0\left[(\chi_M\circ S_x)\, \mathring{\wp}_{t-s}(X_s)\right] = \tag{4.17}$$

$$= \mathsf{E}_0\left[(\chi_{S_x^{-1}(M)}\, \mathring{\wp}_{t-s}(X_s)\right] = \mu^{t,0}_{0,0}\left(S_x^{-1}(M)\right) = \left[\mu^{t,0}_{0,0}\circ S_x^{-1}\right](M).$$

□

5. Proof of the bound for $N(0;V)$

In this Section we will consider $A=0$ and we shall work only with a potential $V = V_+ - V_-$ satisfying the properties:

- $V_\pm \geq 0$,
- $V_+ \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$,
- $V_- \in L^d(\mathbb{R}^d)\cap L^{d/2}(\mathbb{R}^d)$.

We shall use the notations $H := H_0 \dot{+} V$, $H_+ := H_0 \dot{+} V_+$, $H_- := H_0 \dot{+} (-V_-)$ for the operators associated to the sesquilinear forms $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{q}_V$, $\mathfrak{h}_+ = \mathfrak{h}_0 + \mathfrak{q}_{V_+}$, $\mathfrak{h}_- = \mathfrak{h}_0 - \mathfrak{q}_{V_-}$.

Due to the results of Proposition 3.5 we have $\sigma_{\mathsf{ess}}(H) = \sigma_{\mathsf{ess}}(H_+) \subset \sigma(H_+) \subset [0, \infty)$ and $\sigma_{\mathsf{ess}}(H_-) = \sigma_{\mathsf{ess}}(H_0) = \sigma(H_0) = [0, \infty)$.

For any potential function W verifying the same conditions as V above, we denote by $N(W)$ the number of strictly negative eigenvalues (counted with their multiplicity) of the operator $H_0 \dot{+} W$. The following result reduces our study to the case $V_+ = 0$.

LEMMA **5.1** *The following inequality is true:*

$$N(V) \;\le\; N(-V_-).$$

In particular we have that $N(V) = \infty$ implies that $N(-V_-) = \infty$.

Proof. We apply the *Min-Max* principle (see Theorem XIII.2 in [47]) noticing that $\mathcal{D}(\mathfrak{h}_-) = \mathcal{D}(\mathfrak{h}_\circ) \supset \mathcal{D}(\mathfrak{h})$ and $\mathfrak{h}_- \le \mathfrak{h}$ and we deduce that the operator H_- has at least $N(V)$ strictly negative eigenvalues. □

Thus we shall suppose from now on that $V_+ = 0$.

5.1. Reduction to smooth, compactly supported potentials

In this subsection we shall prove that we can suppose $V_- \in C_0^\infty(\mathbb{R}^d)$. This will be done by approximation, using a result of the type of Theorem 4.1 from [50].

LEMMA **5.2** *Let V and V_n $(n \ge 1)$ functions as in Proposition* 3.4. *In addition, $V_+ = V_{n,+} = 0$ for all $n \ge 1$ and $\lim_{n\to\infty} V_{n,-} = V_-$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_{n,-}$ are uniformly $\mathfrak{h}_0$-bounded with relative bound < 1. We set $H_n := H_A \dot{+} V_n$. Then $H_n \to H$ when $n \to \infty$ in strong resolvent sense.*

Proof. We denote by $\mathfrak{h}_n$ the quadratic form associated to H_n, i.e. $\mathfrak{h}_n = \mathfrak{h}_A - \mathfrak{q}_{n,-}$, where $\mathfrak{q}_{n,-}$ is associated to $V_{n,-}$ by (3.7). We have $D(h_n) = D(h_A) \subset D(q_{n,-})$, and according to Proposition 3.4 there exist $\alpha \in (0, 1)$ and $\beta > 0$ such that

$$\mathfrak{q}_{n,-}(v) \le \alpha \mathfrak{h}_A(v) + \beta \parallel v \parallel, \quad \forall v \in D(\mathfrak{h}_A),\ \forall n \ge 1. \tag{5.1}$$

It follows that $\mathfrak{h}_n$ are uniformly lower bounded and the norms defined on $D(\mathfrak{h}_A)$ by $\mathfrak{h}_A$ and $\mathfrak{h}_n$ are equivalent, uniformly with respect to $n \ge 1$. Moreover, $C_0^\infty(\mathbb{R}^d)$ is a core for H_A, thus for $\mathfrak{h}_A$, $\mathfrak{h}$ and $\mathfrak{h}_n$ also.

Let $f \in L^2(\mathbb{R}^d)$ and $u_n := (H_n + i)^{-1} f \in D(H_n) \subset D(\mathfrak{h}_A)$, $n \geq 1$. We have clearly

$$\| u_n \| \leq \| f \|, \quad |\mathfrak{h}_n(u_n)| = |(H_n u_n, u_n)| \leq \| f \|, \quad \forall n \geq 1. \tag{5.2}$$

From (5.1), the subsequent comments and (5.2) it follows that the sequence $(u_n)_{n\geq 1}$ is bounded in $D(\mathfrak{h}_A)$, while the sequence $\left(V_{n,-}^{1/2} u_n\right)_{n\geq 1}$ is bounded in $L^2(\mathbb{R}^d)$. Let $u \in L^2(\mathbb{R}^d)$ be a limit point of the sequence $(u_n)_{n\geq 1}$ with respect to the weak topology on $L^2(\mathbb{R}^d)$. By restricting maybe to a subsequence, we may assume that there exist $\psi, \eta \in L^2(\mathbb{R}^d)$ such that $H_A^{1/2} u_n \underset{n\to\infty}{\to} \psi$ and $V_{n,-}^{1/2} u_n \underset{n\to\infty}{\to} \eta$ in the weak topology of $L^2(\mathbb{R}^d)$. For $g \in D\left(H_A^{1/2}\right)$ we have

$$\left(H_A^{1/2} g, u\right) = \lim_{n\to\infty} \left(H_A^{1/2} g, u_n\right) = \lim_{n\to\infty} \left(g, H_A^{1/2} u_n\right) = (g, \psi),$$

thus $u \in D(H_A^{1/2})$ and $H_A^{1/2} u = \psi$. Then $u \in D(\mathfrak{q}_-)$ and for any $g \in C_0^\infty(\mathbb{R}^d)$

$$(\eta, g) = \lim_{n\to\infty} \left(V_{n,-}^{1/2} u_n, g\right) = \lim_{n\to\infty} \left(u_n, V_{n,-}^{1/2} g\right) = \left(u, V_-^{1/2} g\right) = \left(V_-^{1/2} u, g\right),$$

implying $V_-^{1/2} u = \eta$.

It follows that for every $g \in C_0^\infty(\mathbb{R}^d)$ we have

$$(g, f) = (g, (H_n + \mathrm{i}) u_n) = \mathfrak{h}_n(g, u_n) - \mathrm{i}(g, u_n) =$$

$$= \left(H_A^{1/2} g, H_A^{1/2} u_n\right) - \left(V_{n,-}^{1/2} g, V_{n,-}^{1/2} u_n\right) - \mathrm{i}(g, u_n) \to \mathfrak{h}(g, u) - \mathrm{i}(g, u).$$

Consequently, $u \in D(H)$ and $(H + \mathrm{i})u = f$. Thus the sequence $(u_n)_{n\geq 1}$ has the single limit point $u = (H + \mathrm{i})^{-1} f$ for the weak topology of $L^2(\mathbb{R}^d)$. It follows that $(H_n \pm \mathrm{i})^{-1} f \to (H \pm \mathrm{i})^{-1} f$ weakly in $L^2(\mathbb{R}^d)$ for $n \to \infty$.

By the resolvent identity we get

$$\| (H_n + \mathrm{i})^{-1} f \|^2 = \frac{\mathrm{i}}{2} \left((f, (H_n - \mathrm{i})^{-1} f) - (f, (H_n + \mathrm{i})^{-1} f)\right) \to \| (H + \mathrm{i})^{-1} f \|^2,$$

therefore $(H_n + \mathrm{i})^{-1} f \to (H + \mathrm{i})^{-1} f$ in $L^2(\mathbb{R}^d)$. □

A direct consequence of Lemma 5.2 and Theorem VIII.20 from [47] is

COROLLARY **5.1** *Under the hypothesis of Lemma* 5.2*, for any function f bounded and continuous on $\mathbb{R}$ and any $u \in L^2(\mathbb{R}^d)$, we have $f(H_n) u \to f(H) u$.*

Approximating V_- is done by the standard procedures: cutoffs and regularization. The first of the lemmas below is obvious.

LEMMA **5.3** *Let $V_- \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $V_- \geq 0$ and assume that its associated sesquilinear form is $\mathfrak{h}_0$-bounded with relative bound strictly less then 1. Let $\theta \in C_0^\infty([0,\infty))$ satisfy the following: $0 \leq \theta \leq 1$, θ is a decreasing function, $\theta(t) = 1$ for $t \in [0,1]$ and $\theta(t) = 0$ for $t \in [2,\infty)$.*

If we denote by $\theta^n(x) := \theta(|x|/n)$ and $V_-^n = \theta^n V_-$, then $V_-^n \to V_-$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $0 \leq V_-^n \leq V_-^{n+1}$ and the sesquilinear forms associated to V_-^n are $\mathfrak{h}_0$-bounded with relative bound strictly less then 1, uniformly in $n \in \mathbb{N}^*$.

Moreover, if we denote by $\mathfrak{h}^n$ the sesquilinear form associated to the operator $H_A \dot{+} (-V_-^n)$, we have $\mathfrak{h}^{(n)} \geq \mathfrak{h}^{(n+1)} \geq \mathfrak{h}$ and $\mathfrak{h}^{(n)}(u) \underset{n\to\infty}{\to} \mathfrak{h}(u)$ for any $u \in \mathcal{D}(\mathfrak{h}_A)$.

If, in addition, $V_- \in L^p(\mathbb{R}^d)$, $p \geq 1$, then $V_-^n \in L^p_{\mathrm{comp}}(\mathbb{R}^d)$, $\|V_-^n\|_{L^p} \leq \|V_-\|_{L^p}$ for any $n \geq 1$, and $V_-^n \to V_-$ in $L^p(\mathbb{R}^d)$.

LEMMA **5.4** (a) *Let $V_- \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $V_- \geq 0$ and $\mathfrak{h}_0$-bounded with relative bound < 1. Let $\theta \in C_0^\infty(\mathbb{R}^d)$, $\theta \geq 0$ and $\int_{\mathbb{R}^d} \theta = 1$. We set $\theta_n(x) := n^d\theta(nx)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}^*$ and $V_{n,-} := V_- * \theta_n \in C_0^\infty$. In particular, $V_{n,-} \in C_0^\infty(\mathbb{R}^d)$ if $V_- \in L^1_{\mathrm{comp}}(\mathbb{R}^d)$.*

Then $V_{n,-} \to V_-$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $n \to \infty$ and the functions $V_{n,-}$ are non-negative and uniformly h_0-bounded, with relative bound < 1. Moreover, $\mathfrak{h}_n(u) \to \mathfrak{h}(u)$ for any $u \in D(\mathfrak{h}_A)$, where $\mathfrak{h}_n$ is the quadratic form associated to $H_n := H_A \dot{+} (-V_n)$.

(b) *If, in addition, $V_- \in L^p(\mathbb{R}^d)$ with $p \geq 1$, then $V_{n,-} \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, $\| V_{n,-} \|_{L^p} \leq \| V_- \|_{L^p}$, $\forall n \geq 1$ and $V_{n,-} \to V_-$ in $L^p(\mathbb{R}^d)$.*

Proof. (a) We have for any $x \in \mathbb{R}^d$

$$V_{n,-}(x) = \int_{\mathbb{R}^d} \mathrm{d}y\, \theta_n(y) V_-(x-y) = \int_{\mathbb{R}^d} \mathrm{d}y\, \theta(y) V_-(x - n^{-1}y). \tag{5.3}$$

By the Dominated Convergence Theorem, for any compact $K \subset \mathbb{R}^d$

$$\int_K \mathrm{d}x\, |V_{n,-}(x) - V_-(x)| \leq \int_{\mathbb{R}^d} \mathrm{d}y\, \theta(y) \int_K \mathrm{d}x\, |V_-(x - n^{-1}y) - V_-(x)| \to 0,$$

hence $V_{n,-}$ converges to V_- in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ when $n \to \infty$.

If V_- is relatively small with respect to $\mathfrak{h}_0$, we use the fact that $H_0^{1/2}$ is a convolution operator (hence it commutes with translations) and using the comments after inequality (5.1), we deduce that for any $u \in C_0^\infty(\mathbb{R}^d)$ there exists $\alpha \in (0,1)$ and $\beta \geq 0$ such that

$$\int_{\mathbb{R}^d} \mathrm{d}x\, V_{n,-}|u|^2 = \int_{\mathbb{R}^d} \mathrm{d}y\, \theta_n(y) \int_{\mathbb{R}^d} \mathrm{d}z\, V_-(z)|u(z+y)|^2 \leq$$

$$\leq \int_{\mathbb{R}^d} \mathrm{d}y\, \theta_n(y) \left[\alpha \parallel H_0^{1/2}u(\cdot+y) \parallel^2 + \beta \parallel u(\cdot+y) \parallel^2\right] =$$

$$= \alpha \parallel H_0^{1/2}u \parallel^2 + \beta \parallel u \parallel^2 .$$

(b) From (5.3) it follows that

$$\parallel V_{n,-} \parallel_{L^p} \leq \int_{\mathbb{R}^d} \mathrm{d}y\, \theta_n(y) \parallel V_-(\cdot - y) \parallel_{L^p} \leq \parallel V_- \parallel_{L^p} .$$

Also, using the Dominated Convergence Theorem, we infer that

$$\parallel V_{n,-} - V_- \parallel_{L^p} \leq \int_{\mathbb{R}^d} \mathrm{d}y\, \theta(y) \parallel V_-(\cdot) - V_-(\cdot - n^{-1}y) \parallel_{L^p} \to 0.$$

□

Thus Lemmas 5.3 and 5.4 imply, for a potential function V_- satisfying the hypothesis of the Lemma, the existence of a sequence $(V_{n,-})_{n\geq 1} \subset C_0^\infty(\mathbb{R}^d)$ such that $V_{n,-} \geq 0$, $\parallel V_{n,-} \parallel_{L^p} \leq \parallel V_- \parallel_{L^p}$, $\forall n \geq 1$, $V_{n,-} \to V_-$ in $L^p(\mathbb{R}^d)$ (for $p = d$ and $p = d/2$) when $n \to \infty$ and the functions $V_{n,-}$ are uniformly $\mathfrak{h}_0$-bounded with relative bound < 1.

LEMMA **5.5** *Assume that there exists a constant $C > 0$, such that the inequality*

$$N(-V_{n,-}) \leq C \left(\int_{\mathbb{R}^d} \mathrm{d}x\, |V_{n,-}(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\, |V_{n,-}(x)|^{d/2} \right) \tag{5.4}$$

holds for any $n \geq 1$. Then one also has

$$N(-V_-) \leq C \left(\int_{\mathbb{R}^d} \mathrm{d}x\, |V_-(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\, |V_-(x)|^{d/2} \right). \tag{5.5}$$

Proof. We set $H_{n,-} := H_0 \dot{+} (-V_{n,-})$; $(E_{n,-}(\lambda))_{\lambda\in\mathbb{R}}$ will be the spectral family of $H_{n,-}$ and $(E_-(\lambda))_{\lambda\in\mathbb{R}}$ the spectral family of H_-. For $\lambda < 0$, we denote by $N_\lambda(W)$ the number of eigenvalues of $H_0 \dot{+} W$ which are strictly smaller than λ (for any potential function W satisfying the hypothesis at the begining of this section). It suffices to show that for any $\lambda < 0$ not belonging to the spectrum of H_-, one has the inequality

$$N_\lambda(-V_-) \leq C\left(\int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^{d/2}\right). \tag{5.6}$$

Since $V_{n,-}$ converges to V_- in $L^1_{\text{loc}}(\mathbb{R}^d)$, cf. Lemma 5.2, $H_{n,-}$ will converge to H_- in strong resolvent sense. By [29], Ch. VIII, Th. 1.15, this implies the strong convergence of $E_{n,-}(\lambda)$ to $E_-(\lambda)$ for any $\lambda \notin \sigma(H_-)$. By Lemmas 1.23 and 1.24 from [29], Ch. VII, for $\lambda < 0$ such that $\lambda \notin \sigma(H_-)$, one also has $\| E_{n,-}(\lambda) - E_-(\lambda) \| \to 0$. Let us suppose that there exists some $\lambda < 0$ not belonging to $\sigma(H_-)$ and such that for it the inequality (5.6) is not verified. Thus for the given $\lambda < 0$ we have $\forall n \geq 1$:

$$N(-V_{n,-}) \leq C\left(\int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^{d/2}\right) < N_\lambda(-V_-).$$

But for n large enough, one has $N_\lambda(-V_-) = N_\lambda(-V_{n,-})$ and thus

$$\begin{aligned} N_\lambda(-V_-) = N_\lambda(-V_{n,-}) \;\leq\; N(-V_{n,-}) \leq \\ \leq C\left(\int_{\mathbb{R}^d} \mathrm{d}x\,|V_{n,-}(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\,|V_{n,-}(x)|^{d/2}\right) \leq \\ \leq C\left(\int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^d + \int_{\mathbb{R}^d} \mathrm{d}x\,|V_-(x)|^{d/2}\right) \end{aligned}$$

that is a contradiction with our initial hypothesis. □

5.2. Proof of the Theorem 1.1 without magnetic field

We shall assume from now on that $V_+ = 0$ and $0 \leq V_- \in C_0^\infty(\mathbb{R}^d)$. We check a Birman-Schwinger principle. For $\alpha > 0$ we set $K_\alpha := V_-^{1/2}(H_0 + \alpha)^{-1}V_-^{1/2}$; it is a positive compact operator on $L^2(\mathbb{R}^d)$.

LEMMA **5.6**

$$N_{-\alpha}(-V_-) \leq \#\,\{\mu > 1 \mid \mu \text{ eigenvalue of } K_\alpha\}. \tag{5.7}$$

Proof. We introduce the sequence of functions $\mu_n : [0, \infty) \to (-\infty, 0]$, $n \geq 1$, where $\mu_n(\lambda)$ is the n'th eigenvalue of $H_0 - \lambda V_-$ if this operator has at least n strictly negative eigenvalues and $\mu_n(\lambda) = 0$ if not. Cf. [47], §XIII.3, μ_n is continuous and decreasing (even strictly decreasing on intervals on which it is strictly negative). Obviously, we have $N_{-\alpha}(-V_-) \leq \# \{n \geq 1 \mid \mu_n(1) < -\alpha\}$. Now fix some n such that $\mu_n(1) < -\alpha$ and recall that $\mu_n(0) = 0$. The function μ_n is continuous and injective on the interval $[\epsilon_n, 1]$, where $\epsilon_n := \sup\{\lambda \geq 0 \mid \mu_n(\lambda) = 0\}$, therefore it exists a unique $\lambda \in (0,1)$ such that $\mu_n(\lambda) = -\alpha$. Thus

$$N_{-\alpha}(-V_-) = \# \{\lambda \in (0,1) \mid \exists n \geq 1 \ s.t. \ \mu_n(\lambda) = -\alpha\} =$$

$$= \# \{\lambda \in (0,1) \mid \exists \varphi \in D(H_0) \setminus \{0\} \ s.t. \ (H_0 - \lambda V_-)\varphi = -\alpha\varphi\} \leq$$

$$\leq \# \{\lambda \in (0,1) \mid \exists \psi \in L^2(\mathbb{R}^d) \setminus \{0\} \ s.t. \ K_\alpha \psi = \lambda^{-1}\psi\},$$

where for the last inequality we set $\psi := V_-^{1/2}\varphi$, noticing that the equality $(H_0 + \alpha)\varphi = \lambda V_- \varphi$ implies $\psi \neq 0$. □

Lemma **5.7** *Let $F : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $F(0) = 0$. Then $F(K_\alpha)$ is a positive compact operator and the next inequality holds:*

$$N_{-\alpha}(-V_-) \leq F(1)^{-1} \sum_{F(\mu) \in \sigma[F(K_\alpha)], F(\mu) > F(1)} F(\mu).$$

Proof. The first part is obvious. Using (5.7) and F's monotony, we get

$$N_{-\alpha}(-V_-) \leq \sharp\{\mu > 1 \mid \mu \in \sigma(K_\alpha)\} = \# \{F(\mu) \mid \mu > 1, F(\mu) \in \sigma[F(K_\alpha)]\} =$$

$$= \sum_{\mu > 1, F(\mu) \in \sigma[F(K_\alpha)]} \frac{F(\mu)}{F(\mu)} \leq F(1)^{-1} \sum_{\mu > 1, F(\mu) \in \sigma[F(K_\alpha)]} F(\mu).$$

□

So, we shall be interested in finding functions F having the properties in the statement above, such that $F(K_\alpha) \in B_1$ (the ideal of trace-class operators in $L^2(\mathbb{R}^d)$) and such that $\mathrm{Tr}\,[\mathrm{F}(\mathrm{K}_\alpha)]$ is conveniently estimated.

Using an idea from [48], we are going to consider functions of the form

$$F(t) := t \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s} g(ts), \quad t \geq 0,$$

where $g : [0,\infty) \to [0,\infty)$ is continuous, bounded and $g \not\equiv 0$. Plainly, $F : [0,\infty) \to [0,\infty)$ is continuous, $F(0) = 0$, satisfies $F(t) \leq Ct$ for some $C > 0$ and the identity

$$F(t) = \int_0^\infty \mathrm{d}r \, \mathrm{e}^{-rt^{-1}} g(r)$$

implies that F is strictly increasing. We shall use the notations $F = \Phi(g)$, $\tilde{g}(t) := tg(t)$.

In particular, $g_\lambda(t) = \mathrm{e}^{-\lambda t}$, $\lambda > 0$ leads to $F_\lambda(t) = t(1+\lambda t)^{-1}$. In the sequel, relations valid for this particular case will be extended to the following case, that we shall be interested in:

$$g_\infty : [0,\infty) \to [0,\infty), \quad g_\infty(t) = 0 \text{ if } 0 \le t \le 1, \quad g_\infty(t) = 1 - 1/t \text{ if } t > 1, \tag{5.8}$$

by using an approximation that we now introduce. The first lemma is obvious.

LEMMA **5.8** *Let* g_∞ *be given by* (5.8). *For* $n \ge 1$ *we define* $g_n : [0,\infty) \to [0,1]$, $g_n(t) = g(t)$ *for* $0 \le t \le n$, $g_n(t) = \frac{2n-1}{t} - 1$ *for* $n \le t \le 2n-1$, $g_n(t) = 0$ *for* $t \ge 2n-1$. *Then* $g_n \in C_0((0,\infty))$, $0 \le g_n \le g_{n+1} \le g_\infty$, $\forall n$ *and* $g_n \to g_\infty$ *when* $n \to \infty$ *uniformly on any compact subset of* $[0,\infty)$.

LEMMA **5.9** *Let* f *be a nonnegative continuous function on* $[0,\infty)$ *such that* $\lim_{t\to\infty} f(t) = 0$. *There exists a sequence* $(f^k)_{k\ge1}$ *of real functions on* $[0,\infty)$ *with the properties*

(a) *Every* f^k *is a finite linear combination of functions of the form* g_λ, $\lambda > 0$.

(b) $f^k \ge f^{k+1} \ge f \ge 0$ *on* $[0,\infty)$, $\forall k \ge 1$,

(c) $f^k \to f$ *uniformly on* $[0,\infty)$ *when* $k \to \infty$.

Proof. We define the function $h : [0,1] \to [0,\infty)$, $h(s) := f(-\ln s)$ for $s \in (0,1]$, $h(0) := 0$. It follows that $h \in C([0,1])$. We can chose now two sequences of positive numbers $\{\epsilon_k\}_{k\ge1}$ and $\{\delta_k\}_{k\ge1}$ verifying the properties: $\lim\limits_{k\to\infty}(\epsilon_k + \delta_k) = 0$ and $\delta_k - \epsilon_k \ge \epsilon_{k+1} + \delta_{k+1} > 0, \forall k \ge 1$ (for example we may take $\delta_k = (k+2)^{-1}$ and $\epsilon_k = (k+2)^{-3}$). Using the Weierstrass Theorem we may find for any $k \ge 1$ a real polynomial P_k' such that $\sup\limits_{s\in[0,1]} |h(s) - P_k'(s)| \le \epsilon_k$ and let us denote by $P_k := P_k' + \delta_k$. We get:

$$\sup_{s\in[0,1]} |h(s) - P_k(s)| \le \epsilon_k + \delta_k \underset{k\to\infty}{\to} 0,$$

$$h \le h + \delta_{k+1} - \epsilon_{k+1} \le P'_{k+1} + \delta_{k+1} = P_{k+1} \le h + \delta_{k+1} + \epsilon_{k+1} \le$$
$$\le h + \delta_k - \epsilon_k \le P_k' + \delta_k = P_k$$

on $[0,1]$. Thus $f^k(t) := P_k(\mathrm{e}^{-t})$ defined on $[0,\infty)$ for $k \ge 1$ have the required properties. □

PROPOSITION **5.1** *Let $F_\infty := \Phi(g_\infty)$. The operator $F_\infty(K_\alpha)$ is self-adjoint, positive and compact on $L^2(\mathbb{R}^d)$. It admits an integral kernel of the form*

$$[F_\infty(K_\alpha)](x,y) = \tag{5.9}$$

$$= V_-^{1/2}(x)V_-^{1/2}(y)\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t}\int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega) g_\infty\left(\int_0^t \mathrm{d}s\, V_-(X_s)\right),$$

which is continuous, symmetric, with $[F_\infty(K_\alpha)](x,x) \geq 0$.

Proof. The first part is clear. To establish (3.27), we treat first the operator $B_\lambda := F_\lambda(K_\alpha)$, $\lambda > 0$. We have

$$B_\lambda = K_\alpha(1+\lambda K_\alpha)^{-1} \implies B_\lambda = K_\alpha - \lambda B_\lambda K_\alpha. \tag{5.10}$$

The second resolvent identity gives

$$(H_0+\alpha)^{-1} - (H_0+\lambda V_- + \alpha)^{-1} = \lambda(H_0+\lambda V_-+\alpha)^{-1}V_-(H_0+\alpha)^{-1}.$$

Multiplying by $V_-^{1/2}$ to the left and to the right and taking into account (5.10) and the definition of K_α, one gets

$$B_\lambda = V_-^{1/2}(H_0+\lambda V_-+\alpha)^{-1}V_-^{1/2} = V_-^{1/2}\left[\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t}\mathrm{e}^{-t(H_0+\lambda V_-)}\right]V_-^{1/2}.$$

By Proposition 4.2 and its consequences, for any $u \in C_0(\mathbb{R}^d)$, $u \geq 0$, we have

$$[F_\lambda(K_\alpha)u](x) = \tag{5.11}$$

$$= V_-^{1/2}(x)\int_0^\infty \mathrm{d}t e^{-\alpha t}\int_{\mathbb{R}^d}\mathrm{d}y\left[\int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega)\, g_\lambda\left(\int_0^t \mathrm{d}s\, V_-(X_s)\right)\right]V_-^{1/2}(y)u(y).$$

Since Φ maps monotonous convergent sequences into monotonous convergent sequences, by applying Lemmas 5.8 and 5.9 and the Monotonous Convergence Theorem (B. Levi), we get (5.11) for $\lambda = \infty$, for the couple (g_∞, F_∞).

We introduce the notation

$$G_\lambda(t;x,y) := \int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega)\, g_\lambda\left(\int_0^t \mathrm{d}s\, V_-(X_s)\right), \tag{5.12}$$

for $t > 0$, $x, y \in \mathbb{R}^d$, $0 < \lambda \leq \infty$. By the consequences of Proposition 4.2, for any $0 < \lambda < \infty$ the function G_λ is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and symmetric in x, y. To obtain the same properties for $\lambda = \infty$, we approximate g_∞ by using once again Lemmas 5.8 and 5.9. So it exists a sequence $(f_n)_{n\geq 1}$ of real continuous functions on $[0,\infty)$, each one being a finite linear combination of functions of the form g_λ, such that f_n converges to g_∞ uniformly on any compact subset of $[0,\infty)$. On the other hand, if $M>0$ is an upper bound for V_-, we have

$$0 \le \int_0^t \mathrm{d}s\, V_-(X_s) \le Mt,$$

and $\mu_{0,x}^{t,y}(\Omega) = \mathring{\wp}_t(x-y)$. It follows that G_∞ is, uniformly on compact subsets of $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, the limit of a sequence of continuous functions, which are symmetric in x,y. Thus G_∞ has the same properties. Moreover, since $0\le g_\infty\le 1$ and $g_\infty(t)=0$ for $0\le t\le 1$, we have $G_\infty(t;x,y)=0$ for $t\le 1/M$. Using (2.4) and (2.3), there is a constant $C>0$ such that

$$0 \le G_\infty(t;x,y) \le C, \quad \forall t>0,\ \forall x,y\in\mathbb{R}^d. \tag{5.13}$$

From (5.11) for $\lambda=\infty$, we infer that $F_\infty(K_\alpha)$ has an integral kernel of the form

$$\left[F_\infty(K_\alpha)\right](x,y) = V_-^{1/2}(x)V_-^{1/2}(y)\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\alpha t}G_\infty(t;x,y), \tag{5.14}$$

so (3.27) is verified. The continuity of $F_\infty(K_\alpha)$ follows from the Dominated Convergence Theorem and from (5.13). The symmetry is obvious, and the last property of the statement follows from $F_\infty(K_\alpha)\ge 0$. ☐

REMARK **5.1** *By a lemma from* [47]*, §XI.4, $F_\infty(K_\alpha)\in B_1$ if the function $\mathbb{R}^d\ni x\mapsto\left[F_\infty(K_\alpha)\right](x,x)$ is integrable and one has*

$$\mathrm{Tr}\left[F_\infty(K_\alpha)\right] = \int_{\mathbb{R}^d}\mathrm{d}x\ \left[F_\infty(K_\alpha)\right](x,x). \tag{5.15}$$

Setting $D_\infty(t;x) := V_-(x)G_\infty(t;x,x)$, $t>0, x\in\mathbb{R}^d$, we have

$$\left[F_\infty(K_\alpha)\right](x,x) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\alpha t}D_\infty(t;x). \tag{5.16}$$

To check the integrability of this function, one introduces

$$\Psi_\infty : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}_+,$$

$$\Psi_\infty(t;x) := t^{-1}\int_\Omega \mu_{0,x}^{t,x}(\mathrm{d}\omega)\,\tilde{g}_\infty\left(\int_0^t \mathrm{d}s\,V_-(X_s)\right),$$

where $\tilde{g}_\infty(t) := tg_\infty(t)$. The role of this function is stressed by

LEMMA **5.10** *For $d \geq 3$ consider the following constant depending only on d:*

$$\overline{C}_d := C\left(\int_1^\infty \mathrm{d}s\, s^{-d}\, g_\infty(s) \vee \int_1^\infty \mathrm{d}s\, s^{-d/2}\, g_\infty(s)\right) = C\int_1^\infty \mathrm{d}s\, s^{-d/2}\, g_\infty(s)$$

where C is the constant verifying (2.6). One has

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x\, \Psi_\infty(t;x) \leq \overline{C}_d \left(\int_{\mathbb{R}^d} \mathrm{d}x\, V_-^d(x) + \int_{\mathbb{R}^d} \mathrm{d}x\, V_-^{d/2}(x)\right). \tag{5.17}$$

Proof. The function $\tilde{g}_\infty$ is convex and $\frac{\mathrm{d}s}{t}$ is a probability on $(0,t)$; thus by the Jensen inequality we obtain

$$\tilde{g}_\infty\left(\int_0^t \mathrm{d}s\, V_-(X_s)\right) \leq \int_0^t \frac{\mathrm{d}s}{t}\, \tilde{g}_\infty\left(t\, V_-(X_s)\right).$$

Let us also remark that for the constant $\overline{C}_d$ to be finite we have to ask that $d \geq 3$ for the factor $s^{-d/2}$ to be integrable at infinity, because the convexity condition on $\tilde{g}_\infty$ rather implies that g_∞ cannot vanish at infinity.

Then

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x\, \Psi_\infty(t;x) \leq$$
$$\leq \int_0^\infty \mathrm{d}t\, t^{-2}\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_\Omega \mu_{0,x}^{t,x}(\mathrm{d}\omega) \int_0^t \mathrm{d}s\, \tilde{g}_\infty\left(tV_-(X_s)\right)\right].$$

Using now Proposition 4.3, the last expression is equal to:

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t\, t^{-2}\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_\Omega \mu_{0,0}^{t,0}(\mathrm{d}\omega) \int_0^t \mathrm{d}s\, \tilde{g}_\infty\left(tV_-(x+\omega(s))\right)\right] = \\
&= \int_0^\infty \mathrm{d}t\, t^{-2}\, \mathrm{e}^{-\alpha t} \left[\int_\Omega \mu_{0,0}^{t,0}(\mathrm{d}\omega) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x\, \tilde{g}_\infty\left(tV_-(x)\right)\right] = \\
&= \int_0^\infty \mathrm{d}t\, t^{-1}\, \mathrm{e}^{-\alpha t} \left[\int_\Omega \mu_{0,0}^{t,0}(\mathrm{d}\omega)\right] \int_{\mathbb{R}^d} \mathrm{d}x\, \tilde{g}_\infty\left(tV_-(x)\right) = \\
&= \int_0^\infty \mathrm{d}t\, t^{-1}\, \mathrm{e}^{-\alpha t} \mathring{\wp}_t(0) \int_{\mathbb{R}^d} \mathrm{d}x\, \tilde{g}_\infty\left(tV_-(x)\right) \leq \\
&\leq C \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_0^\infty \mathrm{d}t\, t^{-d-1}(1+t^{d/2})\tilde{g}_\infty\left(tV_-(x)\right)\right] \leq \\
&\leq \overline{C}_d \left(\int_{\mathbb{R}^d} \mathrm{d}x\, V_-^d(x) + \int_{\mathbb{R}^d} dx\, V_-^{d/2}(x)\right),
\end{aligned}$$

where we have used the fact that $s < 1$ implies $g_\infty(s) = 0$. □

The next result gives the connection between D_∞ and Ψ_∞:

PROPOSITION **5.2**

$$\int_{\mathbb{R}^d} \mathrm{d}x\, D_\infty(t,x) \;=\; \int_{\mathbb{R}^d} \mathrm{d}x\, \Psi_\infty(t,x).$$

Proof. First let us verify the following identity for any $t > 0$:

$$\int_{\mathbb{R}^d} \mathrm{d}x\, D_\lambda(t,x) \;=\; \int_{\mathbb{R}^d} \mathrm{d}x\, \Psi_\lambda(t,x), \quad \text{for } \lambda \in (0,\infty) \tag{5.18}$$

where D_λ and Ψ_λ are defined in terms of g_λ in the same way that D_∞ and Ψ_∞ are defined in terms of g_∞. Let us point out that both D_λ and Ψ_λ are positive measurable functions on $(0,\infty) \times \mathbb{R}^d$ but only the integral on the left hand side of (5.18) is evidently finite by what we have proven so far. For simplifying the writing we shall take $\lambda = 1$. For any $r \in [0,t]$ we denote by

$$S_r := \mathrm{e}^{-r(H_0+V_-)} V_- \mathrm{e}^{-(t-r)(H_0+V_-)}.$$

Following the remarks after Proposition 4.2 above, for $r \in (0,t)$, both exponentials appearing in the above right hand side are integral operators with non-negative continuous integral kernels; thus S_r will also be an integral operator with non-negative continuous kernel that we shall denote by K_r, and we can compute it explicitely as follows. For a non-negative $u \in C_0(\mathbb{R}^d)$, using Proposition 4.1 with $A = 0$ gives

$$(S_r u)(x) = \mathsf{E}_x \left\{ \mathrm{e}^{-\int_0^r V_-(X_\rho)\mathrm{d}\rho} V_-(X_r) \mathsf{E}_{X_r} \left[\mathrm{e}^{-\int_0^{t-r} V_-(X_\sigma)\mathrm{d}\sigma} u(X_{t-r}) \right] \right\}$$

and using the Markov property (4.8) we obtain

$$\mathsf{E}_{X_r} \left[\mathrm{e}^{-\int_0^{t-r} V_-(X_\sigma)\mathrm{d}\sigma} u(X_{t-r}) \right] = \mathsf{E}_x \left[\mathrm{e}^{-\int_0^{t-r} V_-(X_\sigma \circ \theta_r)\mathrm{d}\sigma} u(X_t) \mid \mathfrak{F}_r \right] =$$

$$= \mathsf{E}_x \left[\mathrm{e}^{-\int_r^t V_-(X_\sigma)\mathrm{d}\sigma} u(X_t) \mid \mathfrak{F}_r \right].$$

As the function $\mathrm{e}^{-\int_0^r V_-(X_\rho)\mathrm{d}\rho} V_-(X_r) : \Omega \to \mathbb{R}$ is evidently $\mathfrak{F}_r$-measurable, we get (using the property (4.4) of conditional expectations)

$$(S_r u)(x) = \mathsf{E}_x \left\{ \mathsf{E}_x \left(V_-(X_r) \mathrm{e}^{-\int_0^t V_-(X_\sigma)\mathrm{d}\sigma} u(X_t) \mid \mathfrak{F}_r \right) \right\}.$$

We use now the property (4.3) and Proposition 4.2 taking $F := V_-(X_r)$ in order to get

$$(S_r u)(x) = \mathsf{E}_x \left\{ V_-(X_r) \mathrm{e}^{-\int_0^t V_-(X_\sigma)\mathrm{d}\sigma} u(X_t) \right\} =$$

$$= \int_{\mathbb{R}^d} \mathrm{d}y \left\{ \int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega) V_-(X_r) \mathrm{e}^{-\int_0^t V_-(X_\sigma)\mathrm{d}\sigma} \right\} u(y).$$

In conclusion for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ we have

$$K_r(x, y) = \int_\Omega \mu_{0,x}^{t,y}(\mathrm{d}\omega) V_-(X_r) \mathrm{e}^{-\int_0^t V_-(X_\sigma)\mathrm{d}\sigma}. \tag{5.19}$$

Using Proposition 4.3 we obtain

$$\int_{\mathbb{R}^d} \mathrm{d}x\, K_r(x, x) \leq \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_\Omega \mu_{0,x}^{t,x}(\mathrm{d}\omega) V_-(\omega(r)) \right] =$$

$$\int_{\mathbb{R}^d} \mathrm{d}x \left[\int_\Omega \mu_{0,0}^{t,x}(\mathrm{d}\omega) V_-(x + \omega(r)) \right] = \mathring{\wp}_t(0) \int_{\mathbb{R}^d} \mathrm{d}x\, V_-(x) < \infty, \quad \forall t > 0.$$

Thus, for any $r \in [0, t]$ the operator S_r is trace class. Moreover, due to the properties of the trace we have $\mathrm{Tr} S_r = \mathrm{Tr} S_0$, $\forall r \in [0, t]$. We have:

$$\mathrm{Tr} S_0 = \frac{1}{t} \int_0^t \mathrm{d}r\, (\mathrm{Tr} S_0) = \frac{1}{t} \int_0^t \mathrm{d}r\, (\mathrm{Tr} S_r) = \frac{1}{t} \int_0^t \mathrm{d}r \left[\int_{\mathbb{R}^d} \mathrm{d}x\, K_r(x, x) \right] =$$

$$= \frac{1}{t} \int_{\mathbb{R}^d} \mathrm{d}x \left[\int_\Omega \mu_{0,x}^{t,x}(\mathrm{d}\omega) \tilde{g}_1 \left(\int_0^t \mathrm{d}s\, V_-(X_s) \right) \right] = \int_{\mathbb{R}^d} \mathrm{d}x \Psi_1(t, x)$$

In particular, for any $t > 0$, $\Psi_1(t; \cdot)$ is integrable on $\mathbb{R}^d$.

On the other hand

$$\mathrm{Tr} S_0 = \int_{\mathbb{R}^d} K_0(x, x) \mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}x\, V_-(x) \int_\Omega \mu_{0,x}^{t,x}(\mathrm{d}\omega) \mathrm{e}^{-\int_0^t \mathrm{d}\rho\, V_-(X_\rho)}$$

$$= \int_{\mathbb{R}^d} \mathrm{d}x\, V_-(x) G_1(t; x, x) = \int_{\mathbb{R}^d} \mathrm{d}x\, D_1(t; x).$$

One uses the approximation properties contained in Lemmas 5.8 and 5.9 as well as the Monotone Convergence Theorem. □

Proof. of Theorem 1.1 *for* $B = 0$

We can assume $V_+ = 0$ and $V_- \in C_0^\infty(\mathbb{R}^d)$. Lemma 5.7 implies that for any $\alpha > 0$ one has

$$N_{-\alpha}(-V_-) \leq F_\infty(1)^{-1} \mathrm{Tr}\left[F_\infty(K_\alpha) \right].$$

Using (5.15), (5.16), we obtain

$$\mathrm{Tr}\left[F_\infty(K_\alpha) \right] = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x\, D_\infty(t; x) =$$

$$= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t} \int_{\mathbb{R}^d} \mathrm{d}x\, \Psi_\infty(t;x). \tag{5.20}$$

Inequality (6.1) for $B = 0$ follows from (5.20) and Lemma 5.10. In addition $C_d = F_\infty(1)^{-1}\overline{C}_d$. □

6. Proof of the bounds in the magnetic case

Proof. of Theorem 1.1 *for* $B \neq 0$.

Analogously to Section 5, we can assume $V_+ = 0$ and $V_- \in C_0^\infty(\mathbb{R}^d)$. For $\alpha > 0$ one sets $K_\alpha(A) := V_-^{1/2}(H_A + \alpha)^{-1}V_-^{1/2}$. By inequality (3.4) for $r = 1$ and also using Pitt's Theorem [45], $K_\alpha(A)$ is a positive compact operator, and the same can be said about $F_\infty\left[K_\alpha(A)\right]$. We show that $F_\infty\left[K_\alpha(A)\right] \in B_1$ and we estimate the trace-norm. Repeating the arguments from the beginning of the proof of Proposition 5.1,

$$F_\lambda\left[K_\alpha(A)\right] = V_-^{1/2} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t}\mathrm{e}^{-t(H_A+\lambda V_-)}V_-^{1/2}. \tag{6.1}$$

By using Proposition 4.1, we get for any $u \in C_0(\mathbb{R}^d)$, $u \geq 0$

$$\left[F_\lambda\left[K_\alpha(A)\right]u\right](x) = \tag{6.2}$$

$$= V_-^{1/2}(x) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\alpha t} E_x\left[u(X_t)V_-^{1/2}(X_t)\mathrm{e}^{-\mathrm{i}S_A(t,X)} g_\lambda\left(\int_0^t \mathrm{d}s\, V_-(X_s)\right)\right].$$

Approximating g_∞ by means of Lemmas 5.8 and 5.9 and using the Monotone Convergence Theorem, we see that (6.2) also holds for the pair (g_∞, F_∞). The next inequality follows:

$$\left|F_\infty\left[K_\alpha(A)\right]u\right| \leq F_\infty(K_\alpha)|u|, \quad \forall u \in L^2(\mathbb{R}^d). \tag{6.3}$$

By Lemma 15.11 from [48], we have $F_\infty\left[K_\alpha(A)\right] \in B_1$ and

$$\mathrm{Tr}\left(F_\infty\left[K_\alpha(A)\right]\right) \leq \mathrm{Tr}\left(F_\infty\left[K_\alpha\right]\right). \tag{6.4}$$

Denoting by $N_{-\alpha}(B, -V_-)$ the number of eigenvalues of $H_A - V_-$ strictly less than $-\alpha$, analogously to Lemmas 5.6 and 5.7, we deduce that

$$N_{-\alpha}(B, -V_-) \leq F_\infty(1)^{-1}\mathrm{Tr}\left(F_\infty\left[K_\alpha\right]\right). \tag{6.5}$$

Inequality (6.1) follows from (6.5) by using the estimations at the end of Section 5. The constant C_d is the same as for the case $B = 0$. □

Proof. of Corollary 1.1. The idea of the proof is standard (cf. [48] for instance), but one has to use parts of the arguments from the proof of Theorem 1.1 in the case $B = 0$.

1. We show that it is enough to treat the case $V_+ = 0$.

We denote by N (resp. N_-) the number of strictly negative eigenvalues of $H_A \dot{+} V$ (resp. $H_A \dot{+} (-V_-)$). We have $N, N_- \in [0, \infty]$ and the min-max principle shows that $N \leq N_-$. In addition, if $H_A \dot{+} V$ has strictly negative eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$, then $H_A \dot{+} (-V_-)$ has strictly negative eigenvalues $\lambda_1^- \leq \lambda_2^- \leq \dots$ and $\lambda_j^- \leq \lambda_j$, $j \geq 1$. Therefore, one has $\sum_{j\geq 1} |\lambda_j|^k \leq \sum_{j\geq 1} |\lambda_j^-|^k$.

2. We show that treating compactly supported V_- is enough (remark that this property implies that $V_- \in L^p(\mathbb{R}^d)$ for any $p \in [1, d+k]$).

We take into account the approximation sequence defined in Lemma 5.3. The sequence of forms $(\mathfrak{h}^n)_{n\geq 1}$ satisfies the hypothesis of Theorem 3.11, Ch. VIII from [29]. If we denote by $\lambda_1 \leq \lambda_2 \leq \dots$ the strictly negative eigenvalues of $H_A \dot{+} V$ and by $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots$ the strictly negative eigenvalues of $H^{(n)} := H_A \dot{+} V^{(n)}$, once again by Theorem 3.15, Ch. VIII from [29], we have $\lambda_j^{(n)} \geq \lambda_j$, $\forall j, n \in \mathbb{N}^*$ and $\lambda_j^{(n)}$ converges to λ_j. So it will be sufficient to prove (6.1) for the operators $H^{(n)}$.

3. We assume from now on that $V = -V_-$, $V_- \in L^{d+k}(\mathbb{R}^d)$ $(k > 0)$ and that $\operatorname{supp}(V_-)$ is compact. Let $\beta_0 > 0$ and for $\beta \in (0, \beta_0]$ let

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{N_{-\beta}} < -\beta$$

be the eigenvalues of $H = H_A \dot{+} (-V_-)$ strictly smaller than $-\beta$ and let

$$\overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \dots \leq \overline{\lambda}_{M(\beta)} < -\beta$$

be the distinct eigenvalues with m_j the multiplicity of $\overline{\lambda}_j$, $1 \leq j \leq M(\beta)$. We have $N_{-\alpha} := N_{-\alpha}(B, -V_-)$. Using the definition of the Stieltjes integral and integration by parts, we get

$$\sum_{j=1}^{N_{-\beta}} |\lambda_j|^k = \sum_{j=1}^{M(\beta)} m_j |\overline{\lambda}_j|^k = \sum_{j=1}^{M(\beta)} |\overline{\lambda}_j|^k \left(N_{\overline{\lambda}_{j+1}} - N_{\overline{\lambda}_j} \right) = \int_{\lambda_1}^{-\beta} |\lambda|^k \mathrm{d}N_\lambda =$$

$$= |\beta|^k N_{-\beta} + k \int_{\lambda_1}^{-\beta} |\lambda|^{k-1} N_\lambda \, \mathrm{d}\lambda. \tag{6.6}$$

We denote by I the last integral and use (6.5) and (5.20) and the arguments in the proof of Lemma 5.10 to estimate I:

$$I=\int_{\beta}^{-\lambda_1}\alpha^{k-1}N_{-\alpha}\mathrm{d}\alpha=[F_\infty(1)]^{-1}\int_{\beta}^{-\lambda_1}\alpha^{k-1}\mathrm{Tr}F_\infty(K_\alpha)\mathrm{d}\alpha=$$

$$=[F_\infty(1)]^{-1}\int_{\mathbb{R}^d}\mathrm{d}x\int_0^\infty \mathrm{d}t\,\Psi_\infty(t,x)\int_\beta^{-\lambda_1}\mathrm{d}\alpha\,\alpha^{k-1}\mathrm{e}^{-\alpha t}\leq$$

$$\leq[F_\infty(1)]^{-1}\int_{\mathbb{R}^d}\mathrm{d}x\int_0^\infty \mathrm{d}t\,t^{-1}\overset{\circ}{\wp}_t(0)\tilde g_\infty(tV_-(x))\int_\beta^{-\lambda_1}\mathrm{d}\alpha\,\alpha^{k-1}\mathrm{e}^{-\alpha t}\leq$$

$$\leq C\,[F_\infty(1)]^{-1}\int_{\mathbb{R}^d}\mathrm{d}x\int_0^\infty \mathrm{d}t\left(t^{-d-1}+t^{-d/2-1}\right)\tilde g_\infty(tV_-(x))\int_\beta^{-\lambda_1}\mathrm{d}\alpha\,\alpha^{k-1}\mathrm{e}^{-\alpha t}.$$

The α integral may be bounded by

$$\int_0^\infty \mathrm{d}\alpha\,\alpha^{k-1}\mathrm{e}^{-\alpha t}=t^{-k}\int_0^\infty \mathrm{d}s\,s^{k-1}\mathrm{e}^{-s}\leq Ct^{-k}.$$

Recalling that $\tilde g_\infty(t)=0$ for $t\leq 1$ and $\tilde g_\infty(t)=t-1$ for $t>1$, we get that $\tilde g_\infty(tV_-(x))=0$ for $V_-(x)=0$ and for $V_-(x)>0$

$$\int_0^\infty \mathrm{d}t\,t^{-k}\left(t^{-d-1}+t^{-d/2-1}\right)\tilde g_\infty(tV_-(x))=$$

$$=[V_-(x)]^{d+k}\int_1^\infty s^{-d-k-1}(s-1)\mathrm{d}s\,+\,[V_-(x)]^{d/2+k}\int_1^\infty s^{-d/2-k-1}(s-1)\mathrm{d}s,$$

the integrals being convergent for $d\geq 2$.

Using these estimations in (6.6) we conclude that

$$\sum_{j=1}^{N_{-\beta}}\left(|\lambda_j|^k-|\beta|^k\right)\leq C\left\{\int_{\mathbb{R}^d}[V_-(x)]^{d+k}\,\mathrm{d}x\,+\,\int_{\mathbb{R}^d}[V_-(x)]^{d/2+k}\,\mathrm{d}x\right\},$$

thus

$$\sum_{j=1}^{N_{-(\beta_0)}}\left(|\lambda_j|^k-|\beta|^k\right)\leq C\left\{\int_{\mathbb{R}^d}[V_-(x)]^{d+k}\,\mathrm{d}x\,+\,\int_{\mathbb{R}^d}[V_-(x)]^{d/2+k}\,\mathrm{d}x\right\},$$

with the constant C not depending on β or β_0. Taking the limit $\beta\searrow 0$ ends the proof. □

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