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On geometric (quasi-renewal) processes

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Our aim is to approach geometrical (quasi-renewal) processes in their general form. As example we consider generalization of Poisson process

1. Introduction

Geometric (quasi-renewal) processes (G(Q-R)P) provides an appealing alternative to many existing models for describing the behavior of reparable equipment in Reliability Theory [Lam, 1988, Wang and Pham, 1996]. Notion of geometric process it was introduced in 1988 by Yeh Lam.

Definition 1 (Lam, 1988). *A stochastic process* $(X_n)_{n\geqslant 1}$ *is a geometric* (*quasi-renewal*) *process* (G(Q-R)P)*, if there exists some* $\alpha > 0$ *such that* $(\alpha^{n-1}X_n)_{n\geqslant 1}$ *forms a renewal process. The number* α *is called the ratio of the geometric process.*

The majority of repair models assume that, when a system fails, the repair action can restore the system to conditions it was in just before the failure occurrence. The real situation, however, may be different: the system after repair is newer or older than before the failure occurred. In this case, the failure intensity of the system after repair is different from that before failure occurrence; it is reasonable therefore to use a process such a G(Q-R)P to deal with these situations because the failure intensity patterns after each repair can be differentiated in G(Q-R)P models.

With different values of parameter α G(Q-R)P are used to describe the stochastic decrease of life time of a component after repairs and the stochastic increase of repair time after failures. The G(Q-R)P models, however, can only describe a scenario where the failure intensity of a system is monotonously increasing or decreasing with the operating time, they cannot be appropriately applied when a non-monotonic or a complicated trend in the failure data is observed.

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2. Renewal equations for geometric (quasi-renewal) processes

Because a renewal process (RP) is a stochastic process $Z = (Z_n)_{n\geqslant 1}$ such that $Z_1, Z_2, ...$ are independent identically distributed (i.i.d.) non-negative random variables (r.v.) we may formulate the following equivalent definition of G(Q-R)P.

Definition 2. *A stochastic process* $(X_n)_{n\geqslant 1}$ *is a geometric* (*quasi-renewal*) *process* (G(Q-R)P)*, if there exists some* $a > 0$ *such that* $X_n = a^{n-1}Z_n, n \geq 1$ *, where* $Z = (Z_n)_{n\geq 1}$ *is a renewal process.*

So, relation between number α from Definition 1 and number a from Definition 2 is $\alpha = 1/a$.

Clearly, a G(Q-R)P is stochastically increasing if the ratio $a > 1$; it is stochastically decreasing if the ratio $0 < a \leq 1$. A G(Q-R)P become a RP if $a = \alpha = 1$. Thus, the G(Q-R)P is a simple monotone process and it is a generalization of renewal process.

Now, let's consider for G(Q-R)P $X = (X_n)_{n\geq 1}$, the moment S_n of the n-th renewal, i.e.,

$$S_n = \sum_{i=1}^{n} X_i,\ n \geqslant 1,\ S_0 \equiv 0$$

and its counting G(Q-R)P, i.e., $N(t) = \max\{n : S_n \leq t\}$.

Since a G(Q-R)P is defined if it is known the parameter α and probability distribution of the random vector $(Z_1, Z_2,...,Z_n)$, $\forall n \geqslant 1$, we may calculate probability distribution of the moments S_n, $n \geqslant 1$ and distribution $\mathbf{P}_i(t) = \mathbf{P}(N(t) = i)$, $i = 0$, 1, 2, ..., $\forall t \in [0, +\infty)$.

Proposition 1. *If* $0 < a \leq 1$, *then does exists* $\theta_0 \in (0, +\infty)$ *such that mean value*

$$\mathbb{E}\exp\{\theta N(t)\} < +\infty,\ \forall t \geqslant 0, \theta \leq \theta_0.$$

Particularly, $\mathbb{E}(N(t))^r < +\infty, \forall t \geqslant 0, r > 0.$

Now, let's consider $H(t) = \mathbb{E}N(t)$ called renewal function. Using indicators

$$I_{\{S_n \leq t\}} = \begin{cases} 0, & \text{if } S_n > t, \\ 1, & \text{if } S_n \leq t, \end{cases}$$

superposition formula and distribution functions F_{X_n}, $n \geqslant 1$, $F = F_{X_1}$, we observes that

$$\mathbb{E}N(t) = \mathbb{E}\sum_{n\geqslant 1} I_{\{S_n \leq t\}} = \sum_{n\geqslant 1}\mathbf{P}(X_1 + X_2 + ... + X_n \leq t) =$$

$$\sum_{n\geqslant 1} F_{X_1} * ... * F_{Xn}(t) = F_{X_1}(t) + \sum_{n\geqslant 2} F_{X_1} * ... * F_{Xn}(t) =$$

$$F_{X_1}(t) + F_{X_1} * \sum_{n\geqslant 2}\mathbf{P}(X_2 + ... + X_n \leq t) =$$

$$F(t) + F * \sum_{n\geqslant 2}\mathbf{P}(aZ_2 + ... + a^{n-1}X_n \leq t) =$$

$$F(t) + F * \sum_{n\geqslant 2}\mathbf{P}(Z_2 + ... + a^{n-2}X_n \leq t/a) =$$

$$F(t) + F * \sum_{n\geqslant 2}\mathbf{P}(X_2 + ... + X_{n-1} \leq t/a) =$$

$$F(t) + F * \sum_{n\geqslant 1}\mathbf{P}(X_1 + ... + X_n \leq t/a) = F(t) + \int\limits_0^{t/a} H(t/a-u)\mathrm{d}F(u) =$$

$$= F(t) + \int\limits_0^{t/a} F(t/a-u)\mathrm{d}H(u).$$

In this way we obtain two forms of renewal equations for renewal function:

$$H(t) = F(t) + \int\limits_0^{t/a} H(t/a-u)\mathrm{d}F(u), H(t) = F(t) + \int\limits_0^{t/a} F(t/a-u)\mathrm{d}H(u).$$

3. Quasi-renewal Poisson processes

As a particular case of G(Q-R)P let's consider a quasi-renewal Poisson Process (QRPP), i.e., G(Q-R)P $X = (X_n)_{n\geqslant 1} = (a^{n-1}Z_n)_{n\geqslant 1}$, when $(a^{n-1}Z_n)_{n\geqslant 1}$ i.i. exponentially distributed r.v. with parameter λ, $\lambda > 0$.

We are interested to find probability distribution of $N(t)$ and renewal function $H(t) = \mathbb{E}N(t)$. Observing that for QRPP $X = (X_n)_{n\geqslant 1}$ r.v. $X_n \sim \exp\{\lambda/a^{n-1}\}, n \geqslant 1$, where $a > 0$, we may interpret $N(t)$ as a pure Birth process with birth intensities $\lambda_k = \lambda/a^k$, $k = 0, 1, 2, ...$. As a consequence we have

Proposition 2. *Probability distribution* $(\mathbf{P}_k(t))_{k\geqslant 1}$, *where* $\mathbf{P}_0(t) = \exp\{-\lambda t\}$, $\mathbf{P}_k(t) = \mathbf{P}(N(t) = k)$, $k = 0, 1, 2, ...$ *is a unique solution of the system of recurrent differential equations*

$$\mathbf{P}'_k(t) = \lambda_{k-1}\mathbf{P}_{k-1}(t) - \lambda_k\mathbf{P}_k(t),$$

with initial conditions $\mathbf{P}_k(0) = 0$, $k = 0, 1, 2, ...$.

Corollary 1. Probability distribution of counting QRPP $N(t)$ with parameters λ and a, $\lambda, a > 0$, for $n = 0, 1, 2, ...$ is given by formulas

$$\mathbf{P}_0(t) = \mathbf{P}(N(t) = 0) = \exp\{-\lambda t\},$$

$$\mathbf{P}_n(t) = \mathbf{P}(N(t) = n) =$$

$$\sum_{i=1}^{n} C_{i,n}(1 - \exp\{-\lambda a^{-(i-1)}t\}) - \sum_{i=1}^{n+1} C_{i,n+1}(1 - \exp\{-\lambda a^{-(i-1)}t\}),$$

where

$$C_{i,k} = \prod_{j=1, j\neq i}^{k} \frac{a^{i-1}}{a^{i-1} - a^{j-1}}, i = \overline{1,k}$$

Corollary 2. Renewal function $H(t)$ of counting QRPP $N(t)$ is equal to

$$H(t) = \mathbb{E}N(t) = \sum_{n=1}^{+\infty} \sum_{i=1}^{n} C_{i,n}(\exp\{-\lambda a^{-(i-1)}t\} - 1).$$

References

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[2] Wang, H., Pham, H., *A quasi renewal process and its applications in imperfect maintenance*, International Journal of System Science, **27**(10) (1991), 1055–1062.