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Discrete Approximation of Nonlinear Diffusion Equation ♮

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The paper deals with the approximation of some nonlinear diffusion equations with source terms and nonhomogeneous Dirichlet boundary conditions and initial conditions. The approximation scheme consists in the discretization of space derivative operators while the time differentiation is kept continous. As result the solution of the partial differential equations is approximate by the solution of a system of ordinary differential equations. We provide the bounds for the solutions of the discrete model that are independent of the mesh size of triangulation.

1. Introduction

In this paper we develop a numerical approximation scheme for a class of parabolic nonlinear diffusion equations.

The mathematical model is given by

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}\left(\kappa(u)\nabla u + \boldsymbol{f}(u)\right) = g(t,x,u), & t>0, x\in\Omega, \\ u = u_D, & t>0, x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$, $u(t,x)$ is the scalar unknown function, b and κ are real constitutive functions, $\boldsymbol{f}$ is a vector function which models the convective flux and g

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is a real function which models the "mass" production. The derivative operators div and ∇ are taken with with respect to $x \in \mathbb{R}^n$.

The mathematical model (1) covers a large field of physical phenomena such that: heat transfer, infiltration of a fluid through porous media, transport of contaminant in porous media, etc.

A particular case of the model problem (1) is the linear caloric equation:

$$\frac{\partial u}{\partial t} = \mathrm{div}(\kappa \nabla u), \tag{2}$$

where u models the temperature and $\kappa = const. > 0$ represents the thermal conductivty. Here it is supposed that the caloric flux obeys the Fourier law $q = -\kappa \nabla T$ and the thermal conductivty is independent of temperature. The condition $\kappa > 0$ reflects the fact that heat propagates from high to lower temperature. If the temperature of the body is high enuogh one must consider the radiation effects and the temperature dependece of thermal conductivty. For example, if the power radiated by a a body to environment follows the Stefan-Boltzmann law of the fourth powers of both the body and the medium temperature the heat equation becomes [6]

$$\frac{\partial u}{\partial t} = \mathrm{div}(\kappa(u) \nabla u) - k_r(u^4 - u_e^4). \tag{3}$$

The unsaturated water flow through porous media is described by the well known Richards' equations [3]

$$\frac{\partial \theta(h)}{\partial t} - \mathrm{div}(K(h)\nabla h + \boldsymbol{e}_3 K(h)) = 0, \tag{4}$$

where θ represents the relative volumetric water content, h represents the pressure, K is the hydraulic conductivity and $\boldsymbol{e}_3$ is the upward vertical versor. The function $\theta(h)$ is a continous positive function and it is strictly increasing on the interval $(-\infty, 0]$ and constant on $h > 0$. Also the hydraulic conductivity is a continous positive function strictly increasing on $(-\infty, 0]$ and a constant function on the set $h > 0$. The hydraulic conductivity becomes zero as h approaches $-\infty$.

The transport of contaminant in porous media is governed by an equation of the form [9], [8]

$$\frac{\partial\,(C + \lambda C^p)}{\partial t} + \boldsymbol{v} \cdot \nabla C = \mathrm{div}(D \nabla C) + g(x, C), \tag{5}$$

where C represent the mass concetration of the contaminant, $\boldsymbol{v}$ denotes the velocity of the fluid flow, supposed to be constant. The term $\lambda C^p, \lambda \geq 0$ takes into account the adsorbtion reaction by means of Freundlich isoterm. The absorbtion reactions is described by the term $g(x, C)$ that usualy is given by

$$g = -\alpha C^q \tag{6}$$

with $\alpha > 0$, $q > 0$ (the order of the reaction).

In this paper we deal with a discrete version of the equation (1). To have an idea of what kind of solution we approximate we think it will be worthy to make few remarks about the solvabilty of the problem.

Due to the nonlinear parabolic term $b(u)$ and nonlinear diffusion coefficient $\kappa(u)$ the problem (1) can be a degenerate problem and consequently there exists no classical solutions. To have a solvable problem the concept of the strong solution has been weakned in the sense that a function is a solution if it satisfies the equation in weak sense. In the new framework one deals with *weak soltions*, *weak entropy solutions*, *very weak solutions* etc.

The notion of weak solution for the problem of the type (1) was introduced by Alt and Luckhaus in [1]. By imposing some proper conditions on the constitutive functions, boundary data and initial conditions, the authors where able to prove the existence of the weak solution in the case of the parabolic-elliptic degeneration, $b(u)$ is a constant function on some interval of positive measure and the elliptic term $\kappa > 0$. Also F. Otto in [17] proves the existence of the L^1-contraction principle of the weak solutions.

Carrillo [5] extrapolates the concept of entropy solution introduced by Kruzhkov in theory of hyperbolic PDE [12]. The new concept *weak entropy solutions* answers to the question of the solvabilty of the problem, with homogeneous boundary data $u_D = 0$, in the case of parabolic-hyperbolic degeneration.

In the case of nonhomogeneous Dirichlet conditions one supplementary difficulty is to give a sense to boundary conditions. The main problem is that the region of parabolicity and hyperbolicity are glued together in a way that depends on the solution itself [14], [15] . We present here the framework introduced by C. Mascia, A. Porreta and A. Terracina in the paper [14]. Let us introduce the notations:

$$K(u) = \int_0^u \kappa(s)\mathrm{d}s,$$

$$\mathcal{E}(u,v) = \nabla\,|K(u) - K(v)| + \operatorname{sgn}(u-v)(f(u) - f(v)),$$

$$\mathcal{B}(u,v,w) = \mathcal{E}(u,v) + \mathcal{E}(u,w) - \mathcal{E}(v,w).$$

Regarding the domain Ω we suppose that there exists a C^2–covering of $\partial\Omega$, $\mathcal{A} = \{U_i\}_{i=1,m}$ made of open sets such that $\partial\Omega \subset \cap \overline{U}_i$ and , in some local coordinates $x = (x', x_n)$, there exists a C^2 function $x_n = \alpha_i(x')$ such that $U_i \cup \partial\Omega = \{x_n = \alpha_i(x')\}$, $U_i \cup \Omega = \{x_n < \alpha_i(x'\}$.

Definition 1. Boundary layer sequence. *A sequence $\{\vartheta_\delta\}$ of the $C^2(\Omega)\bigcap C^0(\overline{\Omega})$ functions is named a boundary layer sequence if*

$$\lim_{\delta\to 0^+} \vartheta_\delta = 1, \text{ pointwise in } \Omega,\ 0 \le \vartheta_\delta \le 1, \vartheta_\delta = 0 \text{ on } \partial\Omega.$$

Definition 2. Weak Entropy Solutions. Nonhomogeneous case(Mascia et al.) *A function $u \in L^\infty((0,T)\times\Omega)$ is an entropy solution of* (1) *if*

(1) (regularity) *there holds:*

$$K(u) \in L^2((0,T):W^{1,2}(\Omega))$$

and for any $U \in \mathcal{A}$, and any positive $\psi \in C_0^\infty(U)$ we have

$$\Big(-|u-u_D|\psi, \mathcal{E}(u,u_D)\psi\Big) \in \mathcal{DM}(Q)_2$$

where $\mathcal{DM}(Q)_2$ is the set of divergence-measure vector fields in Q.

(2) (entropy condition in interior of Q_T)

$$\int_{Q_T} \left\{|b(u)-b(s)|\frac{\partial v}{\partial t} - \mathcal{E}(u,s)\nabla v + gv\right\} \mathrm{d}x\mathrm{d}t \geq 0$$

for any $v \in W_0^{1,2}(Q_T)$ and $v \geq 0$ and $s \in \mathbb{R}$.

(3) (initial condition)

$$\lim_{t\to 0^+} \int_\Omega |u(t,x)-u_0(x)|\,\mathrm{d}x = 0.$$

(4) (boundary conditions) *in sense of trace in $L^2((0,T):W^{1,2}(\Omega))$ we have*

$$K(u) = K(u_D) \quad t>0,\ x \in \partial\Omega,$$

and for any boundary layer sequence ϑ_δ, and for any $U \in \mathcal{A}$, and any positive $\psi \in C_0^\infty(U)$ we have

$$\liminf_{\delta\to 0} \int_{Q_T} \mathcal{B}(u,s,u_D)\nabla\vartheta_\delta \xi\psi \mathrm{d}x\mathrm{d}t \geq 0, \forall s \in \mathbb{R},$$

for any $\xi \in L^2((0,T):W^{1,2}(\Omega))$, $\xi \geq 0$.

In the next section we introduce our discretization scheme of the problem (1). The main thing in building up the discrete form is the monotonocity of the approximations of the diffusion coeficient and convective flux, respectively.

We assume the following hypotheses:

Assumptions on constitutive functions:

A1	$b:\mathbb{R}\to\mathbb{R}$,	is a continous and nondecreasing function,
	$\kappa:\mathbb{R}\to\mathbb{R}_+$,	is a positive, continous and nondecreasing function,
	$f:\mathbb{R}\to\mathbb{R}^n$,	is a local Lipschitz vector function,
	$g:\mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$,	is a Caratheodory function.

Assumptions on boundary data and initial conditions:

A2 $u_D \in L^2((0,T):W^{1,2}(\Omega)) \cap L^\infty((0,T)\times\Omega)$,
$u_0 \in L^\infty(\Omega)$.

Assumptions on the domain Ω:

A3 $\Omega \in \mathbb{R}^n$, is an open, bounded and connected set.

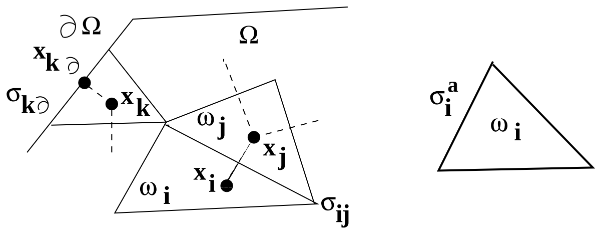

Fig. 1. Triangulation of polygonal domain in $\mathbb{R}^2$.

2. Discrete Approximation

By the method of lines (MOL), one can associate an ordinary differential system of equations (ODE) to a parabolic partial differential equation. The MOL consists in the discretization of the space variable using one of the standard methods as finite element, finite differences or finite-volume method (FVM). The FVM fits very well to conservative equations and there exist a large literature devoted to the method, we recall here the papers that deal with Dirichlet problem, [7] for hyperbolic PDE, [10], [11], [15] for nonlinear parabolic PDE.

The FVM deals with a decomposition of the domain Ω into a small polygonal domains ω_i and a net of inner knots x_i. The assembly $\{\omega_i, x_i\}$ defines a triangulation of the domain and it is an admissible meshe if it satisfies the following conditions, [11]

Definition 3. Admissible meshes. *The triangulation $\mathcal{T} = \{(\omega_i, x_i)\}_{i\in I}$ is called an admissible meshes if it satisfies:*

$$\mathbf{A4}\ \left\|\ \begin{array}{l} \omega_i \text{ is an open poligonal set} \subseteq \Omega,\ x_i \in \overline{\omega}_i,\ \forall i \in I \\ (1)\ \overline{\cup_{i\in I}\omega_i} = \overline{\Omega} \\ (2)\ \forall i \neq j \in I \text{ and } \overline{\omega}_i \cap \overline{\omega}_j \neq \Phi, \text{either } \mathcal{H}_{n-1}(\overline{\omega}_i \cap \overline{\omega}_j) = 0 \text{ or} \\ \qquad \sigma_{ij} := \overline{\omega}_i \cap \overline{\omega}_j \text{ is a common } (n-1) - \text{face of } \omega_i \text{ and } \omega_j \\ (\forall \sigma_{ij}, \quad [x_i, x_j] \perp \sigma_{ij} \end{array}\right.$$

Here $\mathcal{H}_{n-1}$ is the $(n-1)$ - dimensional Hausdorff measure. For each volume ω_i that has a common $(n-1)$ - face with the boundary $\partial\Omega$ one defines an external volume $\omega_{i_b} \in C\Omega$ by the reflection of the ω_i with respect to the face $\sigma_{i_b} = \omega_i \cap \partial\Omega$. Let us define the discrete values u_{i_b}

$$u_{i_b} = \frac{1}{\mathrm{m}(\sigma_{i_b})} \int\limits_{\sigma_{i_b}} u_D \mathrm{d}a. \tag{7}$$

Denotes by $\mathcal{T}^b$ the collection of all external volumes $\{(\omega_{i_b}, x_{i_b})\}$ and by I^b the set of their indices. Let $\mathcal{T}^e = \mathcal{T} \cup \mathcal{T}^b$ and $I^E = I \cup I^b$. We say that the volumes ω_i, $\omega_j \in \mathcal{T}^e$ are neighbors if they share a common $n-1$–face and we denote by $\boldsymbol{n}_{i,j}$ the unit normal vector to the face σ_{ij} that point to ω_j.

The space discretizated equations are derived from the integral form of (1) for each control volume ω_i

$$\int_{\omega_i} \frac{\partial b(u)}{\partial t}\mathrm{d}x - \int_{\partial\omega_i} (\kappa(u)\nabla u + \boldsymbol{f}(u)) \cdot \boldsymbol{n}\mathrm{d}a = \int_{\omega_i} g(t,x,u)\mathrm{d}x,\ \forall i \in I. \tag{8}$$

We introduce a numerical diffusion coefficient $\widetilde{\kappa} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\widetilde{\kappa}(u,v) = \max(\kappa(u), \kappa(v)). \tag{9}$$

It is easy to show that numerical diffusion coefficient satisfies

A5	$\widetilde{\kappa}(u,v) = \widetilde{\kappa}(v,u),$	symmetry,
	$(\widetilde{\kappa}(u_1,v) - \widetilde{\kappa}(u_2,v))(u_1 - u_2) > 0,$	monotonicity,
	$\widetilde{\kappa}(u,u) = \kappa(u),$	consistency.

Corresponding to each face σ_{ij} we introduce a numerical flux function $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following properties [7]

A6	$\widetilde{f}_{i,j}(u,v) = -\widetilde{f}_{j,i}(v,u),$	conservation,
	$(\widetilde{f}_{i,j}(u_1,v) - \widetilde{f}_{i,j}(u_2,v))(u_1 - u_2) \leq 0,$	monotonicity,
	$(\widetilde{f}_{i,j}(u,v_1) - \widetilde{f}_{i,j}(u,v_2))(v_1 - v_2) \geq 0.$	
	$\widetilde{f}_{i,j}(u,u) = \boldsymbol{f}(u) \cdot \boldsymbol{n}_{i,j},$	consistency.

The space discrete finite volume approximation of the problem (1) is defined from the above numerical diffusion coefficient function and the numerical flux functions by the following system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \displaystyle\sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u_i,u_j)\frac{u_j - u_i}{d_{ij}} + \widetilde{f}_{i,j}(u_i,u_j)\right] + g_i(t,u_i) \\ u_i\big|_{t=0} = u_{0i}, \end{cases} \tag{10}$$

for $t > 0$ and for any $i \in I$. $\mathcal{N}(i)$ denotes all neighbours in $\mathcal{T}^e$ of ω_i, $\mathrm{m}(\omega_i)$ represent the volume of polyhedron ω_i and $\mathrm{m}(\sigma_{ij})$ represent the $n-1$-dimensional measure of the face σ_{ij}.

The source terms $g_i(t,u)$ are defined by

$$g_i(t,u) = \frac{1}{\mathrm{m}(\omega_i)} \int_{\omega_i} g(t,x,u)\mathrm{d}x, \tag{11}$$

and the initial datum u_{0i} are given by

$$u_{0i} = \frac{1}{\mathrm{m}(\omega_i)} \int_{\omega_i} u_0(x)\mathrm{d}x, \tag{12}$$

for all polyhedra ω_i. Let us introduce the numerical global flux functions

$$\mathcal{F}_i(\boldsymbol{u};\boldsymbol{u}_D) = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)} \left[\widetilde{\kappa}(u_i,u_j)\frac{u_j - u_i}{d_{ij}} + \widetilde{f}_{i,j}(u_i,u_j)\right]. \tag{13}$$

Then the ODE approximation read as

$$\frac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \mathcal{F}_i(\boldsymbol{u};\boldsymbol{u}_D) + g_i(t,u_i). \tag{14}$$

The boundary conditions are taken into account by the volume elements next to boundary $\partial\Omega$. For such element the contribution of the boundary values to the $\mathcal{F}_i$ is given by

$$\frac{\mathrm{m}(\sigma_{i_b})}{\mathrm{m}(\omega_i)}\left[\widetilde{\kappa}(u_{i_b},u_j)\frac{u_{i_b}-u_i}{d_{i_b}} + \widetilde{f}_{i,i_b}(u_i,u_{i_b})\right]$$

For shortness denotes by $\boldsymbol{u}^e = (\boldsymbol{u}|\boldsymbol{u}_D)$

Lemma 1. *Assume* **A5** *and* **A6**. *Then*
a)

$$\mathcal{F}_i(\boldsymbol{u}^e) = 0 \tag{15}$$

for any constant state $u_i = u, \forall i \in I^e$.
b) $\boldsymbol{\mathcal{F}}$ *verifies Kamke conditions, that is*

$$\mathcal{F}_i(\boldsymbol{v}^e) \geq \mathcal{F}_i(\boldsymbol{w}^e),\ \forall i \in I \tag{16}$$

for any two vectors that satisfy $v_k \geq w_k, \forall k \in I^e$ *and* $v_i = w_i$.

Proof of Lemma **1**. To prove (15) we have

$$\mathcal{F}_i(\boldsymbol{u}^e) = \sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)}\boldsymbol{f}(u)\cdot\boldsymbol{n}_{ij} = 0$$

To prove the Kamke conditions we have

$$\mathcal{F}_i(\boldsymbol{v}^e) - \mathcal{F}_i(\boldsymbol{w}^e) =$$

$$\sum_{j\in\mathcal{N}(i)} \frac{\mathrm{m}(\sigma_{ij})}{\mathrm{m}(\omega_i)}\left[\widetilde{\kappa}(u,v_j)\frac{v_j-u}{d_{ij}} + \widetilde{f}_{i,j}(u,v_j) - \widetilde{\kappa}(u,w_j)\frac{w_j-u}{d_{ij}} - \widetilde{f}_{i,j}(u,w_j)\right]$$

and from (9) and the monotonicity property of **A6** the affirmation results.

We want to prove that the solutions of ODE (14) are bounded with bounds independent of the mesh size. For that we need the supplementary conditions on source term g and accumulation function b.

Supplementary assumptions on constitutive functions:

A1′
- There exists some real numbers $\underline{\alpha} < \alpha < \beta < \overline{\beta}$ such that
- (1)$b \in C^1((\underline{\alpha},\overline{\beta}))$ and $b' > 0$ on $(\underline{\alpha},\overline{\beta})$
- There exists two Lipschitz functions $\overline{g},\underline{g} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that
- (2) $\underline{g}(t,u) \leq g(t,x,u) \leq \overline{g}(t,u), \forall u \in (\underline{\alpha},\overline{\beta})$
- (3) $\underline{g}(t,\alpha) \leq 0, \overline{g}(t,\beta \geq 0$

Theorem 1 (Boundedness of discrete solutions)**.** *Consider the Cauchy problem* (10)*. Assume* **A1***,* **A1′***,* **A2***,* **A5***,* **A6***. We suppose also that initial conditions and boundary data satisfy*

$$\alpha \leq u_0(x) \leq \beta, \text{a.e } x \in \Omega, \alpha \leq u_D(t,x) \leq \beta, \text{ a.e } (t,x) \in (0,T) \times \Omega. \tag{17}$$

Let $\underline{u}(t)$ *be the solution of the problem*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} = \underline{g}(t,u), \\ \\ u|_{t=0} = \alpha, \end{cases} \tag{18}$$

and $\overline{u}(t)$ *be the solution of the problem*

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} = \overline{g}(t,u), \\ \\ u|_{t=0} = \beta \end{cases} \tag{19}$$

on the interval $(0,T)$*,* [2], [4]*. Let* $T_{sup} = \inf\{\sup\{t|\underline{u}(t) > \underline{\alpha}, \overline{u}(t) < \overline{\beta}\}, T\}$*.*

Then the solution $\boldsymbol{u}(t)$ *of the Cauchy problem is bounded by* $\underline{u}$ *and* $\overline{u}$ *on the interval* $[0, T_{sup}]$ *i.e.*

$$\underline{u}(t) \leq u_i(t) \leq \overline{u}(t)\, \forall i \in I, \forall t \in [0, T_{sup}] \tag{20}$$

Proof of theorem. The essential tool in the proof is the Nickel' theorem that provides the monotony of the solution of the quasimonotone ODE. The Kamke condition assures us that we deal with quassimonoton system.

Observe that the conditions **A1′**–3 guaranty that

$$\underline{\alpha} \leq \underline{u}(t) \leq \alpha,\ \beta \leq \overline{u}(t) \leq \overline{\beta}. \tag{21}$$

Define

$$\underline{\mathcal{F}}_i(\boldsymbol{u}) = \mathcal{F}_i(\boldsymbol{u}; \underline{\boldsymbol{u}}), \overline{\mathcal{F}}_i(\boldsymbol{u}) = \mathcal{F}_i(\boldsymbol{u}; \overline{\boldsymbol{u}}).$$

From (7), (11), (16), (21) and the conditions **A1′**–2 one obtains

$$\underline{\mathcal{F}}_i(\boldsymbol{u}) + \underline{g}(t,u) \leq \mathcal{F}_i(\boldsymbol{u}; \boldsymbol{u}_D) + g_i(t,u) \leq \overline{\mathcal{F}}_i(\boldsymbol{u}) + \overline{g}(t,u)$$

Since $u_i^{sup}(t) = \overline{u}(t), \forall i \in I$ is a solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \overline{\mathcal{F}}_i(\boldsymbol{u}) + \overline{g}(t,u_i), \\ u_i|_{t=0} = \beta, \end{cases} \tag{22}$$

$u_i^{inf}(t) = \underline{u}(t), \forall i \in I$ is a solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}b(u_i)}{\mathrm{d}t} = \underline{\mathcal{F}}_i(\boldsymbol{u}) + \underline{g}(t,u_i), \\ u_i|_{t=0} = \alpha, \end{cases} \tag{23}$$

and $\alpha \leq u_{0i} < \beta$ one can applies the Nickel's theorem and one obtains

$$u_i^{inf}(t) \leq u_i(t) \leq u_i^{sup}(t),$$

which is (20).

Infiltration model

In the case of Richards' equations one can defines the numeric flux functions by

$$\widetilde{f}_{i,j}(u,v) = \frac{1}{2}\left(\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j} + |\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j}|\right) K(v) + \frac{1}{2}\left(\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j} - |\boldsymbol{e}_3 \cdot \boldsymbol{n}_{i,j}|\right) K(u). \qquad (24)$$

Appendix 1. Nickel's Theorem [13], [16]. *Assume that* $\underline{\mathbf{g}}, \mathbf{g}, \overline{\mathbf{g}}$ *are quasi-monotone* $\underline{\mathbf{g}} \leq \mathbf{g} \leq \overline{\mathbf{g}}$ *and* $\underline{\mathbf{u}_0} \leq \mathbf{u}_0 \leq \overline{\mathbf{u}_0}$. *Let be*

$$\begin{array}{llll} \underline{\mathbf{u}} & \text{the solution of} & \dot{\mathbf{v}} = \underline{\mathbf{g}}(\mathbf{v}), & \mathbf{v}(0) = \underline{\mathbf{u}_0}, \\ \mathbf{u} & \text{the solution of} & \dot{\mathbf{v}} = \mathbf{g}(\mathbf{v}), & \mathbf{v}(0) = \mathbf{u}_0, \\ \overline{\mathbf{u}} & \text{the solution of} & \dot{\mathbf{v}} = \overline{\mathbf{g}}(\mathbf{v}), & \mathbf{v}(0) = \overline{\mathbf{u}_0}, \end{array} \qquad (25)$$

then $\underline{\mathbf{u}} \leq \mathbf{u} \leq \overline{\mathbf{u}}$.

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