Accordo di cooperazione scientifica CNR/RA

Report for the project

Control and stabilization problems for phase field and biological systems

Project period: 2017-2019

Team of the project:

Italian part:

Pierluigi Colli (director of project of the Italian part), full professor at the University of Pavia and Research Associate at the IMATI of CNR in Pavia, Dipartimento di Matematica, Universita' di Pavia, Via Ferrata 1, 27100 Pavia, Italy. E-mail: pierluigi.colli@unipv.it

Angelo Favini, full professor, University of Bologna and Research Associate at the IMATI of CNR in Pavia. E-mail: angelo.favini@unibo.it

Alberto Gandolfi, Research Associate, Istituto di Analisi dei Sistemi ed informatica "A. Ruberti" - CNR, Via dei Taurini, 19, 00185 Roma. E-mail: alberto.gandolfi@iasi.cnr.it

Gianni Gilardi, full professor at the University of Pavia and Research Associate at the IMATI of CNR in Pavia. E-mail: gianni.gilardi@unipv.it

Mimmo Iannelli, Research Associate, IMATI of CNR in Pavia. E-mail: iannelli@science.unitn.it

Elisabetta Rocca, professor, University of Pavia and Research Associate at the IMATI of CNR in Pavia. E-mail: elisabetta.rocca@unipv.it

Cecilia Cavaterra, associate professor, University of Milano and Research Associate at the IMATI of CNR in Pavia. E-mail: cecilia.cavaterra@unimi.it

Alessandro Reali, professor, University of Pavia and Research Associate at the IMATI of CNR in Pavia. E-mail: alereali@unipv.it (last year)

Guillermo Lorenzo, post-doc researcher University of Pavia and Research Associate at the IMATI of CNR in Pavia. E-mail: guillermo.lorenzo@unipv.it (last year)

Romanian part:

Gabriela Marinoschi (director of project of the Romanian part), senior researcher I, Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, Bucuresti, Romania. E-mail: gabriela.marinoschi@acad.ro

Viorel Barbu - senior researcher I, Octav Mayer Institute of Mathematics of the Romanian Academy, Bdul. Carol I, 8 Iasi, Romania. E-mail: vb41@uaic.ro

Ionut Munteanu, researcher, Octav Mayer Institute of Mathematics of the Romanian Academy, Bdul. Carol I, 8 Iasi, Romania. E-mail: ionut.munteanu@uaic.ro

I. Project overview

The project has addressed well-posedness, control and optimization problems related to some classes of nonlinear parabolic equations and systems that describe processes of major interest in phase transition/ separation and biology. Essentially we referred to optimal control problems, feedback stabilization, controllability and sliding mode control and related well-posedness theories. The classes of nonlinear systems we studied consist in two or more nonlinear diffusion equations intrinsically coupled, as in phase transition and separation, or dynamic population (cell growth) models. The topics developed in this project have both a theoretical and practical importance, since it has been oriented towards a research line of current international interest. Dissemination of results have been done by publication in first class journals**.**

II. Objectives of research

From the mathematical viewpoint, we aimed at developing methods for the study of nonlinear PDEs, some of them with nonlocal boundary conditions or describing phenomena with a free boundary evolution. The investigated models gave the possibility of a better understanding of some peculiarities of the concerned physical and biological processes. We obtained efficient procedures for solving in a different way the PDEs and associated control problems, and for devising some numerical methods that lead to the development of computer codes.
In particular, the main objectives were:

1. Relevant optimal control problems in relation with phase field (existence of the control, well-posedness of the direct and dual systems, determination of the necessary optimality conditions);
2. Sliding mode control of dynamical systems involving a phase variable;
3. Feedback stabilization of phase-field systems;
4. Modeling cell growth and numerical computation of the cell growth system of equations in the nonstationary case: discretization, computation of the solutions along characteristics, exploration of simulation scenarios giving importance to the various parameters of the model.

The impact of the project in the main domain of research, that is mathematics, has been ensured by the publication of original results in high impact journals. During the project we have published 6 papers and submitted one, all in Web of Science (WOS) journals, according to the UEFISCDI classification:

- 4 classified in the 25% top journals, Q1 or red area
- 2 classified in the 25% top journals, Q2 or yellow area
- 1 in a WOS journal.

III Achievement of the objectives

1.Relevant optimal control problems in relation with phase field

Phase transitions are thermodynamic processes in which under the action of some external conditions, such as temperature, pressure, or others, a transition between two or multiple phases of the same matter occurs (as solid to liquid, liquid to gaseous and conversely). A phase is a configuration in which matter has uniform physical properties. Phase separation is a process in which a mixture of two or more components spontaneously separates in its components. These models include equations for the order parameter and for energy and/or momentum balance, with initial and boundary conditions. A certain richness of nonlinearities and somehow unpleasant terms in the equations makes the investigation of these problems particularly interesting from the mathematical point of view. In this project we studied optimal control problems and controllability of various models of these types. Controllability is a challenging problem, especially for systems of equations: it involves solvability issues, analysis of eigenvalues/eigenvectors of the linearized system, and accurate estimates. Special (nonlocal) boundary conditions may complicate the study requiring adapted mathematical techniques.

In the paper [p1] in the list below we studied a distributed control problem for a phase-field system of conserved type with a possibly singular potential. We mainly handled two cases: the case of a viscous Cahn-Hilliard type dynamics for the phase variable in case of a logarithmic-type potential with bounded domain and the case of a standard Cahn-Hilliard equation in case of a regular potential with unbounded domain, like the classical double-well potential, for example. Necessary first order conditions of optimality were derived under natural assumptions on the data.

In the paper [p2] we reviewed some results obtained for a distributed control problem regarding a class of phase field systems of Caginalp type with logarithmic potential. The aim of the control problem is forcing the location of the diffuse interface to be as close as possible to a prescribed set. However, due to some discontinuity in the cost functional, we regularized it and solved the related control problem for the approximation. We discussed the necessary optimality conditions.

2. Sliding mode control of dynamical systems involving a phase variable

Sliding mode control (SMC) is one of the fundamental approaches for the systematic design of robust controllers for nonlinear complex dynamic systems that operate under uncertainty and it is considered a classical tool for the control of continuous or discrete time systems in finite-dimensional settings. It has the advantage of controlling the separation of the motion of the overall system in independent partial components of lower dimensions, thus reducing the complexity of the problem. The design of feedback control systems with sliding modes implies the design of suitable control functions enforcing motion along ad-hoc manifolds. The study takes into account the identification of a manifold of lower dimension (called the sliding manifold) where the control goal is fulfilled, such that the original system restricted to this sliding manifold has a desired behavior, and the design of a SMC-law that forces the trajectories of the system to reach the sliding surface and to maintain them on it. This technique works especially for systems of equations describing certain processes, so that it perfectly fits to the models of nonlinear coupled equations we aim to study.

In the paper [p3] the sliding mode control (SMC) problem for a phase-field model of Caginalp type was considered. First we proved the well-posedness and some regularity results for the phase-field type state systems modified by the state-feedback control laws. Then, we showed that the chosen SMC laws force the system to reach within finite time the sliding manifold (that we chose in order that one of the physical variables or a combination of them remains constant in time). We studied three different types of feedback control laws: the first one appears in the internal energy balance and forces a linear combination of the temperature and the phase to reach a given (space dependent) value, while the second and third ones are added in the phase relation and lead the phase onto a prescribed target. While the control law is nonlocal in space for the first two problems, it is local in the third one, i.e., its value at any point and any time just depends on the value of the state.

In the paper [p4] we studied the sliding mode control (SMC) problem for a diffuse interface tumor growth model coupling a viscous Cahn–Hilliard type equation for the phase variable with a reaction–diffusion equation for the nutrient. First, we proved the well-posedness and some regularity results for the state system modified by the state-feedback control law. Then, we showed that the chosen SMC law forces the system to reach within finite time the sliding manifold (that we chose in order that the tumor phase remains constant in time). The feedback control law was added in the Cahn–Hilliard type equation and led the phase onto a prescribed target for the phase in finite time.

In the paper [p5] we dealt with a feedback control design for the action potential of a neuronal membrane in relation with the non-linear dynamics of the Hodgkin-Huxley mathematical model. More exactly, by using an external current as a control expressed by a relay graph in the equation of the potential, we aimed at forcing it to reach a certain manifold in finite time and to slide on it after that. From the mathematical point of view we solved a system involving a parabolic differential inclusion and three nonlinear differential equations via an approximating technique and a fixed point result. The existence of the sliding mode and the determination of the time at which the potential reaches the prescribed manifold were proved by a maximum principle argument. Numerical simulations were presented.

3. Feedback stabilization of phase-field systems

Stabilization is a challenging problem, especially for systems of equations: it involves solvability issues, analysis of eigenvalues/eigenvectors of the linearized system and accurate estimates. Special (nonlocal) boundary conditions may complicate the study requiring adapted mathematical techniques.

In the paper [p6] a first contribution on the problem of boundary stabilisation for the phase field system of Cahn–Hilliard type, which models the phase separation in a binary mixture has been done. The feedback controller designed here is with actuation only on the temperature flow of the system, on one part of the boundary only. Moreover, it is of proportional type, given in an explicit form, expressed only in terms of the eigenfunctions of the Laplace operator, being easy to manipulate from the computational point of view. Furthermore, it ensures that the closed loop nonlinear system exponentially reaches the prescribed stationary solution provided that the initial datum is close enough to it.

4. Modeling cell growth and dynamic population models

We also considered systems of similar types describing cell growth phenomena. We introduced a new model of epidermis growth insisting on the modeling of new particular biological aspects occurring in the basal layer. This is represented by a system of many nonlinear hyperbolic equations with a free boundary and nonlocal boundary conditions.

In the paper [p7] an age-structured model for the cell populations of the epidermis basal layer was proposed, with the aim of quantifying the cell production which makes possible epidermis renewal. The model, that extends a previous model of the same authors, describes the dynamics of proliferating cells, differentiated cells and apoptotic cells under the physical constraint on the total surface cell density imposed by the cell occupancy of basement membrane. To fulfil this constraint during any dynamics, the surface cell density was assumed to control cell migration towards the epidermis suprabasal region, as well as the balance between production of proliferating and of differentiated cells at cell division. The well-posedness of the model was fully studied, and the unique steady state is characterized. Numerical simulations were provided for realistic values of the parameters, giving some insight into the stability of the equilibrium and into the homeostatic properties of the model.

IV. List of published papers

[p1] P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, Optimal control for a conserved phase field system with a possibly singular potential, **Evol. Equ. Control Theory**, 7, 1, 95-116, 2018. WOS journal, Q1, red area in 2018, AIS = 0.942, IF = 1.231; doi: 10.3934/eect.2018006

[p2] P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, Distributed optimal control problems for phase field systems with singular potential, **An. Stiint. Univ. "Ovidius" Constanta Ser. Mat**. 26, 2, 71-85, 2018.
WOS journal, AIS = 0.232, IF = 0.695

[p3] V. Barbu, P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, *Sliding mode control for a nonlinear phase-field system,* **SIAM J. Control Optim**. 55, 3, 2108-2133, 2017
WOS journal, Q1, AIS = 2.231, IF = 1.363; https://doi.org/10.1137/15M102424X

[p4] P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, Sliding mode control for phase field system related to tumor growth, **Appl. Math.Optimiz**. 79, 3, 647–670, 2019.
WOS journal, Q1, red area, AIS = 1.788, IF = 1.301; DOI: 10.1007/s00245-017-9451-z

[p5] C. Cavaterra, D. Enăchescu, G. Marinoschi, Sliding mode control of the Hodgkin-Huxley mathematical model, Evol. Equ. Control Theory, 8, 4, 883-902, 2019
WOS journal, Q2, yellow area in 2019, AIS = 0.834, IF = 1.142; doi:10.3934/eect.2019043

[p6] P. Colli, G. Gilardi & I. Munteanu, Stabilization of a linearized Cahn–Hilliard system for phase separation by proportional boundary feedbacks, Internat. J. Control, to appear (2020)
WOS journal, Q2, yellow area in 2019, AIS = 1.236, IF = 0.764;
doi:10.1080/00207179.2019.1597280

[p7] A. Gandolfi, M. Iannelli, G. Marinoschi, The basal layer of the epidermis: a mathematical model for cell production under a surface density constraint, **SIAM J. Applied Mathematics**, accepted

WOS journal, Q1, red area in 2019, AIS = 1.395, IF = 0.995

Dr. Gabriela Marinoschi

Director of project for the Romanian partner