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Existence / Non-existence of Acceleration Waves in Third Grade Fluids

by
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Abstract

This paper deals, in the first part, with the existence of harmonic waves in polynomial third grade fluids. The main results, based on some older remarks of the author (see Tigoiu [5]), concern the existence / non-existence the propagation of discontinuities, like spherical and cylindrical acceleration waves, all important cases (referring to the signum of the constitutive coefficient α_1). It was proved that, like in the case of linear viscous fluids, acceleration waves (spherical and cylindrical) do not propagate for $\alpha_1 > 0$. It is proved also that, these discontinuities can propagate if the subclass of third grade fluids with $\alpha_1 < 0$ is considered.

Key words and phrases: propagation of singularities, harmonic waves, acceleration waves, third grade fluids
Mathematics Subject Classification: 76A05, 76D33, 76A99

1 Introduction

In this paper we analyze some problems related to wave propagation in non-newtonian third grade fluids.
It is important to remind that, on a part we expect that harmonic waves will propagate, while acceleration waves do not (like in linear viscous fluids, see Truesdell and Toupin [4] and Dragos [2], for instance). Older results of Tigoiu [5] give some ideas concerning the subject. On an other part, due to the presence of higher order time derivatives in flow equations for a third grade fluid, we expect that, acceleration waves can propagate with finite velocities, at least for some subclasses.
We employ the known constitutive law for third grade fluids (see Fosdick and Rajagopal [7], or Tigoiu [5], [6]).

$$\begin{aligned}\mathbf{T}(\mathbf{x},t) = \quad & -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2) + \beta_1\mathbf{A}_3 + \\ & +\beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1,\end{aligned} \tag{1}$$

in the sense given in [5]. Consequently, the constitutive moduli must obey the following restrictions (obtained as direct consequences from Clausius-Duhem's inequality)

$$\beta_1 < 0,\ \beta_1 \quad +2(\beta_2 + \beta_3) \geq 0,\ \mu \geq 0,\ \alpha_1 + \alpha_2 = 0. \tag{2}$$

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We remark also, concerning third grade fluids, that employing the simplified constitutive equation obtained in [7], Fosdick and Straugham have proved in [8] that catastrophic instabilities can occur in flows of such fluids. Recently, Tigoiu [6] proved that for the constitutive equation given in (1) with constitutive restrictions (2), the rest state is asymptotically stable, for any signum of the constitutive modulus α_1, at least for weakly perturbed flows.
In the present paper we employ the constitutive equations (1) and we write flow equations in the case of weakly perturbed flows (as it was written in [6], for instance), that is

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} - \mu \mathrm{div}\mathbf{A_1} - \alpha_1 \frac{\partial}{\partial t}\mathrm{div}\mathbf{A_1} - \beta_1 \frac{\partial^2}{\partial t^2}\mathrm{div}\mathbf{A_1} + \mathrm{grad}p = \rho\mathbf{b}, \\ &\mathrm{div}\mathbf{v} = 0. \end{aligned} \tag{3}$$

On these basis we analyze below the existence or the non-existence of various types of waves and its propagation.

2 Harmonic waves

A first question when we study wave propagation in continuum materials is to see if harmonic (smooth) waves propagate in such a body. For this, we suppose the whole space filled with our fluid and we are interested to see if harmonic plane waves, with known frequency $\omega \in \mathbf{R}_+$ can propagate.
We consider that the propagation has place in the absence of external forces and than, the linearized flow equations can be easily obtained from (3)

$$\begin{aligned} &\rho\frac{\partial v_i}{\partial t} - \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \alpha_1 \frac{\partial}{\partial t}\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \beta_1 \frac{\partial^2}{\partial^2 t}\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial p}{\partial x_i} = 0, \\ &\frac{\partial v_j}{\partial x_j} = 0, \quad i, j = \overline{1,3}. \end{aligned} \tag{4}$$

We follow the classic way (see [1], [2]) to obtain the dissipation relation, that is the relation between the complex wave number $k \equiv k_r + ik_i$ and the frequency ω

$$k = f(\omega) \tag{5}$$

We suppose that waves propagate in the direction of unitary vector $\mathbf{n}$ and are described by

$$v_e = \overline{V}_e exp(i(k\mathbf{n}\cdot\mathbf{x} - \omega t)), \quad p = \overline{p}_0 + \overline{p} exp(i(k\mathbf{n}\cdot\mathbf{x} - \omega t)). \tag{6}$$

A straight and simple calculus proves that

$$(\frac{\partial}{\partial t}, \frac{\partial}{\partial t^2}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j^2}) \longrightarrow (-i\omega, -\omega^2, ikn_j, -k^2). \tag{7}$$

Finally the system (4) has the form

$$\begin{aligned} &\Big[(\mu - i\alpha_1\omega - \beta_1\omega^2)k^2 - i\rho\omega\Big]\overline{v}_e + ikn_e\overline{p} = 0, \\ &ikn_e\overline{v}_e = 0, \quad e = \overline{1,3}. \end{aligned} \tag{8}$$

The condition to have effective waves (non-null wave amplitudes) is given by

$$\left[(\mu - i\alpha_1\omega - \beta_1\omega^2)k^2 - i\rho\omega\right]^2 k^2 = 0. \tag{9}$$

A simple overview on the relation (9) put into evidence two distinct cases
A. k = 0. In this case two situation are to be put into evidence:
A1. $\omega = 0$, that is, the fluid is in a rest state;
or
A2. $\overline{v}_e = 0, j = \overline{1,3}$ and consequently we cannot propagate waves quad in our fluid, but $p = \overline{p}_0 + \overline{p}\ exp(-i\omega t)$ (that is, the constant phase velocity is unbounded).
B. $\left((\mu - i\alpha_1\omega - \beta_1\omega^2)k^2 - i\rho\omega\right) = 0$ and $k \neq 0$. In this case, from (8) we have $p = \overline{p}_0$ and $\mathbf{v} \cdot \mathbf{n} = 0$. Consequently there are transverse waves to the **n** direction propagating in our fluid. More, the dissipation relation (5) becomes

$$k^2\left[(\mu - \beta_1\omega^2 - i\alpha_1\omega)\right] = i\rho\,\omega. \tag{10}$$

This relation leads to a propagation velocity of the constant phase of transverse waves given by

$$c \equiv \frac{\omega}{k_r} = \pm\sqrt{\frac{2\left[\alpha_1\omega + \sqrt{\alpha_1^2\omega^2 + (\mu - \beta_1\omega^2)^2}\right]\left[\alpha_1^2\omega^2 + (\mu - \beta_1\omega^2)^2\right]\omega}{\rho(\mu - \beta_1\omega^2)}} \tag{11}$$

The relation (11) clearly proves that the absolute value of the velocity **c** is an increasing function of the frequency and the dependence is of the order $O(\omega^{3/2})$.

3 Propagation of singularity surfaces. Acceleration waves.

It is well known that flow equations for a linear viscous fluid have a parabolic character and consequently singular (discontinuity) surfaces propagate with an infinite velocity (which is not a correct response from a physical point of view). Due to the presence in the constitutive law (1) of the second time derivative, we hoped that some discontinuities can propagate in such a fluid. The analysis below (considering the cases of spherical and cylindrical acceleration waves) will prove that these waves have the speed of propagation exponentially increasing in time, which is a similar result with the one obtained for the linear viscous fluid.
The present result is based on the restriction due to the employment of Clausius - Duhem's inequality ($\beta_1 < 0$). In our case a similar result is obtained if we employ Müler's modified Clausius - Duhem inequality.

3.1 Spherical waves

In this chapter we suppose that we have a weak perturbation of the rest state and consequently we will consider the linearized form for the constitutive law as well as for flow equations (see (3)). We analysis if through a fluid described by (3_1) can propagate third order acceleration waves. We suppose that we have a regular surface S($\mathbf{x}$, t) = 0 such as across this one, the velocity field as well as its first and second order derivatives are continuous functions and third and forth order derivatives of the velocity field have discontinuities. The domain Ω occupied by the fluid is divided by the surface S into two region denoted with: Ω^+ and Ω^-. We employ classical notations for the jump of a function f across a surface S:

$$[f(\mathbf{x},t)] = f^+\mathbf{x},t) + f^-\mathbf{x},t). \tag{12}$$

We employ also the known geometric and kinematic compatibility relations for the jumps of first and second order derivatives (see [3] and [4]) in order to right down

$$\begin{aligned} &\left[\frac{\partial f}{\partial x^i}\right] = An_i \\ &\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \bar{A}n_i n_j + n_i x_{j|}{}^{\alpha} A|_{\alpha} + n_j x_{i|}{}^{\alpha} A|_{\alpha} - x_{i|}{}^{\alpha} x_{j|}{}^{\beta} b_{\alpha\beta} A \end{aligned} \tag{13}$$

and respectively

$$\begin{aligned} &\left[\frac{\partial f}{\partial t}\right] = -uA \\ &\left[\frac{\partial^2 f}{\partial t \partial x^i}\right] = (-u\bar{A}) + \frac{\delta A}{\delta t} n_i - a^{\alpha\beta} x_{i|\alpha}(uA)|_{\beta} \\ &\left[\frac{\partial^2 f}{\partial t^2}\right] = u^2 \bar{A} - 2u\frac{\delta A}{\delta y} - A\frac{\delta u}{\delta t}, \end{aligned} \tag{14}$$

where we have employed the following notation (in this chapter $i, j = \overline{1,3}$ and $\alpha, \beta = \overline{1,2}$)

$$A \equiv \left[\frac{\partial f}{\partial x^i} n_i\right], \quad \bar{A} \equiv \left[\frac{\partial^2 f}{\partial x^i \partial x^j} n^i n^j\right], \quad \frac{\delta}{\delta t}[f] \equiv \left[\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x^i} n^i\right] \tag{15}$$

In the above formulas x^i, x_i are components of the vector $\mathbf{x}$ in the local basis $\mathbf{e}_i$, or in its reciprocal basis $\mathbf{e}^i$, "$(\cdot)|$" is the covariant derivative, ($a^{\alpha\beta}$) represents the surface metric, ($b_{\alpha\beta}$) is the second fundamental form and u is the propagation velocity, given by

$$u = \frac{\partial x^i}{\partial t} n_i. \tag{16}$$

The equation of the surface S is given (in spherical coordinates) by

$$\mathbf{x}(t) = R(t)[cos\theta\, cos\phi\, \mathbf{i}_1 + sin\theta\, sin\phi\, \mathbf{i}_2 + sin\phi\, \mathbf{i}_3. \tag{17}$$

Space metric, surface metric and the second fundamental form are simply computed as

$$\begin{aligned} g_{ij} &= \delta_{ij}[\delta_{i1} + \delta_{i2}R^2(t)cos^2\phi + \delta_{i3}R^t], \\ a^{\alpha\beta} &= \delta_{\alpha\beta}(\delta_{\alpha 1}\frac{1}{R^2(t)cos^2\phi} + \delta_{\alpha 2}\frac{1}{R^2(t)}), \\ b_{\alpha\beta} &= \delta_{\alpha\beta}(-\delta_{\alpha 1}R(t)cos^2\phi - \delta_{\alpha 2}R(t)). \end{aligned} \tag{18}$$

In formulas (18) the summation index convention has been not employed. We remind that the position vector of a point is given by $\mathbf{x}(t) = R(t)\mathbf{e}_1 =$ $= R(t)\mathbf{e}^1$ and its derivatives are $x_i|^{\ \beta} = a^{\alpha\beta}\frac{\partial}{\partial y^\alpha}(g_{ij}x^j)$. These formulas and (17) lead to: $x_{2|1} = R(t)cos^2\phi,\ x_{3|2} = R^2(t),\ x_{1|1} = x_{1|2} = x_{3|1} = x_{2|2} = 0.$

To give a complete explanation for kinematic jump relations (14), we remark that the velocity is a vector field and consequently scalar amplitudes A and $\bar{A}$ must be replaced with some vectorial amplitudes. This last ones are constructed as follows: we introduce, for simplicity, the vector field

$$\mathbf{f}(\mathbf{x}, t) \equiv \frac{\partial^2 \mathbf{v}}{\partial x^i \partial x^j} \tag{19}$$

and then we define the vector amplitudes $\mathbf{A}$, $\bar{\mathbf{A}}$ by

$$\mathbf{A} \equiv \left[\frac{\partial \mathbf{f}}{\partial x^i}n^i\right] \equiv [(grad\mathbf{f})\mathbf{n}], \quad \bar{\mathbf{A}} \equiv \left[\frac{\partial^2 \mathbf{f}}{\partial x^i \partial x^j}n^i n^j\right]. \tag{20}$$

Having in mind that $\mathbf{f}$ is a continuous field and employing also the previous results we obtain simply $[(grad\mathbf{f})\mathbf{n}] = [f^k;1]\mathbf{e}_k = \left[\frac{\partial f^k}{\partial x_1}\right]\mathbf{e}_k$. Some straight calculi give:
$[(grad^2\mathbf{f}(\mathbf{n}))\mathbf{n}] = [g_{kt}(\frac{\partial^2 f^t}{\partial x^4 \partial x^4} + 2\frac{\partial f^q}{\partial x_1}\Gamma^t_{q1} - \frac{\partial f^t}{\partial x_p}\Gamma^p_{11})]\mathbf{e}^k$. As $\Gamma^2_{21} = \Gamma^3_{31} = 1/R$ and all other are null we obtain finally for the representation of $\bar{\mathbf{A}}$

$$\bar{\mathbf{A}} = \left(\left[\frac{\partial^2 f^1}{\partial x^1 \partial x^1}\right], \left[\frac{\partial^2 f^2}{\partial x^1 \partial x^1}\right] + \frac{2}{R}\left[\frac{\partial f^2}{\partial x^1}\right], \left[\frac{\partial^2 f^3}{\partial x^1 \partial x^1}\right] + \frac{2}{R}\left[\frac{\partial f^3}{\partial x^1}\right]\right). \tag{21}$$

After some long but straightforward calculi we obtain for the second term from $(14)_3$

$$\begin{aligned} \frac{\delta \mathbf{A}}{\delta t} = \Bigg(\left[\frac{\partial^2 f^1}{\partial x^1 \partial x^1}\right], \left[\frac{\partial^2 f^2}{\partial x^1 \partial x^1}\right] &+ \frac{1}{R}\left[\frac{\partial f^2}{\partial t}\right] + \frac{u}{R}\left[\frac{\partial f^2}{\partial x^1}\right], \left[\frac{\partial^2 f^3}{\partial x^1 \partial x^1}\right] + \\ &+ \frac{1}{R}\left[\frac{\partial f^3}{\partial t}\right] + \frac{u}{R}\left[\frac{\partial f^3}{\partial x^1}\right]\Bigg) + u\bar{\mathbf{A}}. \end{aligned} \tag{22}$$

We employ now the flow equations and the definition of the propagation velocity $u = \frac{\partial x^1}{\partial t}$ and after a long calculus we obtain the system of equation which gives the propagation condition

$$(\alpha_1 u + \dot{u}\beta_1)\left[\frac{\partial f^1}{\partial x^1}\right] + \beta_1 u^2\left[\frac{\partial^2 f^1}{\partial x^1 \partial x^1}\right] + 2\beta_1 u\left[\frac{\partial^2 f^1}{\partial x^1 \partial t}\right] = 0,$$

$$(\alpha_1 u + 2\beta_1\frac{u^2}{R} + \dot{u}\beta_1)\left[\frac{\partial f^2}{\partial x^1}\right] + \beta_1 u^2\left[\frac{\partial^2 f^2}{\partial x^1 \partial x^1}\right] + 2\beta_1 u\left[\frac{\partial^2 f^2}{\partial x^1 \partial t}\right] = 0, \quad (23)$$

$$(\alpha_1 u + 2\beta_1\frac{u^2}{R} + \dot{u}\beta_1)\left[\frac{\partial f^3}{\partial x^1}\right] + \beta_1 u^2\left[\frac{\partial^2 f^3}{\partial x^1 \partial x^1}\right] + 2\beta_1 u\left[\frac{\partial^2 f^3}{\partial x^1 \partial t}\right] = 0.$$

We simply remark that if $\left[\frac{\partial f^k}{\partial x^1}\right] = 0$ then from (13), $(14)_1$ and the remark concerning $[(grad\mathbf{f})\mathbf{n}]$ it results that a third order derivatives jumps of $\mathbf{v}$ are null and the problem degenerates. System (23) has three equations and nine unknowns. Consequently for the determination of the propagation velocity we may have one of the following two equations:

$$\alpha_1 u + \beta_1\dot{u} = 0,$$

$$\beta_1\dot{u} + 2\beta_1\frac{u^2}{R} + \alpha_1 u = 0. \quad (24)$$

A. The first one of equations (24) leads to a propagation velocity given by

$$u(t) = u_0\, exp(-\frac{\alpha_1}{\beta_1}t) \quad (25)$$

and consequently as from constitutive restrictions $\beta_1 < 0$, we have the following cases:
A_1: if $\alpha_1 < 0$, (which is the case for some fluids - see [9]) the propagation velocity vanishes when the time tends to infinity.
A_2: if $\alpha_1 > 0$, it results that the wave velocity is an increasing and unbounded function of time, which is a result similar to those for the linear viscous fluid, mentioned above.
B. The second of equations (24) leads to a propagation velocity given by

$$u(t) = \frac{3\,c\,exp(-\frac{\alpha_1}{\beta_1}t)}{\sqrt[3]{c' - 3\,c\frac{\beta_1}{\alpha_1}\,exp(-\frac{\alpha_1}{\beta_1}t)}}, \quad (26)$$

where c, c' have to be obtained from initial data.
Let us remark that in this case the propagation velocity behaves like $exp(-\frac{\alpha_1}{2\beta_1}t)$ when t tends to infinity and consequently there are also two possibilities:
B_1: if $\alpha_1 < 0$, the propagation velocity vanishes when the time tends to infinity.
B_2: if $\alpha_1 > 0$, it results that the wave velocity is an increasing and unbounded function of time.

3.2 Cylindrical waves

In this section we investigate the case of a cylindrical discontinuity surface given by

$$\mathbf{x}(t) = R(t)\,cos\theta\,\mathbf{i}1 + R(t)\,sin\theta\mathbf{i}_2 + z\mathbf{i}_3 = 0 \quad (27)$$

In the local basis $\mathbf{e}_i,\ i=\overline{1,3}$ and in the corresponding surface basis, the coordinates are given by $x^1=R(t),\ x^2=\theta,\ x^3=z$ and respectively by $y^1=\theta,\ y^2=z$. In the same time the space and surface metrics and the second fundamental form are obtained as in formulas (28)

$$
\begin{aligned}
&g_{ij}=\delta_{ij}(\delta_{i1}+R^2\delta_{i2}+\delta_{i3}), \quad a_{\alpha\beta}=\delta_{\alpha\beta}(\delta_{\alpha 1}R^2+\delta_{\alpha 2}),\\
&b_{\alpha\beta}=\delta_{\alpha\beta}(-\delta_{\alpha 1}R).
\end{aligned}
\tag{28}
$$

A simple calculus gives, for the surface gradient components: $x_{2|1}=R(t)$, $x_{3|2}=1,\ x_{1|1}=x_{1|2}=x_{3|1}=x_{2|2}=0$. In order to obtain the right hand side of jump relations (14) we remark that $\Gamma^2_{12}=\Gamma^2_{21}=1/R$ and then we must compute $\bar{\mathbf{A}}$, $\dfrac{\delta\mathbf{A}}{\delta t}$ and $\dfrac{\delta u}{\delta t}$. After some straightforward calculi we obtain a formula similar to those performed in the case of spherical waves: $[(grad^2\mathbf{f}(\mathbf{n}))\mathbf{n}]=[g_{kt}(\dfrac{\partial^2 f^t}{\partial x^4\partial x^4}+2\dfrac{\partial f^q}{\partial x_1}\Gamma^t_{q1}-\dfrac{\partial f^t}{\partial x_p}\Gamma^p_{11})]\mathbf{e}^k$ and the representation, in the local basis, of $\bar{\mathbf{A}}$ is

$$
\bar{\mathbf{A}}=\left(\left[\frac{\partial^2 f^1}{\partial x^1\partial x^1}\right],\left[\frac{\partial^2 f^2}{\partial x^1\partial x^1}\right]+\frac{2}{R}\left[\frac{\partial f^2}{\partial x^1}\right],\left[\frac{\partial^2 f^3}{\partial x^1\partial x^1}\right]\right).
\tag{29}
$$

For the second term from $(14)_3$, we obtain

$$
\frac{\delta\mathbf{A}}{\delta t}=\left(\left[\frac{\partial^2 f^1}{\partial x^1\partial x^1}\right],\left[\frac{\partial^2 f^2}{\partial x^1\partial x^1}\right],\left[\frac{\partial^2 f^3}{\partial x^1\partial x^1}\right]+u\bar{\mathbf{A}}\right).
\tag{30}
$$

Like in the case of the spherical surface, the definition of the propagation velocity leads to $u(t)=\dot{R}(t)$ and consequently the compatibility kinematic relations for second order derivatives will be

$$
\begin{aligned}
\left[\frac{\partial^2\mathbf{f}}{\partial t^2}\right] &= \Bigg(-u^2\left[\frac{\partial^2 f^1}{\partial x^1\partial x^1}\right]-2u\left[\frac{\partial^2 f^1}{\partial t\partial x^1}\right]-\dot{u}\left[\frac{\partial f^1}{\partial x^1}\right],\\
&-u^2\left[\frac{\partial^2 f^2}{\partial x^1\partial x^1}\right]-\frac{2u}{R}\left[\frac{\partial f^2}{\partial x^1}\right]-2u\left[\frac{\partial^2 f^2}{\partial t\partial x^1}\right]-\dot{u}\left[\frac{\partial f^2}{\partial x^1}\right],\\
&-u^2\left[\frac{\partial^2 f^3}{\partial x^1\partial x^1}\right]-2u\left[\frac{\partial^2 f^3}{\partial t\partial x^1}\right]-\dot{u}\left[\frac{\partial f^3}{\partial x^1}\right]\Bigg)
\end{aligned}
\tag{31}
$$

in the local basis $\mathbf{e}_i,\ i=\overline{1,3}$. Some long calculi lead to the system of equation which give the propagation conditions, which are in strong similarity with equations (23) for spherical waves

$$
\begin{aligned}
&(\alpha_1 u+\dot{u}\beta_1)\left[\frac{\partial f^1}{\partial x^1}\right]+\beta_1 u^2\left[\frac{\partial^2 f^1}{\partial x^1\partial x^1}\right]+2\beta_1 u\left[\frac{\partial^2 f^1}{\partial x^1\partial t}\right]=0,\\
&(\alpha_1 u+2\beta_1\frac{u^2}{R}+\dot{u}\beta_1)\left[\frac{\partial f^2}{\partial x^1}\right]+\beta_1 u^2\left[\frac{\partial^2 f^2}{\partial x^1\partial x^1}\right]+2\beta_1 u\left[\frac{\partial^2 f^2}{\partial x^1\partial t}\right]=0,\\
&(\alpha_1 u+\dot{u}\beta_1)\left[\frac{\partial f^3}{\partial x^1}\right]+\beta_1 u^2\left[\frac{\partial^2 f^3}{\partial x^1\partial x^1}\right]+2\beta_1 u\left[\frac{\partial^2 f^3}{\partial x^1\partial t}\right]=0.
\end{aligned}
\tag{32}
$$

Consequently is not surprising that for the determination of the propagation velocity we will obtain the same two differential equations as in the case of spherical waves (24):

$$\begin{aligned} &\alpha_1 u + \beta_1 \dot{u} = 0, \\ &\beta_1 \dot{u} + 2\beta_1 \frac{u^2}{R} + \alpha_1 u = 0. \end{aligned} \tag{33}$$

Finally, the conclusions are similar with those discussed in cases **A** and **B** in the analysis of the propagation of spherical waves.

4 Conclusions

In conclusion we remark that on a part, in a third grade fluid is generally possible to propagate harmonic waves.
Considering the propagation of singular surfaces, on the other part, we remind that it was proved (see for instance [4], or any classical book referring to the propagation of singularities in viscous fluids) that, due to the parabolic behaviour of flow equations, these surfaces do not propagate in linear viscous fluids. A similar result was obtained here for the subclass of third grade fluids characterized by $\beta_1 < 0,\ \alpha_1 > 0$. The last result, in the previous section, put into evidence an other subclass of these fluids, namely: $\beta_1 < 0,\ \alpha_1 < 0$, for which at least spherical and cylindrical accelerations waves can propagate with finite velocities.
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