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Incompressible flow of the molten powder in meniscus zone of continuous casting mold

by
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Abstract

The aim of this work is to propose a mathematical model for describing the lubrication by the molten powder of the mold-strand gap in the vicinity of the meniscus. The molten powder is considered an incompressible, viscous fluid and the flow is modeled by the Reynolds equation. We study here the simplified case when the surface between the powder and the steel is supposed to be rigid. Some existence and uniqueness results are then established using the variational formulation of the problem. Finally, we analyse the pressure distribution of the molten powder film and we study the effect of the powder viscosity on the powder film pressure. We also discuss the relation between the molten powder and the solidifying shell of the strand.

Key words: molten powder, continuous casting mold, Reynolds equation, variational formulation, pressure distribution

1 Introduction

Steel is cast continuously using an oscillating mold and a lubricant, called the powder. Despite its importance, the mechanism of mold lubrication by the molten powder is not fully clarified. The purpose of this paper is to study, from a mathematical point of view, the hydrodynamic behaviour of the molten powder and to analyse the pressure distribution of the powder film with respect to the powder viscosity and to the thickness of the powder film.

This work deals with the simplified case when the surfaces to be lubricated are supposed to be as those of rigid bodies. This model fails to explain the relation between the deformation of the solidifying shell and the pressure of the molten powder film.

In a forthcoming paper we shall study a more complicated problem, the case when the solidifying shell is an elastic body.

We generalize the problem considered in [1] for a more complicated configuration of the flow domain. While in [1] the thickness of the powder film is assumed to change linearly, in this paper we consider an arbitrary shape for the flow domain.

The molten powder film is assumed to be an incompressible, viscous fluid. The temperature distribution of the powder film is not taken into account. Since the powder film is very thin, the flow is well described by the Reynolds equation.

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The next section deals with the variational formulation of the problem. Existence and uniqueness results are then established. Finally, we discuss the pressure distribution of the powder film with respect to the viscosity of the powder and we analyse the action of the molten powder on the solidifying shell with respect to different configurations of the strand.

2 The physical problem

We study the motion of the powder film between two non parallel rigid surfaces, the mold, S_1, which is a plane surface and the strand, S_2. Let Oxz be the plane of the surface S_1 and Oy the axis normal to the mold. We assume that the strand has the same equation in each plane z =constant, $y = h(x)$. The function h gives the thickness of the powder film. We suppose that h is a Lipschitz continuous function on $[0, L]$ and $\delta > 0$, where $\delta = \min_{x\in[0,L]} h(x)$.
We denote by Ω the flow domain, defined by:

$$\Omega = \{(x, y, z) \ / \ x \in (0, L),\ y \in (0, h(x)),\ z \in (0, a)\}, \tag{1}$$

with L a positive given constant and $a \in (0, \infty]$.
The surfaces S_1 and S_2 are given by:

$$\begin{cases} S_1 = \{(x, 0, z) \ / \ x \in (0, L),\ z \in (0, a)\}, \\ S_2 = \{(x, h(x), z) \ / \ x \in (0, L),\ z \in (0, a)\}. \end{cases} \tag{2}$$

Remark **2.1** *When the flow domain is infinite in Oz direction, the problem is not depending on z. In this case we obtain a 1-dimensional problem.*

Usually, the classical Reynolds equation is obtained for a viscous flow in a thin domain bounded by a fixed surface and a moving one, and the body forces are neglected (see e. g. [2], [3], [4]). In our case, both the mold and the strand are moving with the velocities $\mathbf{U} = U\,\mathbf{i}$ and $\mathbf{V} = V_1\,\mathbf{i} + V_2\,\mathbf{j}$, respectively, and we take into account the body forces.
Let $\mathbf{v} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$ be the velocity of the powder film, p its pressure and the positive constants η, ρ and g the viscosity, the density and the acceleration of gravity, respectively.
Under the thin film hypothesis (the gap thickness h is much smaller than the other dimensions, with the variations of h also assumed small), the powder pressure does not depend on the $y-$coordinate.
Taking into account the boundary conditions

$$\begin{cases} \mathbf{v} = \mathbf{U} \text{ on } S_1, \\ \mathbf{v} = \mathbf{V} \text{ on } S_2, \end{cases} \tag{3}$$

the continuity and the Navier-Stokes equations lead to the following Reynolds equation for the pressure:

$$\begin{aligned} &\frac{\partial}{\partial x}\Big(h^3(x)\frac{\partial p}{\partial x}\Big) + \frac{\partial}{\partial z}\Big(h^3(x)\frac{\partial p}{\partial z}\Big) \\ &\qquad = (6\eta(U - V_1) + 3\rho g h^2(x))h'(x) + 12\eta V_2 \text{ in } (0, L) \times (0, a). \end{aligned} \tag{4}$$

After determining the pressure from the above equation, the components of the velocity are given by the following system:

$$\begin{cases} u(x,y,z) = \dfrac{1}{2\eta}\Big(\dfrac{\partial p}{\partial x} - \rho g\Big) y(y-h(x)) + \dfrac{V_1 - U}{h(x)} y + U, \\ w(x,y,z) = \dfrac{1}{2\eta}\dfrac{\partial p}{\partial z} y(y-h(x)), \\ v(x,y,z) = -\displaystyle\int_0^y \Big(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\Big)\,\mathrm{d}y. \end{cases} \tag{5}$$

Remark **2.2** *If S_2 is a fixed surface (i. e. $V_1 = V_2 = 0$) and we neglect the body forces, (4) becomes the classical Reynolds equation.*

Remark **2.3** *The pressure is not uniquely determined from the Reynolds equation (4). We shall need boundary conditions for the pressure.*

In the sequel we shall denote $D = (0, L) \times (0, a)$ for a finite and $D = (0, L)$ for $a = \infty$.
The physical problem can be formulated as follows:

Find p, *the solution of the problem* :

$$\begin{cases} \mathrm{div}(h^3(x)\nabla p) = (6\eta(U - V_1) + 3\rho g h^2(x))h'(x) + 12\eta V_2 \text{ in } D, \\ p = p_0 \text{ on } \partial D. \end{cases} \tag{6}$$

3 The variational formulation of the problem. Existence and uniqueness results

We denote by F the right hand side of (6), which is a known function.
We assume that p_0 is the trace of a function of $H^2(D)$, denoted also p_0 and we introduce the new function

$$q = p - p_0 \tag{7}$$

The problem (6) written for the function q is

$$\begin{cases} \mathrm{div}(h^3(x)\nabla q) = F - \mathrm{div}(h^3(x)\nabla p_0) \text{ in } D, \\ q = 0 \text{ on } \partial D. \end{cases} \tag{8}$$

The variational formulation of (8) is

$$\begin{cases} \textit{Find } q \in H_0^1(D) \textit{ such that} \\ \displaystyle\int_D h^3 \nabla q \cdot \nabla\varphi = -\int_D F\varphi - \int_D h^3 \nabla p_0 \cdot \nabla\varphi, \ \forall\, \varphi \in H_0^1(D). \end{cases} \tag{9}$$

We obtain the following obvious result:

PROPOSITION **3.1** *The problems (8) and (9) are equivalent.*

The last result of this section is given by

THEOREM **3.1** *The problem (9) has a unique solution.*

Proof. We introduce the functions:

$$\begin{cases} A : H_0^1(D) \times H_0^1(D) \mapsto \mathbf{R}, \\ A(q,\varphi) = \int_D h^3 \nabla q \cdot \nabla \varphi, \end{cases}$$

and

$$\begin{cases} l : H_0^1(D) \mapsto \mathbf{R}, \\ l(\varphi) = -\int_D F\varphi - \int_D h^3 \nabla p_0 \cdot \nabla \varphi. \end{cases}$$

The coercivity of A is given by

$$A(\varphi,\varphi) = \int_D h^3 |\nabla\varphi|^2 \geq \delta^3 \|\nabla\varphi\|^2_{(L^2(D))^2}$$

and the result follows as a consequence of the Lax-Milgram theorem.

4 The pressure distribution of the powder film

We begin the study of the pressure distribution with the easier case when the motion of the powder film is the same in each plane z =constant.

4.1 The 1-dimensional case

In this case we determine the pressure as the solution of the following problem:

$$\begin{cases} (h^3(x)p'(x))' = F \text{ in } D, \\ p(0) = p_0,\ p(L) = 0. \end{cases} \tag{10}$$

PROPOSITION **4.1** *The pressure distribution of the powder film in the 1-dimensional case is given by*

$$p(x) = 6\eta(U - V_1)\int_0^x \frac{\mathrm{d}t}{h^2(t)} + \rho g x + 12\eta V_2 \int_0^x \frac{t\ \mathrm{d}t}{h^3(t)} + c_1 \int_0^x \frac{\mathrm{d}t}{h^3(t)} + p_0 \tag{11}$$

where

$$c_1 = -\Big(\int_0^L \frac{\mathrm{d}t}{h^3(t)}\Big)^{-1}\Big(p_0 + 6\eta(U - V_1)\int_0^L \frac{\mathrm{d}t}{h^2(t)} + \rho g L + 12\eta V_2 \int_0^L \frac{t\ \mathrm{d}t}{h^3(t)}\Big)$$

Proof. The expression (11) is obtained integrating twice the differential equation $(10)_1$ and using the boundary conditions $(10)_2$.

REMARK **4.1** *If we consider, as in [1], that h is a linear function on $(0, L_1)$ and on (L_1, L), with $L_1 < L$, we obtain from (11) the same expression for the pressure of the powder film as in [1].*

The expression of the powder film pressure, given by (11), can be written as

$$p(x) = \alpha(x)\eta + \beta(x), \tag{12}$$

which means that the pressure changes linearly with respect to the powder viscosity.
In the sequel we shall obtain the same result in the 2-dimensional case.

4.2 The 2-dimensional case

From the definition (7) of the function q, it is obvious that for different functions p_0 with the same trace we obtain different functions q. However, since the function p_0 does not depend on η (the boundary values of the pressure are independent on the viscosity coefficient), for obtaining the dependence on η we can choose any arbitrary function p_0.
The next result states that in the 2-dimensional case, the expression of the powder film with respect to the viscosity coefficient is of the same type as in the 1-dimensional case.

PROPOSITION **4.2** *The pressure distribution of the powder film is given by:*

$$p(x, z) = \alpha(x, z)\eta + \beta(x, z). \tag{13}$$

Proof. We consider the problems

$Find\ \alpha \in H^1(D),\ satisfying:$

$$\begin{cases} \operatorname{div}(h^3(x)\alpha) = 6(U - V_1)h'(x) + 12V_2 \text{ in } D, \\ \alpha = 0 \text{ on } \partial D \end{cases} \tag{14}$$

and

$Find\ \beta \in H^1(D),\ satisfying:$

$$\begin{cases} \operatorname{div}(h^3(x)\beta) = 3\rho g h^2(x)h'(x) \text{ in } D, \\ \beta = p_0 \text{ on } \partial D. \end{cases} \tag{15}$$

We can prove, as in Section 3, the existence and the uniqueness of the functions α and β.
Multiplying (14) by η and adding with (15) it follows that the function $\alpha(x, z)\eta + \beta(x, z)$ is a solution for the problem (6) and the proof is achieved due to the uniqueness of the solution, obtained before.

The last result of this paper concerns the relation between the molten powder film and the solidifying shell.

4.3 The action of the molten powder on the solidifying shell

In the sequel we consider a reference shape of the strand given by a positive differentiable function h_0. When the solidifying shell of the strand increases, the mold-strand gap becomes thinner. Let $\varepsilon \in [0, \delta_0)$ be a small parameter, where $\delta_0 = \min_{x\in[0,L]} h_0(x)$. The function which defines the mold-strand gap with respect to the reference shape and to the small parameter ε is given by

$$h_\varepsilon(x) = h_0(x) - \varepsilon, \quad x \in [0, L]. \tag{16}$$

It is obvious that the pressure distribution of the powder film which corresponds to the function h_ε given by (16) depends on ε. We shall denote by p_ε the unique solution of the problem (6), corresponding to h_ε.
The action of the powder film on the solidifying shell of the strand is defined as:

$$g(\varepsilon) = \int_D p_\varepsilon. \tag{17}$$

We shall study in the sequel the dependence of g on ε. This study provides important tools for a forthcoming analysis concerning the relation between the molten powder and the solidifying shell, regarded as an elastic body.
We study here the easier case when $\min_{x\in[0,L]} h_\varepsilon(x) > 0$. Let $\alpha < \delta_0$ be a positive fixed constant.

Proposition **4.3** *The function* $g : [0, \delta_0 - \alpha] \mapsto \mathbf{R}$ *is continuous.*

Proof. Let ε, $\varepsilon_0 \in [0, \delta_0 - \alpha]$. Subtracting the problems (6) corresponding to h_ε and to h_{ε_0}, respectively, we obtain the following variational formulation:

$$\begin{aligned}\int_D h_\varepsilon^3 \nabla(p_\varepsilon - p_{\varepsilon_0}) \cdot \nabla\varphi + \int_D (h_\varepsilon^3 - h_{\varepsilon_0}^3)\nabla p_{\varepsilon_0} \cdot \nabla\varphi \\ = 3\rho g \int_D (h_\varepsilon^2 - h_{\varepsilon_0}^2) h_0' \varphi \ \forall \varphi \in H_0^1(D).\end{aligned} \tag{18}$$

For $\varphi = p_\varepsilon - p_{\varepsilon_0} \in H_0^1(D)$ it follows that

$$\alpha^3 \|\nabla(p_\varepsilon - p_{\varepsilon_0})\|_{(L^2(D))^2} \le |\varepsilon - \varepsilon_0|(c_1\|\nabla p_{\varepsilon_0}\|_{(L^2(D))^2} + c_2), \tag{19}$$

with c_1, c_2 independent on ε; hence, $\{p_\varepsilon\}_\varepsilon$ is strongly convergent in $H^1(D)$ to p_{ε_0} when $\varepsilon \to \varepsilon_0$, which yields the assertion of the proposition.

Corollary **4.1** *For any load* $G \in [\min_\varepsilon g(\varepsilon), \max_\varepsilon g(\varepsilon)]$ *there exists an equilibrium position.*

Proof. We say that the system is in an equilibrium position if an external force G which acts on the solidifying shell satisfies $G = g(\varepsilon)$. Hence, the corollary is an obvious consequence of the continuity of the function g.
From the physical point of view, we are interested to find, for a given load G, the configuration of the solidifying shell of the strand which realizes the equilibrium position.
For determining all the values of the load G which realize an equilibrium position we prove the differentiability of the function g.

PROPOSITION **4.4** *The function g is differentiable on $[0, \delta_0 - \alpha]$ and, for any $\varepsilon_0 \in [0, \delta_0 - \alpha]$ we have*

$$g'(\varepsilon_0) = \int_D q_{\varepsilon_0},$$

where $q_{\varepsilon_0} \in H_0^1(D)$ is the unique solution of the problem

$$\mathrm{div}(h_{\varepsilon_0}^3 \nabla q_{\varepsilon_0}) = -3\mathrm{div}(h_{\varepsilon_0}^2 \nabla p_{\varepsilon_0}) - 6\rho g h_{\varepsilon_0} h_0'. \quad (20)$$

Proof. For arbitrary ε, $\varepsilon_0 \in [0, \delta_0 - \alpha]$, $\varepsilon \neq \varepsilon_0$, we introduce the function $q_\varepsilon = \dfrac{p_\varepsilon - p_{\varepsilon_0}}{\varepsilon - \varepsilon_0}$. From (19) we get the boundedness of $\{q_\varepsilon\}_\varepsilon$ in $H_0^1(D)$. Passing to the limit in (18) with $\varepsilon \to \varepsilon_0$ the proof is achieved.

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