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# The mathematical modelling and the stability study of some speed regulators for nonlinear oscillating systems

by
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**Abstract**

The paper analyses two speed regulators for a uniform response in the case of some mechanisms with periodic motion. For the first mechanism with one degree of freedom, the conditions for uniform motion are computed in three cases: I. the vibrating mass is a rigid coupling with the elastic force and the damping force, II. the vibrating mass is a rigid coupling with the hardened elastic force and the damping force, III. the vibrating mass is a rigid coupling with the elastic force, the hardened elastic force and the damping force. For the second mechanism with two degrees of freedom, the vibrating mass is serially linked with the elastic and damping forces. This analysis leads to the study of some Duffing equations. The obtained equations being nonlinear, we apply the averaging method and the Van der Pol method. The stability of solutions in the phases space, the limit cycles for a uniform response of the system and the conditions of resonance are also studied.

*Key words and phrases:* speed regulator, nonlinear dynamical system, averaging method, Van der Pol method, limit cycle, stable point

*Mathematics Subject Classification:* 34C15, 70G60, 74H45, 74K05, 74K10

## 1 Introduction

The importance of the study of nonlinear dynamical systems in the field of vibrating machines and mechanisms is well known (see [1], [2]). Applications of speed regulators and absorbers are met in technics at every step (see [4], [7]). The mathematical model leads to nonlinear dynamical systems, therefore this study is carried out by considering numerical and approximation methods and by using the results of the linear stability (see [3], [6]). In order to get a uniform motion for the mechanisms, we will deduce validity conditions for the geometrical and mechanical parameters by averaging the nonlinear dynamical systems.

In order to find solutions, similarly to the constants variation method, there are considered parameters and coefficients with slow variation which can be adapted to Van der Pol method.

In order to determine the rotary angular speeds in the case of uniform motion, graphic and analytic methods are applied and attraction areas for limit cycles are specified. Generalizations and important applications to transport and machine tools are pointed out (see [1], [4]).

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## 2 The study of the dynamical system with rigid coupling

We consider a rigid body of mass $m$ with one degree of freedom, subject to an articulated link with the connecting rod-crank device as well as to some elastic and hardened elastic links with a damper (Figure 1). The connecting rod and the crank's lengths are $l$, respectively $r$. The coefficient of linear elasticity stands for $k$, the coefficient of nonlinear elasticity stands for $k^*$ and the damping coefficient stands for $c$. The point $A$ has a rotary motion around the $O_1$-axis and the mass $m\,(B)$ performs a vertical motion on the $Oy$-axis. The point $O$ is chosen so that $OO_1 = l$. A flywheel of radius $R$ is fixed on the $O_1$-axis. A rigid body of mass $M$ is hanged up on a wire which is winded on the flywheel. Therefore, the gravity force $M\vec{g}$ is acting upon the flywheel. Note that the crank $A$ and the flywheel have the same angular speed $\dot{\varphi} = \omega$, where $\varphi$ is the angle between $O_1A$ and the horizontal axis $Ox$. We denote by $y = y_B$ the ordinate of $m$ and $\psi = O_1\hat{B}A$.

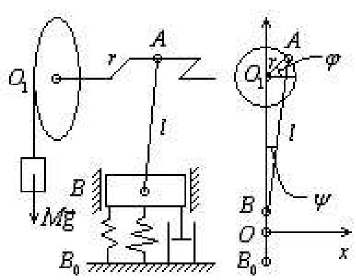


Figure 1: The connecting rod-crank mechanism

By applying the impulse theorem and the theorem of the kinetic moment with respect to the considered coordinate system, we obtain

$$m\ddot{y} + c\dot{y} + ky + k^*y^3 = F \Leftrightarrow \ddot{y} + 2n\dot{y} + p^2y + \alpha y^3 = F/m, \tag{1}$$

$$I\ddot{\varphi} = MgR - M\left(F_1\right), \tag{2}$$

where $2n = c/m$, $p^2 = k/m$, $\alpha = k^*/m$, $\vec{F}_e = -ky\vec{j}$ is the elastic force, $\vec{F}_e^* = -k^*y^3\vec{j}$ is the hardened elastic force, $\vec{F}_a = -c\dot{y}\vec{j}$ is the damping force, $\vec{F}$ is the resultant of these forces together with the inertial force $m\ddot{y}\vec{j}$, $F_1 = F/\cos\psi$, $I = I_0 + MR^2$, $I_0$ is the inertial moment of the flywheel with respect to $O_1$. Note that (1) is a Duffing equation.
Denote $r/l = \lambda < 1$. In $AO_1B$ we have $r\cos\varphi = l\sin\psi$, therefore $\cos\psi = \sqrt{1 - \lambda^2\cos^2\varphi}$. Using the binomial formula $1/\cos\psi = \left(1 - \lambda^2\cos^2\varphi\right)^{-1/2} \cong 1 + \left(\lambda^2/2\right)\cos^2\varphi$, we deduce

$$F_1 = F\left(1 + \frac{\lambda^2}{2}\cos^2\varphi\right). \tag{3}$$

From $M\left(F_1\right) = F_1 r\sin\varphi$, (2) şi (3) we get

$$I\ddot{\varphi} = MgR - F\left(1 + \frac{\lambda^2}{2}\cos^2\varphi\right) r\sin\varphi. \tag{4}$$

The calculus of $y = y_B$ leads to $y = l + r\sin\varphi - l\sqrt{1-\lambda^2\cos^2\varphi}$. This formula, together with the binomial formula $\sqrt{1-\lambda^2\cos^2\varphi} \cong 1 - (\lambda^2/2)\cos^2\varphi$ implies

$$y = r\left[\sin\varphi + \frac{\lambda}{4}(1+\cos 2\varphi)\right]. \tag{5}$$

The statement of problem **1** *It is asked to find the angular speed $\omega = const$ of the flywheel, knowing that the flywheel has a uniform motion, i.e. $\dot\varphi = \omega = const$. It is also asked to find the initial angular speed $\omega_0$ of the flywheel in order to have a stable rotation of the flywheel, i.e. $\omega \to const$ when $t \to \infty$.*

The problem can be generalized by considering that the flywheel is coupled with a rotative motor shaft which generates given moments with some given characteristics (see [2], [4]).
By eliminating $y$ between relations (1) and (5), there remains only one degree of freedom, namely $\varphi \in [0, 2\pi]$, which additionally satisfies $\dot\varphi = \omega$, with $\dot\omega \cong 0$ (i.e. $\omega$ has a slow variation in time). We have

$$\begin{cases} y = r\left[\sin\varphi + \frac{\lambda}{4}(1+\cos 2\varphi)\right],\ \dot y = r\omega\left(\cos\varphi - \frac{\lambda}{2}\sin 2\varphi\right), \\ \ddot y = -r\omega^2(\sin\varphi + \lambda\cos 2\varphi) \end{cases} \tag{6}$$

and so, relation (1) becomes

$$\begin{gathered} F/m = -r\omega^2(\sin\varphi + \lambda\cos 2\varphi) + 2nr\omega\left(\cos\varphi - \frac{\lambda}{2}\sin 2\varphi\right) \\ +p^2 r\left[\sin\varphi + \frac{\lambda}{4}(1+\cos 2\varphi)\right] + \alpha r^3\left[\sin\varphi + \frac{\lambda}{4}(1+\cos 2\varphi)\right]^3. \end{gathered} \tag{7}$$

Using $\ddot\varphi = \dot\omega \cong 0$, relation (4) becomes

$$MgR - Fr\left(\sin\varphi + \frac{\lambda^2}{2}\sin\varphi\cos^2\varphi\right) = 0. \tag{8}$$

We eliminate $F$ between (7) and (8) and then we average the obtained relation using the period $T = 2\pi$; recall that the average of a function $f(x,\lambda)$ on the interval $x \in [0,T]$ is $f(\lambda) = \frac{1}{T}\int_0^T f(x,\lambda)\,dx$. Here, $(x,\lambda) = (\varphi,\omega)$, $\varphi \in [0,2\pi]$. performing the calculations, we obtain

$$-r^2\omega^2\left(\frac{1}{2} + \frac{\lambda^2}{16}\right) + p^2r^2\left(\frac{1}{2} + \frac{\lambda^2}{16}\right) + \alpha r^4\left(\frac{3}{8} + \frac{5\lambda^2}{64} + \frac{15\lambda^4}{4^5}\right) = \frac{MgR}{m}.$$

Hence, using the approximation $\lambda^4 \cong 0$ and denoting $\beta = \frac{16MgR}{mr^2}$, we finally get

$$\omega^2 = \frac{1}{\lambda^2+8}\left[p^2\left(\lambda^2+8\right) + \alpha\left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta\right]. \tag{9}$$

The study of equation (9) is performed in three cases.

**I.** First, we deal with the case when the vibrating mass $m$ is rigidly coupled only with the elastic force and the damping force, i.e. $\alpha = 0$. Equation (9) becomes

$$\omega^2 = p^2 - \frac{\beta}{\lambda^2 + 8}. \tag{10}$$

The following situations appear:
1) If $p^2 - \frac{\beta}{\lambda^2+8} > 0$, then equation (10) has two solutions $\omega_{1,2}^* = \mp\sqrt{p^2 - \frac{\beta}{\lambda^2+8}}$. By noticing that $sgn\dot{\omega} = sgn\left[\omega^2 - \left(p^2 - \frac{\beta}{\lambda^2+8}\right)\right]$ (see (4)), we deduce that $\omega_1^*$ is a stable attractive point and $\omega_2^*$ is an unstable repulsive point. This means that if we start the motion with $\omega_0 \in (\omega_1^* - \varepsilon, \omega_1^* + \varepsilon)$, then $\omega \to \omega_1^*$ when $t \to \infty$, so the flywheel motion becomes stable.
2) If $p^2 - \frac{\beta}{\lambda^2+8} = 0$, then equation (10) has one solution $\omega_1^* = \omega_2^* = 0$ which is an unstable saddle point and the motion is repulsive, accelerated (galloping).
3) If $p^2 - \frac{\beta}{\lambda^2+8} < 0$, then equation (10) has no solution and the motion is repulsive, accelerated (galloping).
Now, we consider the phases space $(X, Y)$, where $X = y$, $Y = \dot{y}$. According to (6), we have

$$\Gamma(\omega) : \begin{cases} X = r\left[\sin\varphi + \frac{\lambda}{4}(1 + \cos 2\varphi)\right] \\ Y = r\omega\left(\cos\varphi - \frac{\lambda}{2}\sin 2\varphi\right) \end{cases}, \quad with\ \varphi = \omega t.$$

If $\omega = \omega_1^*$, then the trajectory $\Gamma(\omega_1^*)$ is closed, periodic of period $T = 2\pi/|\omega_1^*|$ (because $\varphi = \omega_1^* t$) and stable. If $\omega \in (\omega_1^* - \varepsilon, \omega_1^* + \varepsilon)$, then the trajectory $\Gamma(\omega)$ admits as a stable limit cycle the curve $\Gamma(\omega_1^*)$.

REMARK **1** *The stability point $\omega_1^*$ for the angular speed is negative, because the direction of the gravity force $M\vec{g}$ acting upon the flywheel is opposite to the chosen direction for the Oy-axis. In fact, only the absolute value of $\omega_1^*$ is of interest to us.*

**II.** Now, we deal with the case when the vibrating mass $m$ is rigidly coupled only with the hardened elastic force and the damping force, i.e. $p = 0$. Equation (9) becomes

$$\omega^2 = \frac{1}{\lambda^2 + 8}\left[\alpha\left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta\right]. \tag{11}$$

The following situations appear:
1) $\alpha > \beta/\left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, the nonlinear oscillator is hardened and equation (11) has two solutions

$$\omega_{1,2}^* = \mp\sqrt{\frac{1}{\lambda^2 + 8}\left[\alpha\left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta\right]}$$

By observing that $sgn\dot{\omega} = sgn\left\{\omega^2 - \frac{1}{\lambda^2+8}\left[\alpha\left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta\right]\right\}$ (see (4)), we deduce that $\omega_1^*$ is a stable attractive point and $\omega_2^*$ is an unstable repulsive point. This means that if we start the motion with $\omega_0 \in (\omega_1^* - \varepsilon, \omega_1^* + \varepsilon)$, then $\omega \to \omega_1^*$ when $t \to \infty$, so the flywheel motion becomes stable.

2) $\alpha = \beta / \left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, the nonlinear oscillator is hardened, equation (11) has one solution $\omega_1^* = \omega_2^* = 0$ which is an unstable saddle point and the motion is repulsive, accelerated (galloping).
3) $0 < \alpha < \beta / \left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, the nonlinear oscillator is hardened, equation (11) has no solution and the motion is repulsive, accelerated (galloping).
4) $\alpha < 0$. In this situation, the nonlinear oscillator is weakened, equation (11) has no solution and the motion is repulsive, accelerated (galloping).

Remark **2** *The discussion on the phases space $(X, Y)$ is similar to case I.*

**III.** Now, we deal with the case when the vibrating mass $m$ is rigidly coupled only with the elastic force, the hardened elastic force and the damping force, i.e. the general case. Recall equation (9) that we have to study

$$\omega^2 = \frac{1}{\lambda^2 + 8} \left[ p^2 \left(\lambda^2 + 8\right) + \alpha \left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta \right].$$

The following situations appear:
1) $\alpha > \left[\beta - p^2\left(\lambda^2 + 8\right)\right] / \left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, equation (9) has two solutions

$$\omega_{1,2}^* = \mp \sqrt{\frac{1}{\lambda^2 + 8} \left[ p^2 \left(\lambda^2 + 8\right) + \alpha \left(\frac{5r^2\lambda^2}{4} + 6r^2\right) - \beta \right]}.$$

By observing that $sgn\dot{\omega} = sgn\left\{\omega^2 - \frac{1}{\lambda^2+8}\left[p^2\left(\lambda^2 + 8\right) + \alpha\left(\frac{5r^2\lambda^2}{4} + 6r^2\right) \ -\beta\right]\right\}$ (see (4)), we deduce that $\omega_1^*$ is a stable attractive point and $\omega_2^*$ is an unstable repulsive point. This means that if we start the motion with $\omega_0 \in (\omega_1^* - \varepsilon, \omega_1^* + \varepsilon)$, then $\omega \to \omega_1^*$ when $t \to \infty$, so the flywheel motion becomes stable.
2) $\alpha = \left[\beta - p^2\left(\lambda^2 + 8\right)\right] / \left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, equation (9) has one solution $\omega_1^* = \omega_2^* = 0$ which is an unstable saddle point and the motion is repulsive, accelerated (galloping).
3) $\alpha < \left[\beta - p^2\left(\lambda^2 + 8\right)\right] / \left(\frac{5r^2\lambda^2}{4} + 6r^2\right)$. In this situation, equation (9) has no solution and the motion is repulsive, accelerated (galloping).

Remark **3** *In each one of the three sub-cases, the specification that the nonlinear oscillator is hardened or weakened depends now on the sign of the expression $\beta - p^2\left(\lambda^2 + 8\right)$. The discussion on the phases space $(X, Y)$ is similar to case I.*

## 3 The study of the speed regulator with two degrees of freedom

We consider a dynamical system similar to the one from Section 2, the difference being that the viscous damper, the rigid body $m$ and the elastic spring are serially placed (Figure 2(a)). This is the the Buassa-Sarde model (see [5]). This time, the acting forces are: the gravity force $m\vec{g}$, the damping force $\vec{F}_a = -c\dot{y}\vec{j}$ and the elastic force $\vec{F}_e = -k\left(y - r\sin\varphi\right)\vec{j}$. The motion equation is

$$m\ddot{y} + c\dot{y} + k\left(y - r\sin\varphi\right) = mg \Leftrightarrow \ddot{y} + 2n\dot{y} + p^2 y = g + p^2 r \sin\varphi, \qquad (12)$$

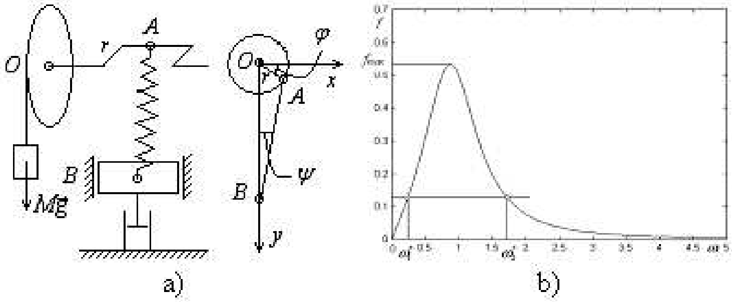

Figure 2: a) The mechanism with two degrees of freedom; b) The graph of $f$

where $2n = c/m$, $p^2 = k/m$, $\alpha = k^*/m$. Taking into account that the stress in the flywheel wire is $T = M(g - R\ddot{\varphi})$, its moment is $TR$, the moment of the elastic force is $k(y - r\sin\varphi)\, r\cos\varphi$ and by applying the theorem of the kinetic moment, we obtain

$$MR(g - R\ddot{\varphi}) + k(y - r\sin\varphi)\, r\cos\varphi = I\ddot{\varphi}. \tag{13}$$

Note that this system has two degrees of freedom: the displacement $y$ and the rotary angle $\varphi$.

THE STATEMENT OF PROBLEM **2** *It is asked to find the angular speed $\omega = const$ of the flywheel, knowing that the flywheel has a uniform motion, i.e. $\dot{\varphi} = \omega = const$. It is also asked to find the initial angular speed $\omega_0$ of the flywheel in order to have a stable rotation of the flywheel, i.e. $\omega \to const$ when $t \to \infty$.*

In order to solve this problem, we will apply the Van der Pol method: the method of parameters with slow variation and the constants variation method.
In the absence of the therm $p^2 r\sin\varphi$, $g/p^2$ is a solution of equation (12). Therefore we will seek for a solution of the type

$$y = A\sin(\varphi + \theta) + \frac{g}{p^2}, \quad with\ \dot{\varphi} = \omega, \tag{14}$$

where $A(t)$, $\theta(t)$ are parameters with slow variation and are to be calculated. From (14), we have $\dot{y} = \dot{A}\sin(\varphi+\theta) + A\dot{\theta}\cos(\varphi+\theta) + A\omega\cos(\varphi+\theta)$. By using the hypothesis of constants variation, we have

$$\dot{A}\sin(\varphi+\theta) + A\dot{\theta}\cos(\varphi+\theta) = 0 \tag{15}$$

and hence

$$\dot{y} = A\omega\cos(\varphi+\theta). \tag{16}$$

From (16) and $\dot{\omega} \cong 0$, we have

$$\ddot{y} = \dot{A}\omega\cos(\varphi+\theta) - A\omega\dot{\theta}\sin(\varphi+\theta) - A\omega^2\sin(\varphi+\theta). \tag{17}$$

By replacing (14), (16), (17) into (12), we obtain

$$\begin{aligned}\dot{A}\omega\cos(\varphi+\theta) - A\omega\dot{\theta}\sin(\varphi+\theta) &= A\omega^2\sin(\varphi+\theta)\\ &-2nA\omega\cos(\varphi+\theta) - p^2A\sin(\varphi+\theta) + p^2 r\sin\varphi.\end{aligned} \tag{18}$$

Relations (15) and (18) constitute a system in unknowns $\dot{A}$, $\dot{\theta}$ which has the solution

$$\begin{aligned}\dot{A}\omega &= A\left(\omega^2-p^2\right)\sin(\varphi+\theta)\cos(\varphi+\theta)-2nA\omega\cos^2(\varphi+\theta)\\ &\quad +p^2r\sin\varphi\cos(\varphi+\theta),\end{aligned} \tag{19}$$

$$\begin{aligned}A\omega\dot{\theta} &= A\left(p^2-\omega^2\right)\sin^2(\varphi+\theta)+2nA\omega\sin(\varphi+\theta)\cos(\varphi+\theta)\\ &\quad -p^2r\sin\varphi\sin(\varphi+\theta).\end{aligned} \tag{20}$$

Similarly, by replacing (14) into (13), we obtain

$$\left(I+MR^2\right)\ddot{\varphi}=MgR+k\left[A\sin(\varphi+\theta)+\frac{g}{p^2}-r\sin\varphi\right]r\cos\varphi. \tag{21}$$

We average the nonlinear equations (19), (20), (21) with respect to $\varphi\in[0,2\pi]$ and we obtain

$$\begin{cases}\dot{A}\omega=-nA\omega-\frac{p^2r}{2}\sin\theta,\ A\omega\dot{\theta}=\frac{A}{2}\left(p^2-\omega^2\right)-\frac{p^2r}{2}\cos\theta,\\ \left(I+MR^2\right)\dot{\omega}=MgR+\frac{kAr}{2}\sin\theta.\end{cases} \tag{22}$$

The nonlinear system (22) in unknowns $A$, $\theta$, $\omega$ allows us to determine the equilibrium points or the uniform static regimes of motion. We consider $\dot{A}\cong 0$, $\dot{\theta}\cong 0$, $\dot{\omega}\cong 0$ and so $A=const$, $\theta=\alpha=const$, $\omega=const$ (i.e. the initial values). The system (22) becomes

$$\begin{cases}nA\omega+\frac{p^2r}{2}\sin\alpha=0,\ \frac{A}{2}\left(p^2-\omega^2\right)-\frac{p^2r}{2}\cos\alpha=0,\\ MgR+\frac{kAr}{2}\sin\alpha=0.\end{cases} \tag{23}$$

We have

$$\begin{cases}tg\alpha=\frac{-2n\omega}{p^2-\omega^2},\\ \sin\alpha=\frac{-2n\omega}{\sqrt{(p^2-\omega^2)^2+4n^2\omega^2}},\ \cos\alpha=\frac{p^2-\omega^2}{\sqrt{(p^2-\omega^2)^2+4n^2\omega^2}},\end{cases} \tag{24}$$

$$A=\frac{p^2r}{\sqrt{\left(p^2-\omega^2\right)^2+4n^2\omega^2}}, \tag{25}$$

$$0=MgR-\frac{kp^2r^2n\omega}{\left(p^2-\omega^2\right)^2+4n^2\omega^2}. \tag{26}$$

By considering the function $f(\omega)=\omega/\left[\left(p^2-\omega^2\right)^2+4n^2\omega^2\right]$, $\omega\geq 0$, and denoting $K=(MgR)/\left(kp^2r^2n\right)$, equation (26) becomes

$$f(\omega)=K. \tag{27}$$

In order to solve it, the graph $f=f(\omega)$ is plotted on the coordinate system $\omega Of$ and the intersection points between this graph and the straight line $f=K$ are studied (Figure 2(b)).
The following cases appear:
1) If $K\in(0,f_{\max})$, then equation (27) has two solutions $0<\omega_1^*<\omega_2^*$ and we derive three regimes of motion: (a) $\omega\in(0,\omega_1^*)$, (b) $\omega\in(\omega_1^*,\omega_2^*)$, (c) $\omega\in(\omega_2^*,\infty)$. Observing that $sgn\dot{\omega}=sgn\left[K-f(\omega)\right]$

(see (22)), we deduce that $\omega_1^*$ is a stable attractive point and $\omega_2^*$ is an unstable repulsive point. This means that if we start the motion with $\omega_0 \in (\omega_1^* - \varepsilon, \omega_1^* + \varepsilon)$, then $\omega \to \omega_1^*$ when $t \to \infty$, so the flywheel motion becomes stable.
2) If $K = f_{\max}$, then equation (27) has one solution $\omega_1^* = \omega_2^*$ which is an unstable saddle point and the motion is repulsive, accelerated (galloping).
3) If $K > f_{\max}$, then equation (27) has no solution and the motion is repulsive, accelerated (galloping).
Once we have found the values for $\omega$, $\theta$, $A$, the general solution of equation (12) can be written using the special solution (14): $y = e^{-nt}(c_1 \cos \delta t + c_2 \sin \delta t) + A \sin(\omega t + \theta) + \frac{g}{p^2}$, where $\delta = \sqrt{p^2 - n^2}$ and $c_1$, $c_2$ can be found from the initial conditions $y_0$, $\dot{y}_0 = v_0$. We have

$$\dot{y} \cong e^{-nt}\left[(-nc_1 + \delta c_2)\cos \delta t + (-nc_2 - \delta c_1)\sin \delta t\right] + A\omega \cos(\omega t + \theta).$$

By considering to the phases space $(X, Y)$, with $X = y$, $Y = \dot{y}$, we denote by $E_i$ $(i = 1, 2)$ the ellipses corresponding to special solutions (14) written for $\omega = \omega_i^*$:

$$E_i : \left(\frac{X - g/p^2}{A(\omega_i^*)}\right)^2 + \left(\frac{Y}{\omega_i^* A(\omega_i^*)}\right)^2 = 1.$$

These ellipses become limit cycles: $E_1$-stable limit cycle, $E_2$-unstable limit cycle. If the motion is started with $\omega$ in regimes (a), (b), then the trajectory $(X, Y)$ will be a spiral line which will tend to $E_1$ when $t \to \infty$. If the motion is started with $\omega$ in regime (c), then the trajectory $(X, Y)$ will be a spiral line which will leave $E_1$ when $t \to \infty$.

Remark **4** *The phenomenon of resonance can also be emphasized herein. For $n \to 0$ and $\omega \to p$ the phenomenon of instability is generated because the amplitude $A$ increases. Consequently, the coincidence zone $\omega \cong p$ must be eliminated and $n$ must be increased.*

## References

[1] Chiriacescu, S.T., *Stability in the Dynamics of Metal Cutting.* Elsevier, 1990.

[2] Chiriacescu, S.T., *Dinamica maşinilor unelte. Prolegomene.* Ed. Tehnică, Bucureşti, 2004.

[3] Lupu, M.; Postelnicu, A.; Deaconu, A., A Study on the Flutter Stability of Dynamic Aeroelastic Systems, *Proceeding Conference CDM*, vol. **I**(2001), 37-47.

[4] Munteanu, M., *Introducere în dinamica maşinilor vibratoare.* Ed. Academiei Române, Bucureşti, 1986.

[5] Panovko, Ya.G.; Gubanova, Y.Y., *Stabilitate şi vibraţiile sistemelor elastice.* Ed. Nauka, Moskva, 1964.

[6] Teodorescu, P.P., *Sisteme mecanice.* Ed. Tehnică, Bucureşti, 2002.

[7] Voinea, R.; Stroie, I., *Introducere în teoria sistemelor dinamice.* Ed. Academiei Române, Bucureşti, 2000.