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Boundary value problems for the Stokes resolvent equations

by

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Abstract

In this paper we survey certain important results related to some boundary value problems of Dirichlet, Neumann and mixed type for the Stokes resolvent system. These properties are obtained by using the potential theory. We give existence and uniqueness results as well as boundary integral representations of classical solutions in the case of certain bounded domains.

Key words and phrases: Stokes resolvent equations, boundary value problems, the potential theory, existence and uniqueness results.

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1 Introduction

Let $n \in \mathbb{N}$, $n \geq 2$, and let D', $D_1 \subset \mathbb{R}^n$ be two bounded domains with connected boundaries Γ' and Γ_1 of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$, such that $\overline{D}_1 \subset D'$. We next consider the bounded domain D given by $D = D' \setminus \overline{D}_1$. Let us assume that the origin of $\mathbb{R}^n$ belongs to the set D_1. The direction of the unit normal $\mathbf{n}$ to the boundary $\Gamma = \Gamma' \bigcup \Gamma_1$ of D is chosen such that $\mathbf{n}$ points outside D.
As it is known, the Stokes resolvent system in D consists of the equations

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} = -\mathbf{f} \text{ in } D, \tag{1.1}$$

where χ^2 is a complex number such that $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \text{ Re } z \leq 0, \text{ Im } z = 0\}$, $\mathbf{u} = (u_1, \dots, u_n)$ is an unknown vector function, q is an unknown scalar function, and $\mathbf{f} = (f_1, \dots, f_n)$ is a given vector function. All occurring functions are complex-valued. Also, ∇ denotes the n-dimensional gradient operator and ∇^2 is the Laplace operator.

2 The potential theory

Let $\mathbf{U}$ be a continuous vector function on the boundary Γ of the domain D. Then the *interior Dirichlet problem* for the unsteady Stokes system (1.1) in the bounded domain D consists in finding a classical solution $(\mathbf{u}, q)$ of this system, which satisfies the boundary condition

$$\mathbf{u} = \mathbf{U} \text{ on } \Gamma. \tag{2.1}$$

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Let us assume that the fields $\mathbf{u}$ and q satisfy the system of equations (1.1). Then the *stress tensor field* $\mathbf{\Sigma}(\mathbf{u})$ associated with the fields $\mathbf{u}$ and q is the second-order tensor field defined by

$$\mathbf{\Sigma}(\mathbf{u}) = -q\mathbf{I}_n + \nabla\mathbf{u} + (\nabla\mathbf{u})^T, \tag{2.2}$$

where $\mathbf{I}_n$ is the $n\times n$ identity matrix and $(\nabla\mathbf{u})^T$ means the matrix transposed to $\nabla\mathbf{u} = (\partial u_i/\partial x_j)_{i,j=1,\ldots,n}$. From the equations (1.1) we deduce that

$$\frac{\partial\Sigma_{ij}(\mathbf{u})}{\partial x_i} = \chi^2 u_j - f_j \text{ in } D, \quad j = 1,\ldots,n, \tag{2.3}$$

where $\Sigma_{ij}(\mathbf{u})$ are the components of $\mathbf{\Sigma}(\mathbf{u})$, $i,j = 1,\ldots,n$. In the equations (2.3) we have used the repeated-index summation convention. From now on, we shall take into account this rule.
Let $\mathbf{T}$ be a continuous vector field on Γ. Then the *interior Neumann problem* for the system (1.1) in the bounded domain D is the boundary value problem consisting of the system of equations (1.1) and the boundary condition of the Neumann type

$$\mathbf{\Sigma}(\mathbf{u})\cdot\mathbf{n} = \mathbf{T} \text{ on } \Gamma. \tag{2.4}$$

2.1 The unsteady Stokeslet

Let us introduce the *free-space Green function* or the *unsteady Stokeslet* $\mathcal{G}^{\chi^2}(\mathcal{G}_{ij}^{\chi^2})$ and the *pressure vector* $\mathbf{\Pi}^{\chi^2}(\Pi_i^{\chi^2})$ such that

$$\frac{\partial\mathcal{G}_{ij}^{\chi^2}(\mathbf{x})}{\partial x_i} = 0, \quad j = 1,\ldots,n, \tag{2.5}$$

$$-\frac{\partial\Pi_j^{\chi^2}(\mathbf{x})}{\partial x_k} + (\nabla^2 - \chi^2)\mathcal{G}_{kj}^{\chi^2}(\mathbf{x}) = -2\varpi_n\delta_{kj}\delta(\mathbf{x}), \quad j,k = 1,\ldots,n, \tag{2.6}$$

where $\delta_{kj} = 1$ for $k = j$, $\delta_{kj} = 0$ for $k \neq j$, ϖ_n is the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, and δ is the Dirac distribution in $\mathbb{R}^n$ $(n \geq 2)$.
The *stress tensor* $\mathbf{S}^{\chi^2}(S_{ijk}^{\chi^2})$, corresponding to the unsteady Stokeslet $\mathcal{G}^{\chi^2}$ and to the pressure vector $\mathbf{\Pi}^{\chi^2}$, is the third order tensor defined by the relations

$$S_{ijk}^{\chi^2}(\mathbf{x}) = -\Pi_j^{\chi^2}(\mathbf{x})\delta_{ik} + \frac{\partial\mathcal{G}_{ij}^{\chi^2}(\mathbf{x})}{\partial x_k} + \frac{\partial\mathcal{G}_{kj}^{\chi^2}(\mathbf{x})}{\partial x_i}, \quad i,j,k = 1,\ldots,n. \tag{2.7}$$

The system of equations (2.5) and (2.6) can be solved by using the method of Fourier transform. For details see [8] p. 81-82, for $n = 2,3$; [14], [15], [16] in the general case $n \geq 2$.
Let $\mathbf{\Lambda}^{\chi^2}$ be the *pressure tensor* associated with the stress tensor $\mathbf{S}^{\chi^2}$. They determine a fundamental solution of the Stokes resolvent system, i.e.,

$$\frac{\partial S_{ijk}^{\chi^2}(\mathbf{y}-\mathbf{x})}{\partial x_j} = 0 \text{ for } \mathbf{x} \neq \mathbf{y}, \quad i,k = 1,\ldots,n, \tag{2.8}$$

$$(\nabla_{\mathbf{x}}^2 - \chi^2) S_{ijk}^{\chi^2}(\mathbf{y} - \mathbf{x}) = \frac{\partial \Lambda_{ik}^{\chi^2}(\mathbf{x} - \mathbf{y})}{\partial x_j} \text{ for } \mathbf{x} \neq \mathbf{y},\ i, j, k = 1, \ldots, n, \tag{2.9}$$

where $\Lambda_{ik}^{\chi^2}$, $i, k = 1, \ldots, n$, are the components of the pressure tensor $\mathbf{\Lambda}^{\chi^2}$ (see [15] p. 61-62 in the general case $n \geq 2$).

2.2 Properties of unsteady hydrodynamic potentials

Let $\mathbf{g} = (g_1, \ldots, g_n)$ and $\mathbf{h} = (h_1, \ldots, h_n)$ be two mappings such that $\mathbf{g}, \mathbf{h} \in C^0(\Gamma)$. Recall that all occurring functions are complex-valued.
The *unsteady hydrodynamic single-layer potential* with density $\mathbf{g}$ is the vector function $\mathbf{V}_{\chi^2,n}(\cdot, \mathbf{g})$ given by

$$\mathbf{V}_{\chi^2,n}(\mathbf{x}, \mathbf{g}) = \int_\Gamma \mathcal{G}^{\chi^2}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{g}(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.10}$$

where $\mathcal{G}^{\chi^2}$ is the unsteady Stokeslet.
The *unsteady hydrodynamic double-layer potential* with density $\mathbf{h}$ is the vector function $\mathbf{W}_{\chi^2,n}(\cdot, \mathbf{h})$ whose j-th component is given by

$$(\mathbf{W}_{\chi^2,n})_j(\mathbf{x}, \mathbf{h}) = \int_\Gamma h_i(\mathbf{y}) S_{ijk}^{\chi^2}(\mathbf{y} - \mathbf{x}) n_k(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.11}$$

$j = 1, \ldots, n$, where $\mathbf{S}^{\chi^2}$ is the stress tensor associated with the unsteady Stokeslet and $\mathbf{n}$ is the unit normal to Γ pointing outside D.
Let $P_{\chi^2,n}^s(\cdot, \mathbf{g})$ and $P_{\chi^2,n}^d(\cdot, \mathbf{h})$ be the functions given by

$$P_{\chi^2,n}^s(\mathbf{x}, \mathbf{g}) = \int_\Gamma \Pi_i^{\chi^2}(\mathbf{x} - \mathbf{y}) g_i(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.12}$$

$$P_{\chi^2,n}^d(\mathbf{x}, \mathbf{h}) = \int_\Gamma h_i(\mathbf{y}) \Lambda_{ik}^{\chi^2}(\mathbf{x} - \mathbf{y}) n_k(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma, \tag{2.13}$$

where $\mathbf{\Pi}^{\chi^2}$ is the pressure vector and $\mathbf{\Lambda}^{\chi^2}$ is the pressure tensor corresponding to the unsteady Stokeslet.
According to the equations (2.5), (2.6), (2.8) and (2.9), we deduce that each of the pairs
$(\mathbf{V}_{\chi^2,n}(\cdot, \mathbf{g}), P_{\chi^2,n}^s(\cdot, \mathbf{g}))$ and $(\mathbf{W}_{\chi^2,n}(\cdot, \mathbf{h}), P_{\chi^2,n}^d(\cdot, \mathbf{h}))$ is a classical solution to the homogeneous Stokes resolvent system in both regions D and $\mathbb{R}^n \setminus \overline{D}$.
For further arguments we need the stress tensor $\mathbf{\Sigma}(\mathbf{V}_{\chi^2,n}(\cdot, \mathbf{g}))$ associated with the single-layer potential $\mathbf{V}_{\chi^2,n}(\cdot, \mathbf{g})$ and defined by the relations

$$\Sigma_{jk}(\mathbf{V}_{\chi^2,n}(\mathbf{x}, \mathbf{g})) = \int_\Gamma S_{jik}^{\chi^2}(\mathbf{x} - \mathbf{y}) g_i(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{2.14}$$

Let us now consider a (scalar, vector, or tensor) field w defined in a domain containing Γ. We use the notations $w^-(\mathbf{x}_0)$ and $w^+(\mathbf{x}_0)$ for the limiting values of w at $\mathbf{x}_0 \in \Gamma$, evaluated from D and $\mathbb{R}^n \setminus \overline{D}$

respectively. In particular, we use the notations $\mathbf{H}^{+}_{\chi^2,n}(\cdot,\mathbf{g})$ and $\mathbf{H}^{-}_{\chi^2,n}(\cdot,\mathbf{g})$ for the limiting values of the normal stress due to the single layer potential $\mathbf{V}_{\chi^2,n}(\cdot,\mathbf{g})$ on each side of Γ. Therefore, we have

$$(\mathbf{H}_{\chi^2,n})^{\pm}_{j}(\mathbf{x},\mathbf{g}) = \Sigma^{\pm}_{jk}(\mathbf{V}_{\chi^2,n}(\mathbf{x},\mathbf{g}))n_k(\mathbf{x}), \quad \mathbf{x}\in\Gamma. \tag{2.15}$$

Theorem 2.1 (see [15] p.66; [8] p. 201) *Let $D\subset\mathbb{R}^n$ $(n\geq 2)$ be a bounded domain with boundary Γ of class $C^{1,\alpha}$ $(0<\alpha\leq 1)$ and let $\chi^2\in\mathbb{C}\setminus\{z\in\mathbb{C}:\ \mathrm{Re}\ z\leq 0,\ \mathrm{Im}\ z=0\}$. Also, let $\mathbf{g}$ and $\mathbf{h}$ be two vector densities such that $\mathbf{g},\mathbf{h}\in C^0(\Gamma)$, and let $\mathbf{V}_{\chi^2,n}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2,n}(\cdot,\mathbf{h})$ and $\mathbf{H}^{\pm}_{\chi^2,n}(\cdot,\mathbf{g})$ be the vector functions given by the relations* (2.10)*,* (2.11) *and* (2.15)*. Then for $\mathbf{x}_0\in\Gamma$ we have*

$$\mathbf{V}^{+}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}) = \mathbf{V}^{-}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}) = \mathbf{V}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}), \tag{2.16}$$

$$\mathbf{W}^{+}_{\chi^2,n}(\mathbf{x}_0,\mathbf{h}) - \mathbf{W}^{*}_{\chi^2,n}(\mathbf{x}_0,\mathbf{h}) = \varpi_n\mathbf{h}(\mathbf{x}_0) = \mathbf{W}^{*}_{\chi^2,n}(\mathbf{x}_0,\mathbf{h}) - \mathbf{W}^{-}_{\chi^2,n}(\mathbf{x}_0,\mathbf{h}), \tag{2.17}$$

$$\mathbf{H}^{+}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}) - \mathbf{H}^{*}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}) = -\varpi_n\mathbf{g}(\mathbf{x}_0) = \mathbf{H}^{*}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}) - \mathbf{H}^{-}_{\chi^2,n}(\mathbf{x}_0,\mathbf{g}), \tag{2.18}$$

where

$$\begin{aligned} (\mathbf{W}^{*}_{\chi^2,n})_j(\mathbf{x}_0,\mathbf{h}) &= \int_{\Gamma}^{PV} h_i(\mathbf{y})S^{\chi^2}_{ijk}(\mathbf{y}-\mathbf{x}_0)n_k(\mathbf{y})d\Gamma(\mathbf{y}) \\ (\mathbf{H}^{*}_{\chi^2,n})_j(\mathbf{x}_0,\mathbf{g}) &= \int_{\Gamma}^{PV} g_i(\mathbf{y})S^{\chi^2}_{jik}(\mathbf{x}_0-\mathbf{y})n_k(\mathbf{x})d\Gamma(\mathbf{y}) \end{aligned} \tag{2.19}$$

and the symbol PV means the principal value.

3 The interior Dirichlet problem

Let $D = D'\setminus\overline{D}_1\subset\mathbb{R}^n$ $(n\geq 2)$ be a bounded domain with boundary Γ of class $C^{1,\alpha}$ $(0<\alpha\leq 1)$ and let $\chi^2\in\mathbb{C}\setminus\{z\in\mathbb{C}:\ \mathrm{Re}z\leq 0,\ \mathrm{Im}z=0\}$. Also, let $\mathbf{U}\in C^0(\Gamma)$ be a vector function which satisfies the condition

$$\int_{\Gamma}\mathbf{U}\cdot\mathbf{n}d\Gamma = 0, \tag{3.1}$$

and let $\mathbf{f}\in C^{\lambda}(D)$ be a Hölder continuous vector function in D with Hölder exponent $\lambda\in(0,1]$. Then the interior Dirichlet problem

$$\nabla\cdot\mathbf{u} = 0, \quad -\nabla q + (\nabla^2-\chi^2)\mathbf{u} = -\mathbf{f} \text{ in } D \tag{3.2}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Gamma \tag{3.3}$$

has at most one classical solution $(\mathbf{u},q)$ (see e.g. [8] Chapter 1).
Next, we try to obtain a classical solution to the boundary value problem (3.2)-(3.3) in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{W}_{\chi^2,n}\left(\mathbf{x},\frac{1}{2\varpi_n}\mathbf{h}\right) + \frac{1}{2\varpi_n}\int_D \mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})d\mathbf{y} \tag{3.4}$$

$$q(\mathbf{x}) = P^{d}_{\chi^2,n}\left(\mathbf{x},\frac{1}{2\varpi_n}\mathbf{h}\right) + \frac{1}{2\varpi_n}\int_D \mathbf{\Pi}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y})d\mathbf{y} \tag{3.5}$$

for $\mathbf{x} \in D$, where $\mathbf{h} \in C^0(\Gamma)$ is an unknown vector density, and $d\mathbf{y} = dy_1 \cdots dy_n$.
The fields $\mathbf{u}$ and q satisfy the equations (3.2). Applying the boundary condition (3.3) to the integral representation (3.4) and using the jump formulas (2.17), we obtain the Fredholm integral equation of the second kind with unknown $\mathbf{h} \in C^0(\Gamma)$

$$-\frac{1}{2}\mathbf{h}(\mathbf{x}_0) + (\mathbf{K}_{\chi^2,n}\mathbf{h})(\mathbf{x}_0) = \mathbf{U}(\mathbf{x}_0) - \frac{1}{2\varpi_n}\int_D \mathcal{G}^{\chi^2}(\mathbf{x}_0 - \mathbf{y}) \cdot \mathbf{f}(\mathbf{y})d\mathbf{y}, \quad \mathbf{x}_0 \in \Gamma. \tag{3.6}$$

In order to prove the existence of solutions to the equation (3.6), we consider the homogeneous equation

$$\left(-\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)\mathbf{h}^0 = \mathbf{0} \text{ on } \Gamma, \tag{3.7}$$

which is the adjoint of the equation

$$\left(-\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right)\Psi = \mathbf{0} \text{ on } \Gamma \tag{3.8}$$

with respect to the inner product $\langle \cdot, \cdot \rangle : C^0(\Gamma) \times C^0(\Gamma) \to \mathbb{C}$ given by

$$\langle \mathbf{g}, \mathbf{h} \rangle \equiv \int_\Gamma \mathbf{g} \cdot \overline{\mathbf{h}} d\Gamma = \int_\Gamma g_i \overline{h}_i d\Gamma, \quad \mathbf{g}, \mathbf{h} \in C^0(\Gamma). \tag{3.9}$$

Theorem 3.1 *The null spaces of the operators*

$$-\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma), \ -\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n} : C^0(\Gamma) \to C^0(\Gamma) \tag{3.10}$$

are one-dimensional. In particular, the null space of the operator $-\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}$ *is given by*

$$\mathcal{N} \equiv \mathcal{N}\left(-\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right) = \left\{\Psi \in C^0(\Gamma) : \ -\frac{1}{2}\Psi + \mathcal{H}_{\overline{\chi}^2,n}\Psi = \mathbf{0} \ \textit{on } \Gamma\right\}$$

$$= \{\gamma\mathbf{n} : \ \gamma \in \mathbb{C}\}. \tag{3.11}$$

Proof. First, we show that $\mathbf{n} \in \mathcal{N}$. For this purpose, we take into account the properties

$$\mathbf{V}_{\overline{\chi}^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\mathbf{n}\right) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^n \tag{3.12}$$

$$P_{\overline{\chi}^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\mathbf{n}\right) = \begin{cases} -1 & \text{if } \ \mathbf{x} \in D \\ 0 & \text{if } \ \mathbf{x} \in \mathbb{R}^n \setminus \overline{D}, \end{cases} \tag{3.13}$$

in view of which we find that

$$\mathbf{H}^+_{\overline{\chi}^2,n}\left(\mathbf{x}_0, \frac{1}{2\varpi_n}\mathbf{n}\right) = \mathbf{0}, \ \mathbf{H}^-_{\overline{\chi}^2,n}\left(\mathbf{x}_0, \frac{1}{2\varpi_n}\mathbf{n}\right) = \mathbf{n}(\mathbf{x}_0), \ \mathbf{x}_0 \in \Gamma. \tag{3.14}$$

On the other hand, from the jump formulas (2.18) and the relations (3.14), we obtain that

$$-\frac{1}{2}\mathbf{n} + \mathcal{H}_{\overline{\chi}^2,n}\mathbf{n} = \mathbf{0} \text{ on } \Gamma, \tag{3.15}$$

i.e., $\mathbf{n} \in \mathcal{N}$.

Next, we prove that for each function Ψ in the set $\mathcal{N}$ there exists a constant $\gamma \in \mathbb{C}$ such that $\Psi = \gamma\mathbf{n}$. To this end, we consider the fields $\mathbf{u}_0$ and q_0 given by

$$\mathbf{u}_0 = \mathbf{V}_{\overline{\chi}^2,n}\left(\cdot, \frac{1}{2\varpi_n}\Psi\right), \quad q_0 = P^s_{\overline{\chi}^2,n}\left(\cdot, \frac{1}{2\varpi_n}\Psi\right) \text{ in } \mathbb{R}^n \setminus \Gamma. \tag{3.16}$$

Since $\Psi \in \mathcal{N}$ it follows that

$$\mathbf{H}^+_{\overline{\chi}^2,n}\left(\mathbf{x}_0, \frac{1}{2\varpi_n}\Psi\right) = -\frac{1}{2}\Psi(\mathbf{x}_0) + \mathcal{H}_{\overline{\chi}^2,n}\Psi(\mathbf{x}_0) = \mathbf{0}, \quad \mathbf{x}_0 \in \Gamma. \tag{3.17}$$

Consequently, the pair $(\mathbf{u}_0, q_0)$ solves the following boundary value problem:

$$\begin{array}{l} \nabla\cdot\mathbf{u}_0 = 0, \quad -\nabla q_0 + (\nabla^2 - \overline{\chi}^2)\mathbf{u}_0 = \mathbf{0} \text{ in } \mathbb{R}^n \setminus \overline{D'} \\ \mathbf{\Sigma}^+(\mathbf{u}_0)\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma' \\ (|\mathbf{u}_0||\nabla\mathbf{u}_0|)(\mathbf{x}) = o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty \\ (|\mathbf{u}_0||q_0|)(\mathbf{x}) = o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty, \end{array} \tag{3.18}$$

as well as the boundary value problems

$$\begin{array}{l} \nabla\cdot\mathbf{u}_0 = 0, \ -\nabla q_0 + (\nabla^2 - \overline{\chi}^2)\mathbf{u}_0 = \mathbf{0} \text{ in } D_1 \\ \mathbf{\Sigma}^+(\mathbf{u}_0)\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma_1. \end{array} \tag{3.19}$$

Recall that the superscript plus applies in (3.18) and (3.19) for the external side of $\Gamma' = \partial D'$ and for the internal side of Γ_1.

In view of the uniqueness of solutions to the boundary value problems (3.18) and (3.19), we deduce that

$$\mathbf{u}_0 = \mathbf{0}, \quad q_0 = 0 \text{ in } \mathbb{R}^n \setminus \overline{D}. \tag{3.20}$$

In addition, the continuity property of the single-layer potential $\mathbf{u}_0$ across Γ yields that the pair $(\mathbf{u}_0, q_0)$ is a solution to the interior Dirichlet problem

$$\nabla\cdot\mathbf{u}_0 = 0, \quad -\nabla q_0 + (\nabla^2 - \overline{\chi}^2)\mathbf{u}_0 = \mathbf{0} \text{ in } D, \tag{3.21}$$

and hence there exists a constant $\gamma \in \mathbb{C}$ such that

$$\mathbf{u}_0 = \mathbf{0}, \quad q_0 = -\gamma \text{ in } D. \tag{3.22}$$

Consequently, we have $\mathbf{\Sigma}^-(\mathbf{u}_0)\cdot\mathbf{n} = \gamma\mathbf{n}$ on Γ, i.e.,

$$\mathbf{H}^-_{\overline{\chi}^2,n}\left(\mathbf{x}_0, \frac{1}{2\varpi_n}\Psi\right) = \gamma\mathbf{n}(\mathbf{x}_0), \quad \mathbf{x}_0 \in \Gamma. \tag{3.23}$$

Now, using the jump formulas (2.18) and the properties (3.17) and (3.23), we find that $\Psi = \gamma \mathbf{n}$. In view of this result and from the fact that $\mathbf{n} \in \mathcal{N}$ it follows the property (3.11), which shows that the dimension of the space $\mathcal{N}$ is equal to one.
Finally, it remains only to apply Fredholm's alternative (see [9]) to deduce that the null spaces of the adjoint operators from (3.10) have the same dimension. This completes the proof of Theorem 3.1. □
Now, using the equation (2.5) and the condition (3.1), we obtain the following result (see [16] in the case of a domain with connected boundary):

Theorem 3.2 *Let* $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ *be a bounded domain with boundary* Γ *of class* $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ *and let* $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re}z \leq 0, \ \mathrm{Im}z = 0\}$. *Also, let* $\mathbf{f} \in C^{\lambda}(D)$ *be a Hölder continuous vector function in* D $(0 < \lambda \leq 1)$ *and let* $\mathbf{U} \in C^0(\Gamma)$ *be a given vector function that satisfies the condition* (3.1). *Then the interior Dirichlet problem* (3.2)-(3.3) *has a unique classical solution* $(\mathbf{u}, q)$, *given by the boundary integral representations* (3.4) *and* (3.5) *in which the density* $\mathbf{h} \in C^0(\Gamma)$ *satisfies the Fredholm integral equation of the second kind* (3.6).

4 The interior Neumann problem

Let $\mathbf{T} \in C^0(\Gamma)$ be a given vector function. We next consider the interior Neumann problem

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} = \mathbf{0} \text{ in } D \tag{4.1}$$

$$\mathbf{\Sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma, \tag{4.2}$$

where $\mathbf{n}$ is the unit normal to Γ pointing outside D.
The Neumann problem (4.1)-(4.2) has at most one classical solution $(\mathbf{u}, q)$ (see e.g. [8] Chapter 1). In order to prove the existence of the solution to this problem, we consider the following boundary integral representations:

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_{\chi^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Psi\right), \ q(\mathbf{x}) = P^s_{\chi^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Psi\right), \ \mathbf{x} \in D, \tag{4.3}$$

where $\Psi \in C^0(\Gamma)$ is an unknown vector density. Applying the boundary condition (4.2) to these boundary integral representations and using the jump formulas (2.18), we obtain the Fredholm integral equation of the second kind with unknown Ψ

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\Psi = \mathbf{T} \text{ on } \Gamma. \tag{4.4}$$

We have the following result:

Lemma 4.1 (see [6]) *Let* $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ *be a bounded domain with boundary* $\Gamma = \Gamma' \cup \Gamma_1$ *of class* $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ *and let* $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re}z \leq 0, \ \mathrm{Im}z = 0\}$. *Then the null spaces of the operators*

$$\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma), \quad \frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n} : C^0(\Gamma) \to C^0(\Gamma) \tag{4.5}$$

are one-dimensional. Moreover, a basis of the space

$$\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right) = \left\{\Psi_0 \in C^0(\Gamma):\ \left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\Psi_0 = \mathbf{0}\ \textit{on}\ \Gamma\right\},$$

is the set $\{\mathbf{N}_1\}$, *where*

$$\mathbf{N}_1(\mathbf{x}) = \begin{cases} \mathbf{n}(\mathbf{x}) & \textit{if}\ \ \mathbf{x} \in \Gamma_1 \\ \mathbf{0} & \textit{if}\ \ \mathbf{x} \in \Gamma', \end{cases} \tag{4.6}$$

and $\mathbf{n}$ *is the unit normal to* Γ *pointing outside* D.

The proof of Lemma 4.1 can be obtained by using similar arguments to those in the proof of Theorem 3.1. For details see [6].
Now, using again Fredholm's alternative, we deduce that the Fredholm integral equation of the second kind (4.4) has a solution $\Psi \in C^0(\Gamma)$ if and only if

$$\int_\Gamma \mathbf{T}\cdot\overline{\mathbf{\Phi}}_0 d\Gamma = 0 \tag{4.7}$$

for all $\mathbf{\Phi}_0 \in \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right)$. This condition is satisfied only in certain particular cases. In order to eliminate this inconvenience, we should complete the integral representations (4.3).
Let $\{\mathbf{\Phi}_1\}$ be a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$. Then $\{\overline{\mathbf{\Phi}}_1\}$ is a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right)$. Also, let $\mathbf{u}_1$ and q_1 be the fields given by

$$\mathbf{u}_1 = \mathbf{W}_{\chi^2,n}\left(\cdot, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right), \quad q_1(\mathbf{x}) = P^s_{\chi^2,n}\left(\cdot, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right) \tag{4.8}$$

for $\mathbf{x} \in \mathbb{R}^n \setminus \Gamma$.
Straightforward computation yields the identity (see [8] Chapter 3):

$$\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} = \int_\Gamma \{\overline{\mathbf{\Sigma}^-(\mathbf{u}_1)\cdot\mathbf{n}}\}\cdot\mathbf{u}_1^- d\Gamma. \tag{4.9}$$

Since $\mathbf{\Phi}_1 \in \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$ it follows that $\mathbf{u}_1^+ = \mathbf{0}$ on Γ, and thus, in view of the jump formulas (2.17), we deduce that $\mathbf{u}_1^- = -\mathbf{\Phi}_1$ on Γ. Therefore, the formula (4.9) becomes

$$\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} = -\int_\Gamma \{\overline{\mathbf{\Sigma}^-(\mathbf{u}_1)\cdot\mathbf{n}}\}\cdot\mathbf{\Phi}_1 d\Gamma. \tag{4.10}$$

On the other hand, from the identity

$$-\frac{1}{2}\mathbf{\Phi}_1 = \mathbf{K}_{\chi^2,n}\mathbf{\Phi}_1 \text{ on } \Gamma$$

and the regularizing properties of the unsteady double-layer integral operator $\mathbf{K}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma)$, we find that (see e.g. [10] in the case $\chi = 0$)

$$\mathbf{\Phi}_1 \in C^{1,\alpha}(\Gamma).$$

Hence the normal stress due to the double-layer potential $\mathbf{u}_1$ has equal limiting values on both sides of Γ (see [8] Theorem 3.4.1, in the case $n = 3$, $\chi = 0$), i.e.,

$$\mathbf{\Sigma}^- \left(\mathbf{W}_{\chi^2,n} \left(\cdot, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) \right) \cdot \mathbf{n} = \mathbf{\Sigma}^+ \left(\mathbf{W}_{\chi^2,n} \left(\cdot, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) \right) \cdot \mathbf{n} \text{ on } \Gamma. \tag{4.11}$$

Further, integrating the equation

$$\nabla \cdot \mathbf{W}_{\chi^2,n} \left(\cdot, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) = 0 \text{ in } D$$

over the domain D, and using the divergence theorem as well as the boundary condition

$$\mathbf{W}^-_{\chi^2,n} \left(\cdot, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) = -\mathbf{\Phi}_1 \text{ on } \Gamma,$$

we obtain the relation

$$\int_\Gamma \mathbf{\Phi}_1 \cdot \mathbf{n} d\Gamma = 0, \tag{4.12}$$

which yields that (see e.g. [8])

$$\mathbf{W}_{\chi^2,n} \left(\mathbf{x}, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) = O(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}| \to \infty. \tag{4.13}$$

This result is sufficient to show that the fields $\mathbf{u}_1$ and q_1 satisfy the far field conditions

$$\begin{aligned} (|\mathbf{u}_1||\nabla \mathbf{u}_1|)(\mathbf{x}) &= o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty \\ (|\mathbf{u}_1||q_1|)(\mathbf{x}) &= o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty. \end{aligned} \tag{4.14}$$

In addition, these fields satisfy the system of equations

$$\nabla \cdot \mathbf{u}_1 = 0, \quad -\nabla q_1 + (\nabla^2 - \chi^2)\mathbf{u}_1 = \mathbf{0} \text{ in } C\overline{D'},$$

as well as the property

$$\mathbf{u}_1^+ = \mathbf{W}^+_{\chi^2,n} \left(\cdot, \frac{1}{2\varpi_n} \mathbf{\Phi}_1 \right) = \mathbf{0} \text{ on } \Gamma'.$$

In view of the uniqueness of the solution to the exterior Dirichlet problem, we thus deduce that

$$\mathbf{u}_1 = \mathbf{0}, \quad q_1 = 0 \text{ in } C\overline{D'} \tag{4.15}$$

and hence

$$\mathbf{\Sigma}^{-}\left(\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot\mathbf{n}=\mathbf{\Sigma}^{+}\left(\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot\mathbf{n}=\mathbf{0}\text{ on }\Gamma'. \tag{4.16}$$

Also, the relation $\mathbf{W}^{+}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)=\mathbf{0}$ on Γ_1 (note that the plus sign applies here for the internal side of Γ_1) together with the uniqueness result of the solution to the interior Dirichlet problem (see [8] Chapter 1) lead to

$$\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)=\mathbf{0},\ P^d_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)=c_1\text{ in }D_1, \tag{4.17}$$

and hence

$$\begin{aligned}\mathbf{\Sigma}^{-}\left(\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot\mathbf{n}&=\mathbf{\Sigma}^{+}\left(\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot\mathbf{n}\\&=-c_1\mathbf{n}\text{ on }\Gamma_1,\end{aligned} \tag{4.18}$$

where $c_1\in\mathbb{C}$.
Now, in view of (4.8), (4.16) and (4.18), the formula (4.10) becomes

$$\begin{aligned}&\int_D(\overline{\chi}^2|\mathbf{u}_1|^2+2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x}\\&=-\int_\Gamma\overline{\left\{\mathbf{\Sigma}^{-}\left(\mathbf{W}_{\chi^2,n}\left(\cdot,\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot\mathbf{n}\right\}}\cdot\mathbf{\Phi}_1d\Gamma=\overline{c_1}\int_{\Gamma_1}\mathbf{\Phi}_1\cdot\mathbf{n}d\Gamma_1.\end{aligned} \tag{4.19}$$

If

$$\int_D(\overline{\chi}^2|\mathbf{u}_1|^2+2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x}=0,$$

then we have $\mathbf{u}_1=\mathbf{0}$ in D, and hence $\mathbf{u}_1^-=\mathbf{0}$ on Γ. In addition, $\mathbf{u}_1^+=\mathbf{0}$ on Γ, and thus, according to the jump formulas (2.17), we obtain $\mathbf{\Phi}_1\equiv\mathbf{0}$. This result contradicts the property $\mathbf{\Phi}_1\neq\mathbf{0}$ on Γ. Therefore, we must have

$$\int_{\Gamma_1}\mathbf{\Phi}_1\cdot\mathbf{n}d\Gamma_1\neq 0,\quad c_1\neq 0. \tag{4.20}$$

4.1 The completion of the boundary integral representations (4.3)

Recall that the boundary integral representation of the velocity field corresponding to the interior Neumann problem in terms of a single-layer potential without any completion leads to the boundary integral equation (4.4), which admits solutions in $C^0(\Gamma)$ only if the condition (4.7) holds.
Let us now consider the boundary integral representations

$$\mathbf{u}(\mathbf{x})=\mathbf{V}_{\chi^2,n}\left(\mathbf{x},\frac{1}{2\varpi_n}\mathbf{\Psi}\right)+\beta_1\mathbf{W}_{\chi^2,n}\left(\mathbf{x},\frac{1}{2\varpi_n}\mathbf{\Phi}_1\right),\quad\mathbf{x}\in D, \tag{4.21}$$

$$q(\mathbf{x}) = P_{\chi^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\mathbf{\Psi}\right) + \beta_1 P^d_{\chi^2,n}\left(\mathbf{x}, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right), \quad \mathbf{x} \in D, \tag{4.22}$$

where $\beta_1 \in \mathbb{C}$ is an unknown constant, $\mathbf{\Psi} \in C^0(\Gamma)$ is an unknown vector density, and the set $\{\mathbf{\Phi}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$.
Applying the boundary condition (4.2) to the boundary integral representations (4.21) and (4.22), and using the jump formulas (2.18), we obtain the following Fredholm integral equation of the second kind with unknown density $\mathbf{\Psi}$:

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\mathbf{\Psi} = \mathbf{T} - \beta_1 \mathbf{\Sigma}^-\left(\mathbf{W}_{\chi^2,n}\left(\cdot, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot \mathbf{n} \text{ on } \Gamma. \tag{4.23}$$

Now, according to the properties (4.18) and (4.20), we can choose the number $\beta_1 \in \mathbb{C}$ such that

$$\beta_1 = \left[\int_\Gamma \left\{\mathbf{\Sigma}^-\left(\mathbf{W}_{\chi^2,n}\left(\cdot, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot \mathbf{n}\right\}\cdot \mathbf{\Phi}_1 d\Gamma\right]^{-1} \int_\Gamma \mathbf{T}\cdot \mathbf{\Phi}_1 d\Gamma. \tag{4.24}$$

Therefore, we get the relation

$$\int_\Gamma \left\{\mathbf{T} - \beta_1 \mathbf{\Sigma}^-\left(\mathbf{W}_{\chi^2,n}\left(\cdot, \frac{1}{2\varpi_n}\mathbf{\Phi}_1\right)\right)\cdot \mathbf{n}\right\}\cdot \mathbf{\Phi}_1 d\Gamma = 0, \tag{4.25}$$

which is just the condition required by Fredholm's alternative in order to have a solution of equations (4.23) in the space $C^0(\Gamma)$.
Concluding the above arguments, we obtain the following property:

Theorem 4.2 *Let $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Re}z \leq 0, \text{Im}z = 0\}$. Also, let $\mathbf{T} \in C^0(\Gamma)$ be given. Assume that the set $\{\mathbf{\Phi}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$. Then there exist the uniquely determined constant β_1 such that the Fredholm integral equation of the second kind* (4.23) *has a solution $\mathbf{\Psi} \in C^0(\Gamma)$. Moreover, the boundary integral representations* (4.21) *and* (4.22)*, obtained with the density $\mathbf{\Psi}$ and the constant β_1, determine the unique classical solution of the interior Neumann problem* (4.1)-(4.2).

5 The mixed boundary value problem

Let $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Re}z \leq 0, \text{Im}z = 0\}$. Also, let $\mathbf{T} \in C^0(\Gamma')$ and $\mathbf{U} \in C^0(\Gamma_1)$ be given.
We next refer to the mixed boundary problem

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} = \mathbf{0} \text{ in } D \tag{5.1}$$

$$\mathbf{\Sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma' \tag{5.2}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Gamma_1. \tag{5.3}$$

The mixed boundary value problem corresponding to the steady case ($\chi = 0$) can be consulted in [12] and [1].
In order to prove the existence of solutions to the boundary value problem (5.1)-(5.3) we consider the boundary integral representations

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_{\chi^2,n}\Big|_{\Gamma'}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Psi\right) + \mathbf{W}_{\chi^2,n}\Big|_{\Gamma_1}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Phi\right), \tag{5.4}$$

$$q(\mathbf{x}) = P^s_{\chi^2,n}\Big|_{\Gamma'}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Psi\right) + P^d_{\chi^2,n}\Big|_{\Gamma_1}\left(\mathbf{x}, \frac{1}{2\varpi_n}\Phi\right), \tag{5.5}$$

$\mathbf{x} \in D$.
Now, imposing the boundary condition (5.2) to the boundary integral representations (5.4) and (5.5), and making use of the jump formulas (2.18), we obtain the equation

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\Big|_{\Gamma'}\right)\Psi + \mathbf{\Sigma}^-\left(\mathbf{W}_{\chi^2,n}\Big|_{\Gamma_1}\left(\cdot, \frac{1}{2\varpi_n}\Phi\right)\right)\cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma'. \tag{5.6}$$

Further, applying the boundary condition (5.3) to the boundary integral representation (5.4) and using the jump formulas (2.17), we get the equation

$$\left(-\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\Big|_{\Gamma_1}\right)\Phi + \mathbf{V}_{\chi^2,n}\Big|_{\Gamma'}\left(\cdot, \frac{1}{2\varpi_n}\Psi\right) = \mathbf{U} \text{ on } \Gamma_1. \tag{5.7}$$

Then we have the following result:

Theorem 5.1 (see [5], [6]) *Let* $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ *be a bounded domain with boundary* $\Gamma = \Gamma' \cup \Gamma_1$ *of class* $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ *and let* $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re}z \leq 0,\ \mathrm{Im}z = 0\}$*. Also, let* $\mathbf{T} \in C^0(\Gamma')$ *and* $\mathbf{U} \in C^0(\Gamma_1)$ *be given. Then the system of Fredholm integral equations of the second kind* (5.6) *and* (5.7) *has a unique solution* $(\Psi, \Phi) \in C^0(\Gamma') \times C^0(\Gamma_1)$*, and the boundary integral representations* (5.4) *and* (5.5)*, obtained with the densities* Ψ *and* Φ*, provide the unique classical solution* $(\mathbf{u}, q)$ *of the mixed boundary value problem* (5.1)-(5.3).

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