,, *Caius Iacob"* **Conference on**
Fluid Mechanics&Technical Applications
Bucharest, Romania, November 2005

A numerical study of laminar flow past two circular cylinders in-line at low Reynolds numbers

by
GHEORGHE JUNCU[1]

Abstract

This work presents a computational study of the steady, axisymmetric, viscous flow around two circular cylinders in tandem. The vorticity – stream function formulation of the Navier – Stokes equations was chosen. Numerical solutions have been obtained in bipolar cylindrical coordinates. The finite difference method was used to discretize the model equations. A nested defect correction algorithm was employed to solve the discrete equations. Different cylinders spacing and sizes were considered for the upstream cylinder Reynolds number equal to 2. Vorticity and pressure distributions on the cylinders surfaces and drag coefficients are presented and compared with those calculated for an isolated cylinder.

Keywords: *Laminar flow, two circular cylinders, bipolar coordinates, multigrid, defect correction.*

1 Introduction

Hydrodynamic interaction between two cylinders is an important phenomenon in engineering flows. The arrangement of the cylinders with respect to the free stream flow direction can be classified as:
tandem (or in-line) – the free stream flow direction is parallel with the line of the centers of the cylinders;
transverse (or side-by-side) – the free stream flow direction is perpendicular to the line of the cylinders centers;
staggered.
Significant research has gone, experimentally and numerically, for the understanding of the flow past two cylinders [1], [2]. The flow past two cylinders in transverse arrangement was analysed

[1]POLITEHNICA University Bucharest, Catedra Inginerie Chimica, Polizu 1, 78126 Bucharest, Romania, Fax: +40 21 345 05 96, Phone: + 40 21 345 05 96
E-mail: juncugh@netscape.net, juncu@easynet.ro

numerically by Tokunga et al. [3] and Cheung et al. [4]. Cheung et al. [4], Mittal et al. [5], Mittal and Kumar [6] and Khorrami et al. [7] report computational results for flow past two cylinders in-line and staggered arrangements.

Cheung et al. [4] employed a spectral element method on a Cartesian grid to analyse the interference effects when two cylinders or two spheres are placed in series or parallel in a low Reynolds number flow. It was found that for two equal cylinders with their line of centers parallel to the free stream flow direction, the drag on the trailing body is less than that of leading body, which in turn is less than the drag of an isolated cylinder. Also, the drag on the leading body is significantly greater than that on the trailing cylinder. Mittal et al. [5] used a stabilized finite element formulation on a Cartesian grid to study incompressible flows past a pair of equal-sized cylinders at Reynolds numbers 100 and 1000. Computations are carried out for three sets of cylinder arrangements, two in tandem and one staggered. The flow-induce oscillations of a pair of equal-sized cylinders in tandem and staggered arrangement placed in uniform incompressible flow for Reynolds number 100 is studied by Mittal and Kumar [6]. As in [5], a Cartesian grid was used. Khorrami et al. [7] focused on two-dimensional, time-accurate flow simulations for the tandem cylinder configuration. This setup is viewed to be representative for several component-level flow interactions that occur when air flows over the main landing gear of large civil transports. Results of the unsteady Reynolds Averaged Navier-Stokes computations with a two-equation turbulence model, at a Reynolds number of 0.166×10^6 and a Mach number of 0.166, are presented.

The flow past two cylinders in tandem is used as test problem for new computational methods in [8 – 10]. Young et al. [8] proposed a three-step FEM-BEM computational technique to simulate high-Reynolds number flow past circular cylinders in 2D incompressible viscous flows in external flow fields. Guermond and Lu [9] introduced a domain decomposition method for simulating 2D external, incompressible viscous flows. Russell and Wang [10] used an underlying regular Cartesian grid to solve 2D incompressible viscous flows around multiple moving objects.

The aim of this work is to complete the analysis of the flow past two cylinders in tandem at low Reynolds numbers. The Navier-Stokes equations in stream function – vorticity formulation were solved numerically in the bipolar cylindrical coordinate system. Different cylinders spacing and sizes were considered for the upstream cylinder Reynolds number equal to 2.

2 Statement of the problem

Let us consider two infinitely long cylinders of diameters d_i, i = 1,2, placed in a horizontal flow, parallel with their line of centers, of an incompressible fluid having free stream velocity U_∞ (as illustrated in figure 1). The diameters of the cylinders are assumed considerably higher than the molecular mean free path of the surrounding fluid. The fluid is homogeneous, Newtonian and the flow is steady and laminar. The density ρ and viscosity μ of the fluid are considered constant. Oscillations and rotation of the cylinders do not occur during the flow. Let a system

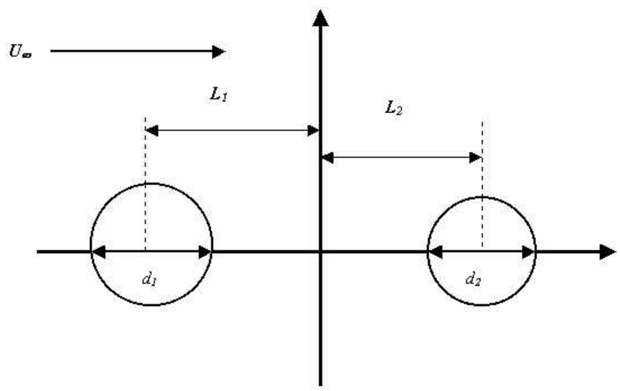

Figure 1: Schematic of the problem.

of Cartesian coordinates $(x, y,\ z)$ be chosen so that the centres of the cylinders lie along the x-axis (see also figure 1). The cylinders being considered infinitely long, the flow does not depend on $z-$ coordinate. Also, for low Reynolds numbers, we can consider the flow symmetric versus the $y-$ axis.
The most convenient coordinate system for a pair of cylinders in tandem is the orthogonal bipolar cylindrical coordinate system. The bipolar cylindrical coordinate system is defined by, [11],

$$x\ =\ \frac{c\ \sin\,\xi}{\cosh\,\eta\ -\ \cos\,\xi}\ ;\quad y\ =\ \frac{c\ \sinh\,\eta}{\cosh\,\eta\ -\ \cos\,\xi};\quad z\ =\ z$$

where c ¿ 0 is a characteristic length. This transformation maps the upper half of the xy – plane (from which the domain occupied by the cylinders is excluded) into the rectangle $\eta_1 \leq \eta \leq \eta_2$, $0 \leq \xi \leq \pi$ (η_1 ¡ 0, η_2 ¿ 0). The surfaces of the cylinders are located at $\eta = \eta_1$ and $\eta = \eta_2$. The relations between η_i, the diameters of the cylinders d_i and the distances L_i of their centers from the origin of the coordinates system are:

$$\frac{d_i}{2}\ =\ \frac{c}{\sinh\,|\,\eta_i|}\ ;\quad L_i\ =\ c\ \coth\,|\,\eta_i|\ ,\ \ i\ =\ 1,\,2$$

The Navier-Stokes equations in the stream-function – vorticity formulation for an axisymmetrical flow field in general orthogonal curvilinear coordinates $\alpha,\ \beta,\ \zeta$, are:

$$\frac{h_\zeta}{h_\alpha h_\beta}\left[\frac{\partial\,\psi}{\partial\alpha}\frac{\partial}{\partial\,\beta}\left(\frac{\omega}{h_\zeta}\right)-\frac{\partial\,\psi}{\partial\,\beta}\frac{\partial}{\partial\,\alpha}\left(\frac{\omega}{h_\zeta}\right)\right]\ =\ \nu\,E^2\ (\,h_\zeta\,\omega\,) \tag{1}$$

$$E^2\,\psi\ =\ h_\zeta\,\omega \tag{2}$$

where

$$E^2 = \frac{h_\zeta}{h_\alpha h_\beta}\left[\frac{\partial}{\partial\,\alpha}\left(\frac{h_\beta}{h_\zeta\,h_\alpha}\frac{\partial}{\partial\,\alpha}\right) + \frac{\partial}{\partial\,\beta}\left(\frac{h_\alpha}{h_\zeta\,h_\beta}\frac{\partial}{\partial\,\beta}\right)\right]$$

and ν is the kinematic viscosity of the fluid. The scale factors (metric coefficients) h_η, h_ξ, h_z for the bipolar cylindrical coordinate system are:

$$h_\eta = h_\alpha = \frac{c}{\cosh\,\eta - \cos\,\xi}\,;\;\; h_\xi = h_\beta = \frac{c}{\cosh\,\eta - \cos\,\xi}\,;\;\; h_z = h_\zeta = 1\,.$$

For convenience, all variables are considered to be non-dimensionalized with respect to the radius of the leading cylinder $d_1\;/\;2$ for length, U_∞ for velocity, $U_\infty d_1\;/\;2$ for stream function and $d_1\;/\;2\;U_\infty$ for vorticity. Also, it is convenient numerically to work with the deviation from the uniform flow ψ^*,

$$\psi^* = \psi - \frac{\bar{c}\,\sin\,\xi}{\cosh\,\eta - \cos\,\xi}\,;\,\bar{c} = \frac{2\,c}{d_1}.$$

After η and ξ are substituted for α and β and the scale factors are expressed explicitly, the dimensionless Navier-Stokes equations are:

$$\frac{(\cosh\,\eta - \cos\,\xi\,)^2}{\bar{c}^2}\left(\frac{\partial^2\,\psi^*}{\partial\,\eta^2} + \frac{\partial^2\,\psi^*}{\partial\,\xi^2}\right) = \omega \tag{3}$$

$$\left[\frac{\partial\,\omega}{\partial\xi}\left(\frac{\partial\,\psi^*}{\partial\,\eta} - \frac{\bar{c}\,\sin\,\xi\,\sinh\,\eta}{(\cosh\,\eta - \cos\,\xi\,)^2}\right) - \frac{\partial\,\omega}{\partial\,\eta}\left(\frac{\partial\,\psi^*}{\partial\,\xi} + \frac{\bar{c}\,(\cosh\,\eta\,\cos\,\xi - 1\,)}{(\cosh\,\eta - \cos\,\xi\,)^2}\right)\right]$$
$$= \frac{2}{Re}\left(\frac{\partial^2\,\omega}{\partial\,\eta^2} + \frac{\partial^2\,\omega}{\partial\,\xi^2}\right) \tag{4}$$

where the Reynolds number Re, based on the leading cylinder diameter d_1, is:

$$\mathrm{Re} = U_\infty d_1\;/\;\nu.$$

The boundary conditions for the dimensionless stream-function and vorticity are:
- cylinders surfaces ($\eta = \eta_i$, i = 1,2)

$$\psi^* = \text{-}\;c\;sin\xi\;/\;(cosh\;\eta\;\text{-}\;cos\;\xi\;) \tag{3a}$$

- free stream ($\eta \to 0$, $\xi \to 0$)

$$\psi^* \to 0,\;\omega \quad \to 0 \tag{3b}$$

- symmetry axis ($\xi = 0$ and $\eta \neq 0$, $\xi = \pi$)

$$\psi^* = \omega = 0 \quad (3c)$$

The pressure coefficients $C_P\ (\xi)$ on the cylinders surfaces and the drag coefficients, C_D, are computed with the relations [12],

$$C_P\ (\xi) = P\ (\xi) - P\ (\xi_{ref}) = \frac{2}{Re} \int\limits_{\xi_{ref}}^{\xi} \left. \frac{\partial \omega}{\partial \eta} \right|_{\eta = \eta_i} d\xi \qquad (5)$$

$$C_D = 2 \int\limits_{\xi_{\min}}^{\xi_{\max}} \frac{\partial y}{\partial \xi} C_P\ (\xi)\ d\xi + \frac{4}{Re} \int\limits_{\xi_{in}}^{\xi_{ax}} \frac{\partial x}{\partial \xi} \omega\, d\xi \qquad (6)$$

The two integrals in (6) are referred as the pressure and friction drag coefficients and are denoted $C_{D,P}$ and $C_{D,F}$, respectively. In relations (5) and (6), for both cylinders, $\xi_{ref} = \xi_{min} = 0$ and $\xi_{max} = \pi$.. For this reason, the sign "–" should be considered for the integrals in (6), when the calculations are carried out for the trailing cylinder.

3 Solution procedure

The Navier-Stokes equations were solved numerically. The two-dimensional region $(\eta_1, \eta_2) \times (0, \pi)$ was transformed into the unit square. The finite difference method was used for discretization. Equation (3) was discretized with the central second order accurate finite difference scheme. A double discretization (upwind and central finite difference schemes), necessary for the defect correction iteration, was used for equation (**??**). Numerical experiments were made with the discretization steps $h = 1 / 64, 1 / 128, 1 / 256$. The algorithm employed is the nested defect-correction iteration, [13, 14]. The method is well described in the references mentioned previously and it is not necessary to reproduce it here. It must be mentioned that in this work the defect-correction iteration was used only locally [15], in the region defined by

$|\ 2x / d_1\ | \leq n\ max\ (\ 2\ L_1 / d_1, 2\ L_2 / d_2\ )$

$$y \leq n\ max\ (\ 2\ L_1 / d_1, 2\ L_2 / d_2\ )$$

The values of n were varied until for two consecutive values, n_i, n_{i+1}, $n_{i+1} - n_i \geq 2$, the relative changes in C_D are lower than 1%. This process was repeated for each $2\ L_i / d_i$ and d_1 / d_2 value.

One of the main problems in solving numerically the Navier-Stokes equations in unbounded regions is the boundary conditions at infinity. For the flow past an isolated cylinder, a reference article in solving this problem is [16]. In this case, i.e. the flow past an isolated cylinder, the free stream must be located at a large but finite distance from the cylinder center. For the present problem, in bipolar cylindrical coordinate system, the infinity of the physical space (x, y) is located in the point $\xi = \eta = 0$. For this reason, the boundary conditions (3b) were used in this work.

4 Results

The dimensionless equations (2a,b) and the boundary conditions (3) depend on three dimensionless parameters: Re, $2L_1/d_1$, and d_1/d_2. The first question discussed in this section is the selection of the numerical values of these parameters.

Only one value of the up-stream cylinder Re number was used: Re = 2. For Re = 2, the hydrodynamic regime of the laminar flow around an isolated circular cylinder is steady flow without separation (Re $\leq$ 5). The geometric quantities $2L_1/d_1$, and d_1/d_2,take the values, $2L_1/d_1 = 1.5$, 2, 3 and $d_1/d_2 = 0.5$, 1, 2. The diameters of the cylinders are the characteristics lengths of the present problem. If $d_1 = d_2 = d$, the values of $2L/d = 1.5$, 2, 3 correspond to a separation gap between cylinders equal to 0.5 d, $d$and $2d$, respectively.

The first task in any numerical work is to validate the code's ability to reproduce published results accurately. Unfortunately, excepting figure 6 from [4], there are no data in literature to verify the accuracy of the present computations.

The first numerical experiments were made considering $d_1 = d_2 = d$. The influence of $2L/d$ on the surface vorticity and surface pressure coefficient is plotted in figures 2 and 3, respectively. In all graphs, the surface's coordinate $\xi\ /\ \pi$ varies from 0 to 1. For the upstream cylinder, the front stagnation point is located at $\xi\ /\ \pi = 0$ while the rear stagnation point at $\xi\ /\ \pi = 1$. For the downstream cylinder, the front stagnation point is located at $\xi\ /\ \pi = 1$ while the rear stagnation point at $\xi\ /\ \pi = 0$. Under these conditions, in each situation, the isolated cylinder data were adapted to be consistent with the present data.

Figures 2 and 3 show that the tandem interaction changes the surface vorticity and coefficient pressure of the two cylinders. As expected, the decrease in $2L/d$increases these effects. These effects are stronger on the trailing cylinder. The surface vorticity and coefficient pressure on the front stagnation zone of the leading cylinder are not strongly influenced by the interaction. For the leading cylinder, the interaction's effects increases on the rear stagnation zone. The strongest interaction effect is a radical change in the surface vorticity and coefficient pressure (especially) of the trailing cylinder. In figure 3b it can be seen that a low pressure zone occurs on the front of the trailing cylinder. Also, it must be mentioned that flow separation occurs for $2L/d$= 1.5, 2.

This behaviour has some similarities with that described in [17] for the flow past two spheres in tandem. For the same hydrodynamic system, i.e. two spheres in tandem, Tsuji et al. [18] showed that the tandem interactions are more pressure drag effects than friction drag effects. This statement seems to apply to the low Re number flow around two circular cylinders in

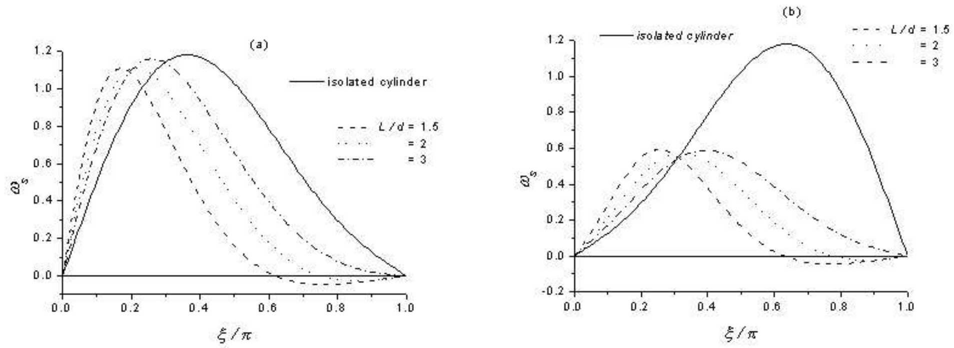

Figure 2: The vorticity distribution over the bodies surface; $d_1 = d_2 = d$; (a) upstream cylinder; (b) downstream cylinder.

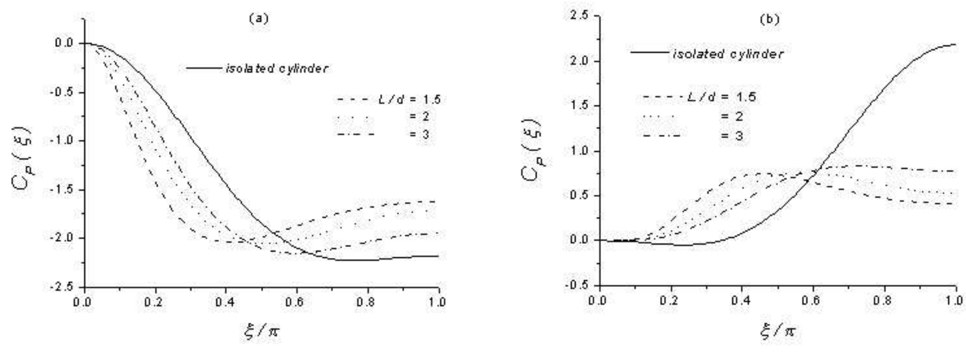

Figure 3: The distribution of the pressure coefficient over the bodies surface; $d_1 = d_2 = d$; (a) upstream cylinder; (b) downstream cylinder.

tandem.

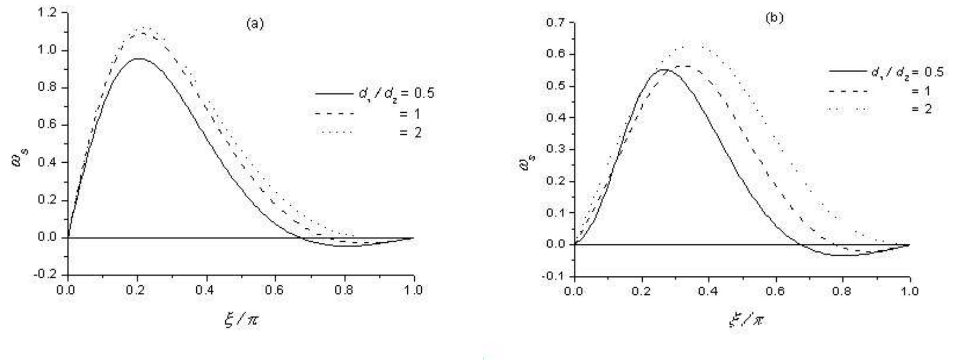

Figure 4: The distribution of the vorticity over the bodies surface; 2 L_1 / d_1 = 2; (a) upstream cylinder; (b) downstream cylinder.

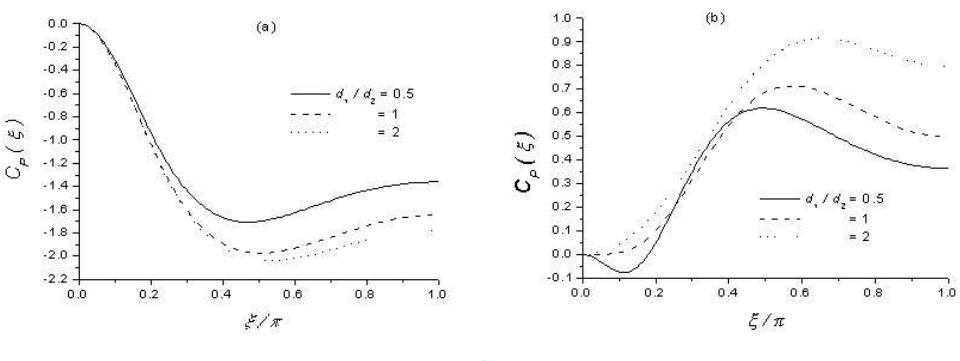

Figure 5: The distribution of the pressure coefficient over the bodies surface; 2 L_1 / d_1 = 2; (a) upstream cylinder; (b) downstream cylinder.

The tandem interaction for cylinders of different sizes is plotted in figure 4 and 5. It can be seen that the general rules of tandem interaction discussed previously remain valid even when the cylinders have different diameters. The variation of the diameters ratio induces only quantitative changes. If d_1 / d_2 is smaller / greater than one, the surface vorticity decreases / increases for both cylinders. The pressure coefficient over the surface of the leading cylinder increases with the decrease in the diameters ratio. For the trailing cylinder, the decrease in the diameters ratio decreases the pressure coefficient over the surface of the body.
Table 1 summarizes the present computations of the drag coefficients. Based on the data presented in table 1, we can make the following observations:
both cylinders experience a lower drag coefficient compared to the isolated cylinder;

for both cylinders, the drag coefficient decreases as the gap decreases;
if d_1 / d_2 ≥ 1, the drag on the trailing cylinder is significantly lower than that of the leading cylinder (slip-streaming phenomenon); this aspect amplifies with the increase in diameters ratio;
if d_1 / d_2 ¡ 1, the trailing cylinder may exhibit a higher drag than the leading cylinder;
for d_1 / d_2 = 1, the agreement between the present λ_1 values and those presented in [4] is very good; our λ_2 values are smaller than those from [4]; note that in [4] the cylinder Reynolds number is considered equal to 1.

Table 1: Drag coefficients for a pair of cylinders in tandem, Re = 2, $\lambda_1 = C_{D1}/C_D$, $\lambda_2 = C_{D2}/C_D$

L_1/d_1	d_1/d_2	C_{D1}	C_{D2}	C_D *isolated cylinder*	λ_1	λ_2
1.5	1	5.14	2.51		0.775	0.379
2	1	5.29	2.62		0.798	0.395
3	1	5.48	2.81	6.629	0.827	0.424
2	0.5	4.46	4.92		0.673	0.742
2	2	5.64	1.61		0.85	0.243

References

M.M. Zdravkovich. Review of flow interference between two circular cylinders in various arrangements. ASME J, Fluids Eng. 99, 618-633 (1977).

S.S. Chen. Flow – Induced Vibrations of Circular Cylindrical Structures. New York, Hemisphere, 1987.

H. Tokunga, T. Tanaka, N. Satofuka. Numerical simulation of viscous flows along multiple bodies by generalized vorticity-stram function formulation. Comput. Fluid. Dyn. J. 1, 58-66 (1992).

A.K.W. Cheung, B.T. Tan, K. Hourigan, M.C. Thompson. Interference drag between spherical and cylindrical particles in Stokes flow. 14^{th} Australasian Fluid Mechanics Conference, Adelaide, Australia, pp. 115 – 118, 2001.

S. Mittal, V. Kumar, A. Raghuvanshi. Unsteady incompressible flows past two cylinders in tandem and staggered arrangements. Int. J. Num. Meth. Fluids 25, 1315-1344 (1997).

S. Mittal, V. Kumar. Flow-induced oscillations of two cylinders in tandem and staggered arrangements. J. Fluids Structures 15, 717-736 (2001).

M.R. Khorrami, M.M. Choudhari, L.N. Jenkins, C.B. McGinley. Unsteady flowfield around tandem cylinders as prototype for component interaction in airframe noise. 26^{th} AIAA Aeroacoustics Conference. Monterey, California, 2005.

D.L. Young, J.L. Huang, T.I. Eldho. Numerical simulation of high-Reynolds number flow around circular cylinders by a three-step FEM-BEM model. J. Eng. Mech. 121, 367-378 (1995).

J.-L. Guermond, H.Z. Lu. A domain decomposition method for simulating advection dominated, external incompressible viscous flows. Comput. & Fluids 29, 525-546 (2000).
D. Russell, Z. J. Wang. A Cartesian grid method for modeling multiple moving objects in 2D incompressible viscous flow. J. Comput. Phys. 191, 177-205 (2003).
E. W. Weisstein. Bipolar cylindrical coordinates. MathWorld – A Wolfram web resource. http://mathworld.wolfram.com/BipolarCylindricalCoordinates.html
F.C. Thames, J.F. Thompson, C.W. Mastin, R.L. Walker. Numerical solutions for viscous and potential flow about arbitrary two-dimensional bodies using body-fitted coordinate systems. J. Comput. Phys. 24, 245-273 (1977).
Gh. Juncu, R. Mihail. Numerical solution of the steady incompressible Navier-Stokes equations for the flow past a sphere by a multigrid defect correction technique. Int. J. Num. Meth. Fluids 11, 379-395 (1990).
Gh. Juncu. A numerical study of steady viscous flow past a fluid sphere. Int. J. Heat Fluid Flow 20, 414-421 (1999).
W. Hackbusch. Local defect correction method and domain decomposition technique. *Computing Supplementum* **5**, Wien, Springer, 89 (1984).
B. Fornberg. A numerical study of steady viscous flow past a circular cylinder. J. Fluid Mech. 98, 819 - 855 (1980).
R. Tal (Thau), D.N. Lee, W.A. Sirignano. Heat and momentum transfer around a pair of spheres in viscous flow. Int. J. Heat Mass Transfer 27, 1953 (1984).
Y. Tsuji, Y. Morikawa, K. Terashima. Fluid-dynamic interaction between two spheres. Int. J. Multiphase Flow 8, 71 (1982).