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Asymptotic thermal flow around a highly conductive suspension by

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Abstract

Radiant spherical suspensions have an ε-periodic distribution in a three dimensional incompressible viscous fluid governed by the Stokes-Boussinesq system. We study the border case when the radius of the spheres is of order ε^3 and the ratio of the solid/fluid conductivities is of order ε^{-6}. We apply a homogenization procedure by adapting the energy method introduced by [1] and developed by [2]-[7]. The macroscopic behavior is described by a nonlocal law of Brinkman-Boussinesq type and two coupled heat equations, where the radiation and a certain capacity of the vanishing suspensions appear. This result completes those obtained for the thermal flow when the volume of the solid matrix is not vanishing as $\varepsilon \to 0$ (see [8]-[9]).

1 Preliminaries

Let $\Omega \subset \mathbf{R}^3$ be a bounded open set and let

$$Y := \left(-\frac{1}{2}, +\frac{1}{2}\right)^3.$$

$$Y_\varepsilon^k := \varepsilon k + \varepsilon Y, \quad k \in \mathbf{Z}^3.$$

$$\mathbf{Z}_\varepsilon := \{k \in \mathbf{Z}^3, \quad Y_\varepsilon^k \subset \Omega\}$$

The reunion of the suspensions is defined by

$$T_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} B(\varepsilon k, r_\varepsilon),$$

where $0 < r_\varepsilon << \varepsilon$ and $B(\varepsilon k, r_\varepsilon)$ is the ball of radius r_ε centered at εk, $k \in \mathbf{Z}_\varepsilon$. The fluid domain is given by

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon.$$

Let $\mathbf{e}^{(3)}$ be the last vector of the canonical basis of $\mathbf{R}^3$, n the normal on $\partial\Omega_\varepsilon$ in the outward direction and $[\cdot]_\varepsilon$ the jump across the interface ∂T_ε. For $a > 0$ (the so-called Rayleigh number), $f \in L^2(\Omega)$, $g \in C(\overline{\Omega})$ and $b > 0$, where $b(\varepsilon/r_\varepsilon)^3$ stands for the ratio of the solid/fluid conductivities, we consider

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the problem corresponding to the non-dimensional Stokes-Boussinesq system governing the thermal flow of an ε-periodic distribution suspension of solid spheres:
To find $(u^\varepsilon, p^\varepsilon, \theta^\varepsilon) \in H^1(\Omega_\varepsilon; \mathbf{R}^3) \times L^2(\Omega_\varepsilon) \times H^1(\Omega)$, solution of

$$\begin{aligned}
\mathrm{div} u^\varepsilon &= 0 \quad \text{in} \quad \Omega_\varepsilon, && (1)\\
-\Delta u^\varepsilon + \nabla p^\varepsilon &= a\theta^\varepsilon \mathbf{e}^{(3)} \quad \text{in} \quad \Omega_\varepsilon, && (2)\\
-\Delta \theta^\varepsilon + u^\varepsilon \nabla \theta^\varepsilon &= f \quad \text{in} \quad \Omega_\varepsilon, && (3)\\
-\Delta \theta^\varepsilon &= g \quad \text{in} \quad T_\varepsilon, && (4)\\
[\theta^\varepsilon]_\varepsilon &= 0 \quad \text{on} \quad \partial T_\varepsilon, && (5)\\
\frac{\partial \theta^\varepsilon}{\partial n} &= b(\varepsilon/r_\varepsilon)^3 \frac{\partial \theta^\varepsilon}{\partial n} \quad \text{on} \quad \partial T_\varepsilon, && (6)\\
u^\varepsilon &= 0 \quad \text{on} \quad \partial \Omega_\varepsilon, && (7)\\
\theta^\varepsilon &= 0 \quad \text{on} \quad \partial \Omega. && (8)
\end{aligned}$$

Introducing

$$V_\varepsilon := \{\ v \in H_0^1(\Omega_\varepsilon; \mathbf{R}^3), \quad \mathrm{div}\, v = 0\},$$

the variational formulation of (1)-(8) reads:

$$\begin{aligned}
\forall (v,q) \in V_\epsilon \times L^2(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla v\, dx &= \ a \int_{\Omega_\varepsilon} \theta^\varepsilon v_3\, dx \\
\int_{\Omega_\varepsilon} q\, \mathrm{div} u^\varepsilon\, dx &= \ 0
\end{aligned}$$

$$\begin{aligned}
\forall \varphi \in H_0^1(\Omega_\varepsilon), \qquad \int_{\Omega_\varepsilon} \nabla \theta^\varepsilon \nabla \varphi\, dx \qquad &+ \ b(\varepsilon/r_\varepsilon)^3 \int_{T_\varepsilon} \nabla \theta^\varepsilon \nabla \varphi\, dx \\
+ \int_{\Omega_\varepsilon} u^\varepsilon \varphi \nabla \theta^\varepsilon\, dx \ &= \int_{\Omega_\varepsilon} f\varphi + b(\varepsilon/r_\varepsilon)^3 \int_{T_\varepsilon} g\varphi dx.
\end{aligned}$$

We define $\mathcal{F}_\varepsilon \in H^{-1}(\Omega)$ by

$$\forall \varphi \in H_0^1(\Omega), \quad \mathcal{F}_\varepsilon(\varphi) := \int_{\Omega_\varepsilon} f\varphi\, dx + b(\varepsilon/r_\varepsilon)^3 \int_{T_\varepsilon} g\varphi dx.$$

Then, for $\gamma > 0$ (we shall choose a suitable value for this parameter later), we can present the variational formulation of the problem (1)–(8):

To find $(u^\varepsilon, \theta^\varepsilon) \in V_\varepsilon \times H_0^1(\Omega)$ such that

$$\langle G(u^\varepsilon, \theta^\varepsilon), (v, \varphi) \rangle = \mathcal{F}_\varepsilon(\varphi) \ \ for\ \ any \ \ (v, \varphi) \in V_\varepsilon \times H_0^1(\Omega), \tag{9}$$

where the mapping $G :\ V_\varepsilon \times H_0^1(\Omega) \to V'_\varepsilon \times H^{-1}(\Omega)$ is defined by

$$\begin{aligned}
\langle G(u,\theta), (v,\varphi) \rangle = \qquad & \gamma \int_{\Omega_\varepsilon} \nabla u \nabla v\, dx \qquad - \gamma\, a \int_{\Omega_\varepsilon} \theta v_3\, dx \\
+ \int_{\Omega_\varepsilon} \nabla \theta \nabla \varphi\, dx \ \ + & \int_{\Omega_\varepsilon} u \varphi \nabla \theta\, dx \ \ + b(\varepsilon/r_\varepsilon)^3 \int_{T_\varepsilon} \nabla \theta \nabla \varphi\, dx.
\end{aligned}$$

In order to prove the existence theorem for problem (9), we make use of the following result of Gossez:

Theorem 1.1 *Let X be a reflexive Banach space and $G: X \to X'$ a continuous mapping between the corresponding weak topologies. If*

$$\frac{\langle G\varphi, \varphi\rangle}{|\varphi|_X} \to \infty \quad as \quad |\varphi|_X \to \infty$$

then G is a surjection.

Acting as in the proof of Theorem 5.2.2 [8] Ch 1, Sec. 5, we find that the existence of the weak solutions of problem (9) is assured if γ is chosen sufficiently small. Moreover, if $(u^\varepsilon, \theta^\varepsilon)$ is a solution of problem (9), then, by using the weak maximum principle, we obtain that $\theta^\varepsilon \in L^\infty(\Omega)$, (see Theorem 3.4 [8] Ch 2, Sec. 3). We do not have a uniqueness result, except if we assume that $a > 0$ is small enough.

2 Basic inequalities

Lemma 2.1 *For every $0 < r_1 < r_2$, consider:*

$$C(r_1, r_2) := \{x \in \mathbf{R}^3, \quad r_1 < |x| < r_2\}.$$

Then, if $u \in H^1(C(r_1, r_2))$, the following estimate holds true:

$$|\nabla u|^2_{C(r_1,r_2)} \geq \frac{4\pi r_1 r_2}{r_2 - r_1} \left| \fint_{\mathbf{S}_{r_2}} u \, d\sigma - \fint_{\mathbf{S}_{r_1}} u \, d\sigma \right|^2, \tag{10}$$

where $\fint_{\mathbf{S}_r} \cdot \, d\sigma := \frac{1}{4\pi r^2} \int_{\mathbf{S}_r} \cdot \, d\sigma$.

Lemma 2.2 *There exists a positive constant $C > 0$ such that, for any $R > 0$, $\alpha \in (0, 1)$ and $u \in H^1(B(0, R))$, the following inequality holds:*

$$\int_{B(0,R)} |u - \fint_{\mathbf{S}_{\alpha R}} u \, d\sigma|^2 \, dx \leq C \frac{R^2}{\alpha} |\nabla u|^2_{B(0,R)}. \tag{11}$$

We denote the domain confined between the spheres of radii $a < b$ by

$$\mathcal{C}(a, b) := \{x \in \mathbf{R}^3, \ a < |x| < b\}$$

and correspondingly

$$\mathcal{C}^k(a, b) := \varepsilon k + \mathcal{C}(a, b),$$

From now on, we consider R_ε to be a radius satisfying

$$r_\varepsilon << R_\varepsilon << \varepsilon \tag{12}$$

and we use the following notations:

$$\mathcal{C}_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} \mathcal{C}^k(r_\varepsilon, R_\varepsilon).$$

$$S^k_{r_\varepsilon} = \partial B(\varepsilon k, r_\varepsilon), \quad S_{r_\varepsilon} := \cup_{k\in \mathbf{Z}_\varepsilon} S^k_{r_\varepsilon},$$
$$S^k_{R_\varepsilon} = \partial B(\varepsilon k, R_\varepsilon), \quad S_{R_\varepsilon} := \cup_{k\in \mathbf{Z}_\varepsilon} S^k_{R_\varepsilon},$$

For any $r \in (0, \varepsilon)$, we define $G_r : H^1_0(\Omega) \to L^2(\Omega)$ by

$$G_r(\theta)(x,t) = \sum_{k\in \mathbf{Z}_\varepsilon} \left(\fint_{\mathbf{S}^k_r} \theta(y,t)\, d\sigma_y \right) 1_{Y^k_\varepsilon}(x). \tag{13}$$

Lemma 2.3 *For every $\theta \in H^1_0(\Omega)$ we have*

$$|\theta - G_{R_\varepsilon}(\theta)|_{L^2(\cup_{k\in\mathbf{Z}_\varepsilon} Y^k_\varepsilon)} \leq C \left(\frac{\varepsilon^3}{R_\varepsilon}\right)^{1/2} |\nabla\theta|_{L^2(\Omega)} \tag{14}$$
$$|\theta - G_{r_\varepsilon}(\theta)|_{L^2(T_\varepsilon)} \leq C r_\varepsilon |\nabla\theta|_{L^2(T_\varepsilon)} \tag{15}$$
$$|G_{R_\varepsilon}(\theta) - G_{r_\varepsilon}(\theta)|_{L^2(\Omega)} \leq C \left(\frac{\varepsilon^3}{r_\varepsilon}\right)^{1/2} |\nabla\theta|_{L^2(\mathcal{C}_\varepsilon)}. \tag{16}$$

Denoting $\fint_{T_\varepsilon} \cdot\, dx = \frac{1}{|T_\varepsilon|}\int_{T_\varepsilon} \cdot dx$, we also have

$$|G_{R_\varepsilon}(\theta)|^2_{L^2(\Omega)} = \fint_{T_\varepsilon} |G_{R_\varepsilon}(\theta)|^2 dx \quad \text{and} \quad |G_{r_\varepsilon}(\theta)|^2_{L^2(\Omega)} = \fint_{T_\varepsilon} |G_{r_\varepsilon}(\theta)|^2 dx. \tag{17}$$

Proposition 2.4 *For any $\theta \in H^1_0(\Omega)$ we have:*

$$\fint_{T_\varepsilon} |\theta|^2\, dx \leq C \max(1, \frac{\varepsilon^3}{r_\varepsilon}) |\nabla\theta|^2_{L^2(\Omega)}, \tag{18}$$

Remark 2.5 *Using the Mean Value Theorem, we easily find that*

$$|G_{r_\varepsilon}(\varphi) - \varphi|_{L^\infty(\mathcal{C}_\varepsilon \cup D_\varepsilon)} \leq 2R_\varepsilon |\nabla\varphi|_{L^\infty(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{19}$$

3 A priori estimates

In the sequel, we denote

$$\gamma_\varepsilon := \frac{r_\varepsilon}{\varepsilon^3} \tag{20}$$

and we assume that

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon = \gamma \in]0, +\infty[. \tag{21}$$

Proposition 3.1 *We have*

$$\mathcal{F}_\varepsilon \rightharpoonup \mathcal{F} \quad in \quad H^{-1}(\Omega), \tag{22}$$

where $\mathcal{F} \in H^{-1}(\Omega)$ is given by

$$\mathcal{F}(\varphi) := \int_\Omega f\varphi\, dx + \frac{4\pi b}{3} \int_\Omega g\varphi\, dx \tag{23}$$

Proposition 3.2 *If $(u^\varepsilon, \theta^\varepsilon) \in V_\varepsilon \times H_0^1(\Omega)$ is a solution of the problem (9), and if $\hat{u}^\varepsilon$ stands for u^ε continued with zero to Ω, then we have*

$$\hat{u}^\varepsilon \quad \textit{and} \quad \theta^\varepsilon \quad \textit{are bounded in } H_0^1(\Omega). \tag{24}$$

Moreover,

$$|\nabla\theta^\varepsilon|^2_{\Omega_\varepsilon} + b(\varepsilon/r_\varepsilon)^3 |\nabla\theta^\varepsilon|^2_{T_\varepsilon} \leq C. \tag{25}$$

Proposition 3.3 *There exist $u \in H_0^1(\Omega; \mathbf{R}^3)$, $\theta \in H_0^1(\Omega)$ and $\tau \in L^2(\Omega)$ such that*

$$\begin{aligned}
\hat{u}^\varepsilon &\rightharpoonup u \quad \textit{in} \quad H_0^1(\Omega; \mathbf{R}^3), && (26)\\
\operatorname{div} u &= 0 \quad \textit{in} \quad \Omega && (27)\\
\theta^\varepsilon &\rightharpoonup \theta \quad \textit{in} \quad H_0^1(\Omega), && (28)\\
G_{R_\varepsilon}(\theta^\varepsilon) &\to \theta \quad \textit{in} \quad L^2(\Omega), && (29)\\
G_{r_\varepsilon}(\theta^\varepsilon) &\rightharpoonup \tau \quad \textit{in} \quad L^2(\Omega). && (30)
\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0} ⨍_{T_\varepsilon} |\theta^\varepsilon - G_{r_\varepsilon}(\theta^\varepsilon)|^2 \, dx = 0 \tag{31}$$

4 The two macroscopic heat equations

The aim of this section is to pass to the limit as $\varepsilon \to 0$ in the variational formulation of the heat equations:

$$\begin{aligned}
\forall \Phi \in H_0^1(\Omega),\ & \int_{\Omega_\varepsilon} \nabla\theta^\varepsilon \nabla\Phi \, dx + b(\varepsilon/r_\varepsilon)^3 \int_{T_\varepsilon} \nabla\theta^\varepsilon \nabla\Phi \, dx + \\
& + \int_{\Omega_\varepsilon} u^\varepsilon \nabla\theta^\varepsilon \Phi \, dx = \mathcal{F}_\varepsilon(\Phi).
\end{aligned} \tag{32}$$

First we consider the unique solution of the following problem:

$$\begin{aligned}
\Delta W^\varepsilon &= 0 \quad \text{in} \quad \mathcal{C}(r_\varepsilon, R_\varepsilon), && (33)\\
W^\varepsilon &= 1 \quad \text{in} \quad r = r_\varepsilon, && (34)\\
W^\varepsilon &= 0 \quad \text{in} \quad r = R_\varepsilon. && (35)
\end{aligned}$$

that is,

$$W^\varepsilon(y) = \frac{r_\varepsilon}{(R_\varepsilon - r_\varepsilon)} \left(\frac{R_\varepsilon}{r} - 1 \right) \quad \text{if} \quad y \in \mathcal{C}(r_\varepsilon, R_\varepsilon) \quad \text{and} \quad r := |y|.$$

Proposition 4.1 *Setting*

$$w^\varepsilon(x) \ := \ \begin{cases} 0 \quad \textit{in} \quad \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon, \\ W^\varepsilon(x - \varepsilon k) \quad \textit{in} \quad \mathcal{C}_\varepsilon^k, \quad k \in \mathbf{Z}_\varepsilon, \\ 1 \quad \textit{in} \quad T_\varepsilon, \end{cases} \tag{36}$$

we have

$$|\nabla w^\varepsilon|_\Omega \leq C \tag{37}$$

Lemma 4.2 *If for any $\varphi, \psi \in \mathcal{D}(\Omega)$ we denote $\Phi^\varepsilon \in H_0^1(\Omega)$ by*

$$\Phi^\varepsilon = (1 - w^\varepsilon)\varphi + w^\varepsilon G_{r_\varepsilon}(\psi), \tag{38}$$

then, as $\lim_{\varepsilon\to 0} |\Phi^\varepsilon - \varphi|_\Omega = 0$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \nabla\theta^\varepsilon \cdot (\nabla\Phi^\varepsilon + \Phi^\varepsilon u^\varepsilon)\ dx \\ =\ &\int_\Omega \nabla\theta \cdot (\nabla\varphi + \varphi u)\ dx + \gamma \int_\Omega (\theta - \tau)(\psi - \varphi)\ dx, \end{aligned}$$

where $(u^\varepsilon, \theta^\varepsilon)$ is a solution of (9).

Using Lemma 4.2, the two heat equations of the homogenized system are easily obtained. They are given by the next corollary.

Corollary 4.3 *The limit (u, θ, τ) verifies*

$$u\nabla\theta - \Delta\theta + 4\pi\gamma(\theta - \tau) = f \quad in \quad \Omega, \tag{39}$$

$$4\pi\gamma(\tau - \theta) = \frac{4\pi b}{3} g \quad in \quad \Omega. \tag{40}$$

5 The homogenized problem

From [7], we find that there exists an extension of the pressure (denoted by $\hat{p}^\varepsilon$) and some $p \in L^2(\Omega)$ such that

$$\hat{p}^\varepsilon \rightharpoonup p \quad \text{in} \quad L^2(\Omega)/\mathbf{R}. \tag{41}$$

We denote by $(w_\varepsilon^k, q_\varepsilon^k) \in H^1(\mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2})) \times L_0^2(\mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}))$ the only solution of the following Stokes problem

$$\begin{aligned} \text{div} w_\varepsilon^k &= 0 \quad \text{in} \quad \mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}), \\ -\Delta w_\varepsilon^k + \nabla q_\varepsilon^k &= 0 \quad \text{in} \quad \mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}), \\ w_\varepsilon^k &= 0 \quad \text{if} \quad r = r_\varepsilon, \\ w_\varepsilon^k &= \mathbf{e}^{(k)} \quad \text{if} \quad r = \frac{\varepsilon}{2}. \end{aligned}$$

Correspondingly, we define

$$v_\varepsilon^k(x) = \begin{cases} 0 \quad \text{if} \quad x \in T_\varepsilon, \\ w_\varepsilon^k(x - \varepsilon i) \quad \text{if} \quad x \in \mathcal{C}^i(r_\varepsilon, \frac{\varepsilon}{2}), \quad i \in \mathbf{Z}_\varepsilon, \\ \mathbf{e}^{(k)} \quad \text{if} \quad x \in \Omega_\varepsilon \setminus \cup_{i\in\mathbf{Z}_\varepsilon} \mathcal{C}^i(r_\varepsilon, \frac{\varepsilon}{2}). \end{cases}$$

Setting $v = \varphi v_\varepsilon^k$ in (9), for some $\varphi \in \mathcal{D}(\Omega)$, and using the energy method like in [1], we find that the limits satisfy the following Brinkman type homogenized equation:

$$-\Delta u + 6\pi\gamma u = -\nabla p + a\theta \mathbf{e}^{(3)} \quad \text{in} \quad \Omega. \tag{42}$$

Finally, the results of Proposition 3.3, Corollary 4.3 and relations (41)-(42) are summarized by our main theorem:

Theorem 5.1 *If $(u^\varepsilon, \theta^\varepsilon)$ is a solution of problem (9), then there exists an extension of the pressure (denoted by $\hat{p}^\varepsilon$) such that the following convergences hold on some subsequence*

$$\hat{p}^\varepsilon \rightharpoonup p \quad in \quad L^2(\Omega)/\mathbf{R}. \tag{43}$$

$$\hat{u}^\varepsilon \rightharpoonup u \quad in \quad H_0^1(\Omega;\mathbf{R}^3), \tag{44}$$

$$\theta^\varepsilon \rightharpoonup \theta \quad in \quad H_0^1(\Omega), \tag{45}$$

$$G_{r_\varepsilon}(\theta^\varepsilon) \rightharpoonup \tau \quad in \quad L^2(\Omega). \tag{46}$$

The limit (p, u, θ, τ) is a solution of the following system

$$\text{div} u = 0 \quad in \quad \Omega. \tag{47}$$

$$-\Delta u + 6\pi\gamma u = -\nabla p + a\theta \mathbf{e}^{(3)} \quad in \quad \Omega, \tag{48}$$

$$u\nabla\theta - \Delta\theta = f + \frac{4\pi b}{3} g \quad in \quad \Omega, \tag{49}$$

$$\tau = \theta + \frac{b}{3\gamma} g \quad in \quad \Omega. \tag{50}$$

Remark 5.2 If we assume that $a > 0$ is small enough, then the homogenized system has a unique solution in the corresponding space and the convergences of Theorem(5.1) hold on the entire sequence.

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