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Domain decomposition method for fixed-point problems

by
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Abstract

In this paper, we prove the convergence of an iterative method for fixed-point problems in a reflexive Banach space. As a particular case, the proposed method is exactly the additive Schwarz domain decomposition method when we use the Sobolev spaces. Also, for the finite element spaces, our result proves that the one-level and multi-level methods (the multigrid method, for instance) can be applied to find the fixed-points of contraction operators.

Key words and phrases: domain decomposition methods, fixed-point problems, non-linear problems, multigrid methods, multi-level methods, finite element methods.

AMS subject classification: 65N55, 65N30, 65J15

1 Introduction

The literature on the domain decomposition methods is very large. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1988 with [4] or those cited in the books [12] and [13]. Naturally, most of the papers dealing with these methods are dedicated to the linear elliptic problems. For the variational inequalities, the convergence proofs refer in general to the inequalities coming from the minimization of quadratic functionals. Also, most of the papers consider the convex set decomposed according to the space decomposition as a sum of convex subsets. To our knowledge, even if sometimes the authors make some remarks in their papers on the nonlinear cases, very few papers really deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Lui [8], [9], Tai and Espedal [14], and Tai and Xu [15], for nonlinear equations, Hoffmann and Zhou [6], Zeng and Zhou in [16] for inequalities having nonlinear source terms, and Badea [1] for the minimization of non-quadratic functionals.
The multilevel or multigrid methods can be viewed as domain decomposition methods and we can cite the results obtained by Kornhuber [7], Mandel [10], [11], Smith, Bjørstad and Gropp [13], Badea, Tai and Wang [2], and Badea[3]. Evidently, this list is not exhaustive and it can be completed with other papers.
As we have already said, few papers deal with the domain decomposition methods for non-linear problems, even if there are many mechanical or engineering problems which are modeled by a non-linear equation which does not come from the minimization of a energy functional. In this paper,

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we prove the convergence of an iterative method for fixed-point problems. The paper is organized as follows. In Section 2, we introduce the framework of our paper. The problem is stated in a reflexive Banach space and it generalizes the well known fixed-point problem in the Hilbert spaces. Section 3 is dedicated to a general subspace correction method for the problem in previous section. We give here an error estimation theorem for the proposed algorithm. Finally, in Section 4, we show that the particular form of proposed method in which the subspaces are associated with a domain decomposition is the multiplicative Schwarz method. Also, we make some remarks concerning the convergence rate (as a function of overlapping and mesh parameters) of the one-level and multi-level methods when the proposed algorithm is applied for problems in finite element spaces.

2 General framework

Let V be a reflexive Banach space and $V_1, \cdots, V_m$, be some closed subspaces of V such that $V = V_1 + \cdots + V_m$. We make the following assumption concerning the decomposition of the space V.

Assumption 2.1. *There exists a constant $C_0 > 0$ such that for any $v \in V$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$v = \sum_{i=1}^{m} v_i, \tag{2.1}$$

and

$$\sum_{i=1}^{m} ||v_i|| \leq C_0 ||v||. \tag{2.2}$$

We consider a Gâteaux differentiable functional $F : V \to R$, and we assume that there exists $p > 1$ such that for any real number $M > 0$ there exist two real numbers $A_M,\ B_M > 0$ for which

$$A_M ||v-u||^p \leq < F'(v) - F'(u), v-u > \tag{2.3}$$

and

$$||F'(v) - F'(u)||_{V'} \leq B_M ||v-u||^{p-1}, \tag{2.4}$$

for any $u, v \in V$ with $||u||, ||v|| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F. It is evident that if (2.3) and (2.4) hold, then for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$A_M ||v-u||^p \leq < F'(v) - F'(u), v-u > \leq B_M ||v-u||^p. \tag{2.5}$$

Following the way in [5], we can prove that for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\begin{aligned} &< F'(u), v-u > + \tfrac{A_M}{p} ||v-u||^p \leq F(v) - F(u) \leq \\ &< F'(u), v-u > + \tfrac{B_M}{p} ||v-u||^p, \end{aligned} \tag{2.6}$$

Finally, let $T : V \to V'$ be a contraction operator in the sense that for any $M > 0$ there exist $0 < \rho_M < 1$ such that

$$||T(v) - T(u)||_{V'} \leq \rho_M ||v-u||^{p-1} \tag{2.7}$$

for any $v, u \in V$, $||v||, ||u|| \leq M$.
We consider the problem

$$u \in V : \; < F'(u), v > - < T(u), v >= 0, \text{ for any } v \in V. \tag{2.8}$$

Since the functional F is convex and differentiable, any solution of problem (2.8) is also a solution for

$$u \in V : F(u) - < T(u), u > \leq F(v) - < T(u), v >, \text{ for any } v \in V. \tag{2.9}$$

Using (2.6), for a given $M > 0$ such that the solution u of (2.8) satisfies $||u|| \leq M$, we get

$$\frac{A_M}{p} ||v - u||^p \leq F(v) - F(u) - < T(u), v - u > \text{ for any } v \in V, \; ||v|| \leq M. \tag{2.10}$$

3 Subspace correction algorithm

We define the following correction algorithm corresponding to the subspaces $V_1, \cdots, V_m$ of the space V.

ALGORITHM 3.1. *We start the algorithm with an arbitrary $u^0 \in V$. At iteration $n+1$, having $u^n \in V$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$ which satisfies the equation*

$$\begin{aligned} &< F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i > - < T(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i >= 0, \\ &\text{for any } v_i \in V_i, \end{aligned} \tag{3.1}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

Evidently, if w_i^{n+1} is a solution of problem (3.1), then it also satisfies

$$\begin{aligned} &F(u^{n+\frac{i-1}{m}} + w_i^{n+1}) - < T(u^{n+\frac{i-1}{m}} + w_i^{n+1}), w_i^{n+1} > \leq \\ &F(u^{n+\frac{i-1}{m}} + v_i) - < T(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i >, \text{ for any } v_i \in V_i. \end{aligned} \tag{3.2}$$

We have the following global convergence result.

Theorem 3.1. *Let V be a reflexive Banach and $V_1, \cdots, V_m$ be some closed subspaces of V such that $V = V_1 + \cdots + V_m$. We assume that F is convex, Gâteaux differentiable and satisfies (2.3) and (2.4). Also, we suppose that the operator T satisfies (2.7). If Assumption 2.1 holds, u is the solution of problem (2.8) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 3.1, then there exists*

$$\rho\text{max} \leq \frac{\frac{A_M}{p}}{2^{|p-2|} m^{p-1}} \tag{3.3}$$

such that for values $0 < \rho_M < \rho\text{max}$ of the contraction constant in (2.7), we have

$$\begin{aligned} &F(u^n) - < T(u), u^n > - F(u) + < T(u), u > \leq \\ &\left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) - < T(u), u^0 > - F(u) + < T(u), u >\right] \end{aligned} \tag{3.4}$$

and

$$||u^n - u||^p \leq C_2 \left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0) - < T(u), u^0 > - F(u) + < T(u), u >\right] \tag{3.5}$$

The constants C_1 and C_2 are given in (3.18) and (3.20), respectively, in which we take for ε_1 and ε_2 their values giving $\rho_{\max}$ in (3.25).

Proof. In this proof, we use (2.3), (2.4) and (2.7) in which u and v are replaced only with the solution of problem (2.8) or its approximations obtained from Algorithm 3.1. Consequently, we are interested in the existence of an $M > 0$ such that $||u^{n+\frac{i}{m}}|| \leq M$, $n \geq 0$, $i = 1, \cdots, m$. To this end, we see that, with a proof by induction on n and i, equation (3.5) can be used to prove the existence of such a M at the same time with (3.4) and (3.5). In the following, for the simplicity of the proof, we prove only (3.4) and (3.5) assuming that there exists an $M > 0$ such that $||u^{n+\frac{i}{m}}|| \leq M$, $n \geq 0$, $i = 1, \cdots, m$, ie. we prove only the general step of the induction process.
From (3.1) and (2.6), for any $n \geq 0$ and $i = 1, \cdots, m$, we get,

$$\begin{aligned} & F(u^{n+\frac{i-1}{m}}) - F(u^{n+\frac{i}{m}}) - < T(u^{n+\frac{i}{m}}), u^{n+\frac{i-1}{m}} > + \\ & < T(u^{n+\frac{i}{m}}), u^{n+\frac{i}{m}} > \geq \frac{A_M}{p} ||w_i^{n+1}||^p, \end{aligned} \tag{3.6}$$

and, using (2.10), for any $n \geq 0$ and $i = 1, \cdots, m$, we have

$$F(u^{n+\frac{i}{m}}) - F(u) - < T(u), u^{n+\frac{i}{m}} > + < T(u), u > \geq \frac{A_M}{p} ||u^{n+\frac{i}{m}} - u||^p. \tag{3.7}$$

From Assumption 2.1, we get a decomposition $u_1, \cdots, u_m$ of $v = u - u^n$ satisfying (2.1)–(2.2). Replacing v_i by u_i in (3.1), and using (2.6) we get

$$\begin{aligned} & F(u^{n+1}) - F(u) - < T(u), u^{n+1} > + < T(u), u > + \frac{A_M}{p} ||u - u^{n+1}||^p \leq \\ & < F'(u^{n+1}), u^{n+1} - u > - < T(u), u^{n+1} > + < T(u), u > = \\ & \sum_{i=1}^{m} < F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > - \\ & \sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i - w_i^{n+1} > - < T(u), u^{n+1} > + < T(u), u > . \end{aligned} \tag{3.8}$$

In the following, using (2.4) and (2.2) for the decomposition $u_1, \cdots, u_m$ of $u - u^n$, we get

$$\begin{aligned}
&\sum_{i=1}^{m} < F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > \leq \\
&\sum_{i=1}^{m} \sum_{j=i+1}^{m} < F'(u^{n+\frac{j-1}{m}}) - F'(u^{n+\frac{j}{m}}), u_i - w_i^{n+1} > \leq \\
&B_M \sum_{i=1}^{m} ||w_i^{n+1}||^{p-1} \sum_{i=1}^{m} ||u_i - w_i^{n+1}|| \leq \\
&B_M \sum_{i=1}^{m} ||w_i^{n+1}||^{p-1} \left(\sum_{i=1}^{m} ||u_i|| + \sum_{i=1}^{m} ||w_i^{n+1}|| \right) \leq \\
&B_M m^{\frac{1}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p \right)^{\frac{p-1}{p}} \left((1 + C_0) \sum_{i=1}^{m} ||w_i^{n+1}|| + C_0 ||u^{n+1} - u|| \right) \leq \\
&B_M m^{\frac{1}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p \right)^{\frac{p-1}{p}} \left[m^{\frac{p-1}{p}} (1 + C_0) \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p \right)^{\frac{1}{p}} + C_0 ||u^{n+1} - u|| \right] = \\
&B_M m (1 + C_0) \sum_{i=1}^{m} ||w_i^{n+1}||^p + B_M m^{\frac{1}{p}} C_0 \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p \right)^{\frac{p-1}{p}} ||u^{n+1} - u||.
\end{aligned}$$

But, for any $\varepsilon > 0$, $p > 1$ and $x, z \geq 0$ we have the inequality

$$zx^{\frac{1}{p}} \leq \varepsilon x + (\frac{z^p}{\varepsilon})^{\frac{1}{p-1}}. \tag{3.9}$$

Applying this inequality to the last term of the above equation, we get

$$\begin{aligned}
&\sum_{i=1}^{m} < F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > \leq \\
&B_M \left(m(1 + C_0) + m^{\frac{1}{p}} \frac{C_0}{\varepsilon_1^{\frac{1}{p-1}}} \right) \sum_{i=1}^{m} ||w_i^{n+1}||^p + B_M m^{\frac{1}{p}} C_0 \varepsilon_1 ||u - u^{n+1}||^p,
\end{aligned} \tag{3.10}$$

for any $\varepsilon_1 > 0$.
Let $u^n = u_1^n + \cdots + u_m^n$, $u_i^n \in V_i$, $i = 1, \cdots, m$, be an arbitrary decomposition of u^n. Using (2.7) and

(2.2) for the above decomposition $u_1, \cdots, u_m$ of $u - u^n$, we get

$$
\begin{aligned}
& -\sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i - w_i^{n+1} > - < T(u), u^{n+1} > + < T(u), u >= \\
& -\sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i^n + u_i > + \sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i^n + w_i^{n+1} > + \\
& -\sum_{i=1}^{m} < T(u), u_i^n + w_i^{n+1} > + \sum_{i=1}^{m} < T(u), u_i^n + u_i >= \\
& \sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}) - T(u), w_i^{n+1} - u_i > \leq \\
& \rho_M \sum_{i=1}^{m} ||u^{n+\frac{i}{m}} - u||^{p-1} ||w_i^{n+1} - u_i|| \leq \\
& \rho_M \left(||u^{n+1} - u|| + \sum_{i=1}^{m} ||w_i^{n+1}|| \right)^{p-1} \sum_{i=1}^{m} ||w_i^{n+1} - u_i|| \leq \\
& \rho_M \left(||u^{n+1} - u|| + \sum_{i=1}^{m} ||w_i^{n+1}|| \right)^{p-1} \sum_{i=1}^{m} (||w_i^{n+1}|| + ||u_i||) \leq \\
& \rho_M \left(||u^{n+1} - u|| + \sum_{i=1}^{m} ||w_i^{n+1}|| \right)^{p-1} \left((1 + C_0) \sum_{i=1}^{m} ||w_i^{n+1}|| + C_0 ||u^{n+1} - u||) \right) \leq \\
& \rho_M (1 + C_0) \left(\sum_{i=1}^{m} ||w_i^{n+1}|| + ||u^{n+1} - u|| \right)^p
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
& -\sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i - w_i^{n+1} > - < T(u, u^{n+1}) + < T(u, u) \leq \\
& \rho_M (1 + C_0)(m+1)^{\frac{p-1}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p + ||u^{n+1} - u||^p \right)
\end{aligned}
\tag{3.11}
$$

From (3.8), (3.10) and (3.11), we get

$$
\begin{aligned}
& F(u^{n+1}) - F(u) - < T(u), u^{n+1} > + < T(u), u > + \\
& \left(\frac{A_M}{p} - B_M m^{\frac{1}{p}} C_0 \varepsilon_1 - \rho_M (1 + C_0)(m+1)^{\frac{p-1}{p}} \right) ||u^{n+1} - u||^p \leq \\
& \left(B_M m + B_M m^{\frac{1}{p}} \frac{C_0}{\varepsilon_1^{\frac{1}{p-1}}} + \rho_M (1 + C_0)(m+1)^{\frac{p-1}{p}} \right) \sum_{i=1}^{m} ||w_i^{n+1}||^p
\end{aligned}
\tag{3.12}
$$

for any $\varepsilon_1 > 0$.

Now, from (3.6) we get

$$\begin{aligned}&\frac{A_M}{p}\sum_{i=1}^{m}||w_i^{n+1}||^p \leq F(u^n) - F(u^{n+1}) - < T(u), u^n > + < T(u), u^{n+1} > - \\ &\sum_{i=1}^{m}[< T(u^{n+\frac{i}{m}}), u^{n+\frac{i-1}{m}} > - < T(u^{n+\frac{i}{m}}), u^{n+\frac{i}{m}} >] + \\ &< T(u), u^n > - < T(u), u^{n+1} >\end{aligned} \tag{3.13}$$

Again, with an arbitrary decomposition $u^n = u_1^n + \cdots + u_m^n$, $u_i^n \in V_i$, $i = 1, \cdots, m$, of u^n, in the same way as for (3.11), we have

$$\begin{aligned}&-\sum_{i=1}^{m}[< T(u^{n+\frac{i}{m}}), u^{n+\frac{i-1}{m}} > - \\ &< T(u^{n+\frac{i}{m}}), u^{n+\frac{i}{m}} >] + < T(u), u^n > - < T(u), u^{n+1} > \leq \\ &-\sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i^n > + \sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}), u_i^n + w_i^{n+1} > + \\ &\sum_{i=1}^{m} < T(u), u_i^n > - \sum_{i=1}^{m} < T(u), u_i^n + w_i^{n+1} > = \\ &\sum_{i=1}^{m} < T(u^{n+\frac{i}{m}}) - T(u), w_i^{n+1} > \leq \rho_M \sum_{i=1}^{m} ||u^{n+\frac{i}{m}} - u||^{p-1}||w_i^{n+1}|| \leq \\ &\rho_M \left(\sum_{i=1}^{m}||w_i^{n+1}|| + ||u^{n+1} - u||\right)^{p-1} \sum_{i=1}^{m}||w_i^{n+1}||\end{aligned}$$

For $x, y \geq 0$, we have $(x + y)^q \leq x^q + y^q$ for $0 < q \leq 1$, and $(x + y)^q \leq 2^{q-1}(x^q + y^q)$ for $1 < q$. Using it and inequality (3.9), we get from the above equation

$$\begin{aligned}&-\sum_{i=1}^{m}[< T(u^{n+\frac{i}{m}}), u^{n+\frac{i-1}{m}} > - \\ &< T(u^{n+\frac{i}{m}}), u^{n+\frac{i}{m}} >] + < T(u), u^n > - < T(u), u^{n+1} > \leq \\ &\rho_M 2^{|p-2|}\left(\sum_{i=1}^{m}||w_i^{n+1}||\right)^p + \rho_M 2^{|p-2|}||u^{n+1} - u||^{p-1}\sum_{i=1}^{m}||w_i^{n+1}|| \leq \\ &\rho_M 2^{|p-2|}(1 + \varepsilon_2)\left(\sum_{i=1}^{m}||w_i^{n+1}||\right)^p + \rho_M \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}}||u^{n+1} - u||^p \leq \\ &\rho_M 2^{|p-2|}(1 + \varepsilon_2)m^{p-1}\sum_{i=1}^{m}||w_i^{n+1}||^p + \rho_M \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}}||u^{n+1} - u||^p\end{aligned} \tag{3.14}$$

for any $\varepsilon_2 > 0$. Consequently, from (3.13) and (3.14), we get

$$\begin{aligned}&\left(\frac{A_M}{p} - \rho_M 2^{|p-2|}(1+\varepsilon_2)m^{p-1}\right)\sum_{i=1}^{m}||w_i^{n+1}||^p \leq \\ &F(u^n) - F(u^{n+1}) - < T(u), u^n > + < T(u), u^{n+1} > + \rho_M \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}}||u^{n+1} - u||^p\end{aligned} \tag{3.15}$$

for any $\varepsilon_2 > 0$.
Let us write

$$C_3 = \frac{A_M}{p} - \rho_M 2^{|p-2|}(1+\varepsilon_2)m^{p-1}. \tag{3.16}$$

For some ρ_M and ε_2 such that $C_3 > 0$, we get from (3.12) and (3.15),

$$\begin{aligned}&F(u^{n+1}) - F(u) - < T(u, u^{n+1}) + < T(u,u) + C_4||u - u^{n+1}||^p \leq \\ &C_1\left[F(u^n) - F(u^{n+1}) - < T(u, u^n) + < T(u, u^{n+1})\right]\end{aligned} \tag{3.17}$$

where

$$C_1 = \tfrac{1}{C_3}\left(B_M m + B_M m^{\frac{1}{p}}\frac{C_0}{\varepsilon_1^{\frac{1}{p-1}}} + \rho_M(1+C_0)(m+1)^{\frac{p-1}{p}}\right) \tag{3.18}$$

and

$$C_4 = \frac{A_M}{p} - B_M m^{\frac{1}{p}} C_0 \varepsilon_1 - \rho_M(1+C_0)(m+1)^{\frac{p-1}{p}} - \rho_M \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}} C_1 \tag{3.19}$$

From (3.17), we easily get (3.4) assuming that $C_4 > 0$. From (3.7) we get that $\frac{A_M}{p}||u^{n+1} - u||^p \leq F(u^{n+1}) - F(u) - < T(u), u^{n+1} > + < T(u), u >$, and using again (3.17), we get

$$[(1 + C_1)\frac{A_M}{p} + C_4]||u^{n+1} - u||^p \leq F(u^n) - F(u) - < T(u, u^n) + < T(u, u).$$

Using (3.4), we get (3.5) with

$$C_2 = \frac{1 + C_1}{(1 + C_1)\frac{A_M}{p} + C_4}. \tag{3.20}$$

Conditions $C_3 > 0$ and $C_4 > 0$ can be written as

$$\frac{A_M}{p} - \rho_M 2^{|p-2|}(1+\varepsilon_2)m^{p-1} > 0 \tag{3.21}$$

and

$$\begin{aligned}&\tfrac{A_M}{p} - B_M m^{\frac{1}{p}} C_0 \varepsilon_1 - \rho_M(1+C_0)(m+1)^{\frac{p-1}{p}} - \\ &\rho_M \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}} \frac{B_M m + B_M m^{\frac{1}{p}}\frac{C_0}{\varepsilon_1^{\frac{1}{p-1}}} + \rho_M(1+C_0)(m+1)^{\frac{p-1}{p}}}{\frac{A_M}{p} - \rho_M 2^{|p-2|}(1+\varepsilon_2)m^{p-1}} > 0\end{aligned} \tag{3.22}$$

respectively. Using (3.21), the above inequality could have a solution $\rho_M > 0$ only if

$$\varepsilon_1 < \frac{\frac{A_M}{p}}{B_M m^{\frac{1}{p}} C_0} \tag{3.23}$$

Moreover, (3.22) can be written as a second order algebraic inequality,

$$\begin{aligned}&\rho_M^2\left[2^{|p-2|}(1+\varepsilon_2)(1+C_0)m^{p-1}(m+1)^{\frac{p-1}{p}} - \frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}}(1+C_0)(m+1)^{\frac{p-1}{p}}\right] - \\ &\rho_M\left[2^{|p-2|}(1+\varepsilon_2)m^{p-1}\left(\frac{A_M}{p} - B_M m^{\frac{1}{p}} C_0\varepsilon_1\right) + (1+C_0)(m+1)^{\frac{p-1}{p}}\frac{A_M}{p} + \right. \\ &\left.\frac{2^{|p-2|}}{\varepsilon_2^{\frac{1}{p-1}}}\left(B_M m + B_M m^{\frac{1}{p}}\frac{C_0}{\varepsilon_1^{\frac{1}{p-1}}}\right)\right] + \frac{A_M}{p}\left(\frac{A_M}{p} - B_M m^{\frac{1}{p}} C_0\varepsilon_1\right) > 0,\end{aligned} \tag{3.24}$$

and we can simply verify that for any ε_1 satisfying (3.23) and $\varepsilon_2 > 0$ there exists a $\rho_{\varepsilon_1\varepsilon_2} > 0$ such that any $0 < \rho_M < \rho_{\varepsilon_1\varepsilon_2}$ is a solution of (3.24). Also, we can verify that the bound of ρ_M obtained from (3.21) do not satisfy (3.24). Consequently,

$$\rho_{\varepsilon_1\varepsilon_2} < \frac{\frac{A_M}{p}}{2^{|p-2|}(1+\varepsilon_2)m^{p-1}} < \frac{\frac{A_M}{p}}{2^{|p-2|}m^{p-1}},$$

and we get (3.3) with

$$\rho\max = \sup_{0<\varepsilon_1<\frac{\frac{A_M}{p}}{B_M m^{\frac{1}{p}} C_0},\ 0<\varepsilon_2} \rho_{\varepsilon_1\varepsilon_2} \tag{3.25}$$

□

4 Multiplicative Schwarz method as a subspace correction method

In the following, we show that the particular form of Algorithm 3.1 in which the subspaces are associated with a domain decomposition is the multiplicative Schwarz method.
Let Ω be an open bounded domain in $\mathbf{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We consider an overlapping decomposition of the domain Ω,

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{4.1}$$

in which Ω_i are open subdomains with Lipschitz continuous boundary.
We take $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$ and associate to the domain decomposition (4.1) the subspaces $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \cdots, m$. In this case, Algorithm 3.1 represents the multiplicative Schwarz method written as a subspace correction method.

To see that Assumption 2.1 holds, let us consider a unity partition associated with decomposition (4.1), $\theta_1, \cdots, \theta_m$, ie.

$$\theta_i \in C^1(\bar{\Omega}), \ \mathrm{supp}\theta_i \subset \Omega_i, i = 1, \cdots, m, \ \text{and} \ \sum_{i=1}^{m} \theta_i = 1 \text{ on } \Omega. \tag{4.2}$$

It is evident that for a $v \in V$, the decomposition $v_i = \theta_i v$, $i = 1, \cdots, m$, satisfies (2.1) and (2.2) in Assumption 2.1. Consequently, the convergence and error estimation given in Theorem 3.1 hold, too. Evidently, the constant C_0 in (2.2) depends on the unity partition (4.2). From error estimations (3.4) and (3.5), we see that the convergence rate essentially depend on this constant C_0. If we use the finite element spaces associated with the above spaces V and V_i, $i = 1, \cdots, m$, following the techniques in [3], we can write the constant C_0 depending on the overlapping and mesh parameters. In this way, we can prove that C_0 depends only on the overlapping parameter in the case of the one-level method, but it is independent of these parameters for the multi-level methods (multigrid methods, for instance).

Remark 4.1. The above spaces V and V_i correspond to Dirichlet boundary conditions. Similar results can be obtained if we consider mixed boundary conditions. We take $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ a partition of the boundary such that meas$(\Gamma_1) > 0$, and we consider the Sobolev space $V = \{v \in W^{1,s}(\Omega) : v = 0 \text{ on } \Gamma_1\}$. This space corresponds to Dirichlet boundary conditions on Γ_1 and Neumann boundary conditions on Γ_2. The subspaces V_i will be defined in this case as $V_i = \{v_i \in W^{1,s}(\Omega) : v_i = 0 \text{ in } \Omega - \bar{\Omega}_i, \ v_i = 0 \text{ in } \partial\Omega_i \cap \Gamma_1\}$, $i = 1, \cdots, m$.
Also, we have considered problems having the solution in $W^{1,s}(\Omega)$, but all the obtained results hold with $[W^{1,s}(\Omega)]^N$, $N \geq 2$, in the place of $W^{1,s}(\Omega)$.

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