



Constructive Approach of the Solution of Riemann Problem for Shallow Water Equations with Topography and Vegetation

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Abstract

We investigate the Riemann Problem for a shallow water model with porosity and terrain data. Based on recent results on the local existence, we build the solution in the large settings (the magnitude of the jump in the initial data is not supposed to be “small enough”). One difficulty for the extended solution arises from the double degeneracy of the hyperbolic system describing the model. Another difficulty is given by the fact that the construction of the solution assumes solving an equation which has no global solution. Finally, we present some cases to illustrate the existence and non-existence of the solution.

1 Introduction

Riemann Problem is a classical topic in the theory of hyperbolic systems, [10, 6, 12, 16] and it is widely used in testing or elaborating numerical schemes, [17], to refer to just a few classical papers. The study of Riemann Problem for non-conservative hyperbolic systems requires a new concept on the definition of the discontinuous solution. In this sense, two main ingredients are introduced: *measure solution*, [3] and *path connection*, [13, 14, 18]. Using a path connection, one can define the Rankine-Hugoniot relations that relate the two side values of a shock solution on the discontinuity curve. The shock solutions depend on the path connection.

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The shallow water equations with topography and vegetation is a widely used mathematical model in environmental sciences to study the flow of water into natural systems, [4, 9, 15]. The model fits into the class of non-conservative hyperbolic systems, where there are known several formulations for jump relations, [1, 2, 7, 11].

In [8], we have introduced the jump relations using a class of path connections that was chosen on the basis of certain physical arguments.

In this paper, we investigate the existence of the solution in the “large”: the Riemann Problem data are not restricted to be “closed enough”. The “constructive” word from the title must be interpreted as follows. First, a problem being given, there is a way to affirm that the problem has or has not a solution. In case of an affirmative answer, there is an algorithm that allows one to build the solution. Secondly, we do not have results that can give general conditions for the existence of the solution.

For an easier understanding of the results, we briefly recall the shallow water model with topography and vegetation. For more details, the readers are referred to the papers [1, 2, 7, 8].

In the absence of the friction terms and if there are not water gain or loss, the 1D shallow water equations with topography and vegetation can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \theta h + \partial_x (\theta h u) &= 0, \\ \frac{\partial}{\partial t} \theta h u + \partial_x (\theta h u^2) + \theta h \partial_x w &= 0, \end{aligned} \tag{1}$$

where $h(t, x)$ is the water height, $u(t, x)$ the water velocity and $z(x)$ the soil surface level. The function $w = g(z + h)$ stands for the potential of the water level and g is gravitational acceleration. The variation of the cover plant density is taken into account through the function $\theta(x)$, the porosity of the plant cover.

The Riemann problem for the shallow water equations with topography and vegetation consists in finding a solution in the class of functions with bounded variation for the equations (1) with the following initial conditions

$$(h, u)_{t=0} = \begin{cases} (h^L, u^L), & x < 0, \\ (h^R, u^R), & x > 0. \end{cases} \tag{2}$$

The terrain data (the porosity θ and the soil surface z) are defined by

$$(\theta, z) = \begin{cases} (\theta^L, z^L), & x < 0, \\ (\theta^R, z^R), & x > 0. \end{cases} \tag{3}$$

2 The Riemann Problem for arbitrary data

A solution of the problem is built by using rarefaction waves and shock waves. The rarefaction waves are smooth solutions of (1) in a domain where the terrain functions are constant.

For a function Ψ , $[[\Psi]]$ stands for the jump $\Psi^+ - \Psi^-$. The shock wave solutions verify the classical Rankine-Hugoniot relations:

$$\begin{aligned} -\sigma [[h]] + [[hu]] &= 0, \\ -\sigma [[hu]] + [[hu^2 + gh^2/2]] &= 0, \end{aligned} \quad (4)$$

in the domains $x < 0$ or $x > 0$ or generalized Rankine-Hugoniot relations

$$\begin{aligned} [[\theta hu]] &= 0, \\ [[\theta hu^2]] + g \int_0^1 \theta(s) h(s) \frac{d(z(s) + h(s))}{ds} ds &= 0, \end{aligned} \quad (5)$$

for a steady shock located at $x = 0$. The integral is evaluated on a path connection curve $\{\theta(s; \theta^L, \theta^R), z(s; z^L, z^R), h(s; h^L, h^R)\}$.

2.1 Riemann Constructor. (z, θ) constant function

Whenever the terrain function are constant the Riemann problem can be solved by using the two kind of the waves, rarefaction waves and shock waves. In the phase space (h, u) one defines a 1-wave curve *issuing from a point* (h_L, u_L)

$$W_1(h; h_L, u_L) := \{(h, u_1(h; h_L, u_L)) | h > 0\}$$

where

$$u_1(h; h_L, u_L) = \begin{cases} u_L + 2\sqrt{gh_L} \left(1 - \sqrt{\frac{h}{h_L}}\right), & h < h_L, \\ u_L + \sqrt{gh_L} \left(1 - \frac{h}{h_L}\right) \sqrt{\frac{1}{2} \left(1 + \frac{h_L}{h}\right)}, & h > h_L \end{cases} \quad (6)$$

and 2-backward wave curve *reaching a point* (h_R, u_R)

$$W_2^B(h; h_R, u_R) := \{(h, u_2^B(h; h_R, u_R)) | h > 0\},$$

where

$$u_2^B(h; h_R, u_R) = \begin{cases} u_R - 2\sqrt{gh_R} \left(1 - \sqrt{\frac{h}{h_R}}\right), & h < h_R, \\ u_R - \sqrt{gh_R} \left(1 - \frac{h}{h_R}\right) \sqrt{\frac{1}{2} \left(1 + \frac{h_R}{h}\right)}, & h > h_R. \end{cases} \quad (7)$$

The interpretation of the two wave curves is as follows:

(a) $W_1(h; h_L, u_L)$. A point (h_L, u_L) being given as the left state in the Riemann problem, the curve $W_1(h; h_L, u_L)$ defines all right states that can be connected to the left state either by a 1-shock wave, $h > h_L$ or by a 1-rarefaction wave, $h < h_L$.

(b) $W_2^B(h; h_R, u_R)$. A point (h_R, u_R) being given as the right state in the Riemann problem, the curve $W_2^B(h; h_R, u_R)$ defines all left states that can be connected to the right state either by a 2-shock wave, $h > h_R$ or by a 2-rarefaction wave, $h < h_R$.

The shock speed on each curve can be calculated by formula

$$\begin{aligned}\sigma_1(h; h_L, u_L) &= u_L - \sqrt{gh} \sqrt{\frac{1}{2} \left(1 + \frac{h}{h_L} \right)}, \quad h > h_L, \\ \sigma_2(h; h_R, u_R) &= u_R + \sqrt{gh} \sqrt{\frac{1}{2} \left(1 + \frac{h}{h_R} \right)}, \quad h > h_R\end{aligned}\tag{8}$$

and the eigenvalues are given by

$$\begin{aligned}\lambda_1(h; h_L, u_L) &= \lambda_1(h_L, u_L) + 3(\sqrt{gh_L} - \sqrt{gh}), \quad h < h_L, \\ \lambda_2(h; h_R, u_R) &= \lambda_2(h_R, u_R) + 3(\sqrt{gh} - \sqrt{gh_R}), \quad h < h_R.\end{aligned}\tag{9}$$

For the case of terrain constant functions, the solution of Riemann Problem for arbitrary data is a composite wave that can be found following two steps:

Riemann Constructor. (z, θ) Constant function

Step 1 Find the intersection point h_* such that,

$$W_2^B(h_*; h_R, u_R) = W_1(h_*; h_L, u_L).$$

Step 2 The composite wave curve of the solution for the Riemann problem is

$$W_2^B(h_R; W_1(h_*; h_L, u_L)).$$

A typical pattern of solution is illustrated in Figure 1.

2.2 Riemann Constructor. Jump in (z, θ)

For the case of jump in the terrain function, there are three types of waves. In addition to the first two waves W_1, W_2 , there is another steady shock wave W_3 that results as a solution of the generalized Rankine-Hugoniot equation, (5).

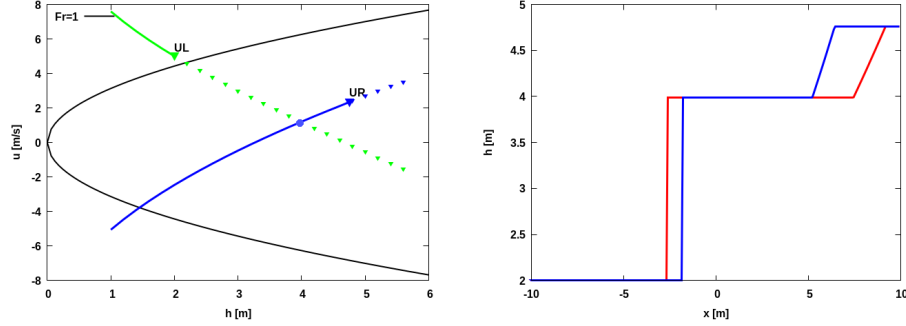


Figure 1: The figure illustrates an application of the construction algorithm to solve a Riemann Problem. The left picture contains the 1-shock curve (green scatter plot), 1-rarefaction wave (green line), 2-backward shock (blue scatter plot), 2-backward rarefaction wave (blue line), and the curve $Fr = 1$ (black line). The right picture contains the graphs of water height at two different moments of time $t = 1s$ (red line) and $t = 0.7s$ (blue line).

To build the third wave, it is necessary to introduce a physical path that connects two arbitrary states $\mathcal{P}^- := (h^-, u^-, z^-, \theta^-)$ and $\mathcal{P}^+ := (h^+, u^+, z^+, \theta^+)$. In [8], we define a physical path by

$$\begin{aligned} h(s; h^-, h^+) &= h^- + \phi(s)(h^+ - h^-), \\ z(s; z^-, z^+) &= z^- + \phi(s)(z^+ - z^-), \\ \theta(s; \theta^-, \theta^+) &= \theta^- + \frac{\theta^- \phi(s)}{\theta^- \phi(s) + (1 - \phi(s))\theta^+}(\theta^+ - \theta^-), \end{aligned}$$

where $\phi(s)$ is an arbitrary smooth and monotone function that satisfies $\phi(0) = 0$ and $\phi(1) = 1$. Based on this path, one can define the W_3 -steady shock curve as follows.

We introduce the notations

$$[|z|] = \frac{z^+ - z^-}{h^-}, \quad \theta = \frac{\theta^+}{\theta^-}, \quad Fr_-^2 = \frac{(u^-)^2}{gh^-},$$

and the function

$$\psi(y; \theta, [|z|], Fr_-) := -b(\theta)y^3 - (a(\theta) - b(\theta)(1 - [|z|]))y^2 + ((1 - [|z|])a(\theta) - Fr_-^2)y + \frac{Fr_-^2}{\theta},$$

where $a(\theta)$ and $b(\theta)$ are given by

$$b(\theta) = \frac{\theta(\theta - 1 - \theta \log \theta)}{(\theta - 1)^2}, \quad a(\theta) = -1 - \frac{b(\theta)}{\theta}.$$

Definition 2.1 (3-wave). *Given the terrain configuration (z^-, z^+) , (θ^-, θ^+) and the left state $U^- = (h^-, u^-)$ a right state $U^+ = (h^+, u^+)$ is defined by*

$$\begin{aligned} h^+ &= hh^-, \\ u^+ &= \frac{u^-}{\theta} \frac{1}{h}, \end{aligned} \quad (10)$$

where h is the solution of the equation

$$\psi(x; \theta, \lfloor |z| \rfloor, \text{Fr}_-) = 0 \quad (11)$$

that minimizes the function

$$\mathcal{E}(x) = \max\{|\theta^{-1} - x|, |1 - \lfloor |z| \rfloor - x|\}. \quad (12)$$

To understand the necessity of minimization criterion required in Definition 2.1, the following remarks are in order.

Since the equation $\psi(x; \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$ can have two positive solutions, it is necessary to introduce a criterion to select a physically admissible solution.

When solving the local problem, the selection of a solution is based on the continuity argument, in the sense that if the ratio θ and the soil surface jump approach 1 and 0, respectively, then the left and right states must be equal to each other. In computations, one can use as selection criterion the comparison of the left Froude number with the unity, (see Theorem 2.1 in [8]).

But when we deal with large data, there is no guarantee that selection based on the Froude number determines a solution that goes to unity when the terrain data become continuous.

Here is an example. Assume that $1 - \lfloor |z| \rfloor = 1/\theta$. In such a case, there are two positive solutions:

$$h_1 = \frac{1}{\theta}, \quad h_2 = \frac{a(\theta) + \sqrt{a(\theta)^2 - 4b(\theta)\text{Fr}^2}}{-2b(\theta)}.$$

Using the identity $a(\theta) + b(\theta)/\theta = -1$, one can prove that if $\text{Fr}^2 < 1/\theta$ then $h_2 < h_1$. Assuming that $\lfloor |z| \rfloor < 0$ and $1 < \text{Fr}^2 < 1/\theta$, then the physical solution is $\beta = h_2$. But

$$\lim_{\theta \rightarrow 1} h_2 = \frac{-1 + \sqrt{1 + 8\text{Fr}^2}}{2} \neq 1$$

On the other hand, the solution h_1 is obtained by a continuous deformation of the solution $h_1 = 1$.

Fr^2	0	$1/\theta + (a(\theta) + 2)(1/\theta - (1 - \lfloor z \rfloor))$	∞
$\psi_2(\text{Fr})$	— ↗	$1/\theta - (1 - \lfloor z \rfloor)$	↘ —
$\partial_{\text{Fr}^2} \psi_2$	+	0	—
	$\tilde{h}_2(\text{Fr}) < 1/\theta$	$\tilde{h}_2(\text{Fr}) = 1/\theta$	$\tilde{h}_2(\text{Fr}) > 1/\theta$

Table 1: The influence of the Froude number on the existence of the solution for the terrain jump equation.

To overcome this problem, we introduce as a selection criterion the minimization of the function $\mathcal{E}(x)$ that is a measure of the magnitude of the discontinuity in free surface and fluid velocity.

There are two major difficulties encountered when building the solution of the problem:

- (a) the terrain jump equation can be unsolvable;
- (b) along the wave curves W_1 and W_2 , the eigenvalues λ_i change their sign, which implies that two different states situated on the same wave curve but having the eigenvalues with opposed sign can not be connected due to the presence of the discontinuity line ($x = 0$) that separates the left and right states.

The behavior of the solutions of the equation $\psi(h, \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$ as functions of Fr is analyzed in the next lemma. If $\theta < 1 - \lfloor |z| \rfloor$, there always are two positive solutions for the terrain jump equation. But in the case $\theta > 1 - \lfloor |z| \rfloor$, there are two critical numbers Fr_* and Fr^* such that if $(\text{Fr}_*)^2 < \text{Fr}^2 < (\text{Fr}^*)^2$, then the solution does not exist.

Lemma 2.1. *Let θ and $\lfloor |z| \rfloor$ be fixed. Let Fr be a variable parameter in the terrain jump equation $\psi(h; \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$. Then, we can state the following properties about the solutions:*

- (a) *If $1/\theta < 1 - \lfloor |z| \rfloor$, then the terrain jump equation has two solutions $h_1 < h_2$, for any Froude number, and*

$$h_1 < \frac{1}{\theta} < 1 - \lfloor |z| \rfloor < h_2. \quad (13)$$

- (b) *If $1/\theta > 1 - \lfloor |z| \rfloor$, then there are two critical Froude numbers Fr_* and Fr^* , such that*

(i) if $\text{Fr}^2 \in (0, (\text{Fr}_)^2) \cup ((\text{Fr}^*)^2, \infty)$, then there are two solutions that satisfy the inequalities*

$$\begin{aligned} h_1 < h_2 < 1 - \lfloor |z| \rfloor < \frac{1}{\theta}, & \quad \text{if } \text{Fr}^2 < (\text{Fr}_*)^2, \\ 1 - \lfloor |z| \rfloor < \frac{1}{\theta} < h_1 < h_2, & \quad \text{if } \text{Fr}^2 > (\text{Fr}^*)^2; \end{aligned} \quad (14)$$

(ii) if $\text{Fr}^2 \in ((\text{Fr}_*)^2, (\text{Fr}^*)^2)$, then there are no solutions.

Proof. The inequalities (13) are consequence of the property that if $1/\theta < 1 - \lfloor |z| \rfloor$, then $\psi(1/\theta; \theta, \lfloor |z| \rfloor, \text{Fr}) < 0$ and $\psi(1 - \lfloor |z| \rfloor; \theta, \lfloor |z| \rfloor, \text{Fr}) < 0$.

To prove (14), we analyze the behavior of the minimum value of $\psi(h; \theta, \lfloor |z| \rfloor, \text{Fr})$. Let $\tilde{h}_2(\theta, \lfloor |z| \rfloor, \text{Fr})$ be the positive solution of the equation $\partial_h \psi(h; \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$. Let $\psi_2(\text{Fr}) := \psi(\tilde{h}_2(\theta, \lfloor |z| \rfloor, \text{Fr}); \theta, \lfloor |z| \rfloor, \text{Fr})$. One has

$$\partial_{\text{Fr}^2} \psi_2(\text{Fr}) = \frac{1}{\theta \tilde{h}_2} - 1.$$

Since $\partial_{\text{Fr}^2} \tilde{h}_2 > 0$, one can draw the variation of $\psi_2(\text{Fr})$ as in the Table 1. In the case $1/\theta > 1 - \lfloor |z| \rfloor$, there is a value Fr_* such that $\psi_2(\text{Fr}_*) = 0$ and $\psi_2(\text{Fr}) < 0$ for any Froude number Fr satisfying $(\text{Fr})^2 < (\text{Fr}_*)^2$. For such a value of the Froude number, there are two solutions that are both smaller than $1/\theta$. Taking into account that $\psi(h; \theta, \lfloor |z| \rfloor, \text{Fr}) > 0$, if $1 - \lfloor |z| \rfloor < h < 1/\theta$, one can conclude the first inequality in (14). Similar arguments can be invoked to prove the second inequality in (14). \square

Lemma 2.2. *Let θ and $\lfloor |z| \rfloor$ be fixed. Let Fr be a variable parameter in the terrain jump equation $\psi(h; \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$ and let Fr_* and Fr^* be the critical values given by Lemma 2.1. Assume that the solutions $h_1 < h_2$ of the equation $\psi(h, \theta, \lfloor |z| \rfloor, \text{Fr}) = 0$ exist. Then we can affirm:*

(a) *If $1/\theta > 1 - \lfloor |z| \rfloor$, then*

$$\begin{aligned} \mathcal{E}(h_2) &< \mathcal{E}(h_1), & \text{if } \text{Fr}^2 < (\text{Fr}_*)^2, \\ \mathcal{E}(h_1) &< \mathcal{E}(h_2), & \text{if } \text{Fr}^2 > (\text{Fr}^*)^2. \end{aligned} \quad (15)$$

(b) *If $1/\theta \leq 1 - \lfloor |z| \rfloor$, then there is a value $\tilde{\text{Fr}}$ with*

$$\tilde{\text{Fr}} = \frac{1}{\theta} + \frac{-b(\theta)}{-b(\theta) + \theta} \left(1 - \lfloor |z| \rfloor - \frac{1}{\theta} \right), \quad (16)$$

such that

$$\begin{aligned} \mathcal{E}(h_2) &< \mathcal{E}(h_1) & \text{if } \text{Fr}^2 < \tilde{\text{Fr}}, \\ \mathcal{E}(h_1) &< \mathcal{E}(h_2) & \text{if } \text{Fr}^2 > \tilde{\text{Fr}}, \\ \mathcal{E}(h_1) &= \mathcal{E}(h_2) & \text{if } \text{Fr}^2 = \tilde{\text{Fr}}. \end{aligned} \quad (17)$$

In addition, let $\text{Fr}_+^2 = (u^+)^2/gh^+$ be the Froude number defined by physical solution of the terrain jump equation. Then,

$$(\text{Fr}^2 - \tilde{\text{Fr}})(\text{Fr}_+^2 - 1) > 0. \quad (18)$$

Proof. The inequalities (15) immediately result from the definition of $\mathcal{E}(h)$ and (14). To prove the inequalities (17), we proceed as follows. We observe that $\partial_{\text{Fr}^2} \mathcal{E}(h_1) < 0$ and $\partial_{\text{Fr}^2} \mathcal{E}(h_2) > 0$. Moreover, there is a value of Fr such that $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. This equality implies that

$$h_1 + h_2 = \frac{1}{\theta} + 1 - [|z|] .$$

The above equality allows us to calculate the negative solution $h_3 = 1/b(\theta)$ of the equation $\psi(h, \theta, [|z|], \text{Fr}) = 0$. Then,

$$\text{Fr}^2 \left(\frac{b(\theta)}{\theta} - 1 \right) + \frac{b(\theta)}{\theta} \left(\frac{1}{b(\theta)} - (1 - [|z|]) \right) = 0.$$

This proves (16).

Note that $\tilde{\text{Fr}}$ satisfies the following inequalities

$$\frac{1}{\theta} \leq \tilde{\text{Fr}} \leq 1 - [|z|] .$$

To prove (18) we observe that if $\text{Fr}^2 < \tilde{\text{Fr}}$, then (17)-1 and (13) imply that the physical solution, h_2 , is greater than $1 - [|z|]$. One has

$$\text{Fr}_+^2(\theta; [|z|]; \text{Fr}) := \frac{\text{Fr}^2}{\theta^2 h_2^3} < \frac{\tilde{\text{Fr}}}{\theta^2 h_2^3} < \frac{h_2}{\theta^2 h_2^3} < \frac{1}{\theta^2 h_2^2} < 1.$$

In the case $\text{Fr}^2 > \tilde{\text{Fr}}$, then (17)-2 and (13) imply that the physical solution, h_1 , is less than $1/\theta$ and inequality (18) can be proven in a similar way. \square

2.3 Dam-break problem

A very interesting Riemann Problem is the dam-break. Here, we consider an extended dam-break problem in which we also have a soil surface jump. It can be formulated as

$$\begin{aligned} u_L &= 0, & u_R &= 0; \\ h_L + z_L &> h_R + z_R. \end{aligned} \tag{19}$$

We seek a solution of the problem that is defined by a composite wave. In the presence of a jump in one of terrain function, the composite wave must include a 3-wave since it is the only wave that supports a jump in a terrain function. Also, in the case of dam-break problem, the composite wave must include a 1-rarefaction wave issuing from the left state U_L and ending at a point U with $\lambda_1(U) \leq 0$ and this U must be an admissible state for a 3-wave. It follows that it is essential to investigate the composite wave $W_3(W_1(h; h_L, u_L),$

where $W_1(h; h_L, u_L)$ is restricted to the segment $\lambda_1(W_1(h; h_L, u_L)) \leq 0$, for $h < h_L$.

For this purpose, we consider that the terrain functions data θ_R , θ_L , z_R and z_L are frozen and we study what happens with the solution when the hydrodynamic data h_L and h_R take different values.

We denote by $h_L^\#$ the value of h where the rarefaction 1-wave issuing from (h_L, u_L) with $\text{Fr}_L^2 < 1$ intersects the curve $\text{Fr}(u, h) = 1$

$$h_L^\# = h_L \frac{(\text{Fr}_L + 2)^2}{9}, \quad u_L^\# = \sqrt{gh_L^\#}. \quad (20)$$

Here, we use the notations $||z|| = z_R - z_L$ and $\theta = \theta_R/\theta_L$. As in Lemma 2.2, we introduce

$$\tilde{\text{Fr}}(h) = \frac{1}{\theta} + \frac{-b(\theta)}{-b(\theta) + \theta} \left(1 - \frac{|z|}{h} - \frac{1}{\theta} \right). \quad (21)$$

The curve $W_3(W_1(h; h_L, u_L))$ can be defined for h close enough to h_L ($\text{Fr}_L = 0!$) but it is questionable whether it can be defined for any $h_L^\# < h < h_L$.

The next proposition provides sufficient conditions on h_L to guaranty that the curve $W_3(W_1(h; h_L, u_L))$ is well defined for any $h_L^\# < h < h_L$ and also describes properties of this curve. It is based on the fact that there are some circumstances that allow to solve the terrain jump equation for any Froude number, see Lemma 2.1.

Proposition 2.1. *Suppose that:*

Case a: $\theta_R > \theta_L$, $||z|| > 0$ and

$$h_L^\# > (z_R - z_L) \frac{\theta_R}{\theta_R - \theta_L}. \quad (22)$$

Case b: $\theta_R > \theta_L$, $||z|| < 0$, (without restrictions on h_L).

Case c: $\theta_R < \theta_L$, $||z|| < 0$ and

$$h_L < (z_R - z_L) \frac{\theta_R}{\theta_R - \theta_L}. \quad (23)$$

Then:

(1) *In all three cases, one can define the curve 3-wave $W_3(W_1(h; h_L, u_L))$, for $h_L^\# \leq h \leq h_L$.*

(2) *$W_3(W_1(h; h_L, u_L))$, for $h_L^\# \leq h \leq h_L$ is a disconnected curve composed by two continuous branches, one with Froude number greater than one and the other one with Froude number smaller than one, in the Case a and in the Case b provided that*

$$\tilde{\text{Fr}}(h_L^\#) < 1. \quad (24)$$

(3) $W_3(W_1(h; h_L, u_L))$, for $h_L^\# \leq h \leq h_L$ is a continuous curve with Froude number smaller than one in Case c and in Case b provided that

$$\tilde{\text{Fr}}(h_L^\#) > 1. \quad (25)$$

Proof. (1). If the conditions (22) and (23) are satisfied, then one can use Lemma 2.1-a. To prove (2) and (3), we use Lemma 2.2-b, the estimation for $\tilde{\text{Fr}}(h_L^\#)$,

$$\tilde{\text{Fr}}(h_L^\#) < 1 - \frac{[|z|]}{h_L^\#} < 1,$$

in Case a and

$$\tilde{\text{Fr}}(h_L^\#) > \frac{1}{\theta} > 1,$$

in Case c, combined with the property that the functions $\tilde{\text{Fr}}(h)$ and $\text{Fr}(h)^2$ have at most only one intersection point on the interval $[h_L^\#, h_L]$. \square

The remaining case, $\theta_R < \theta_L$ and $[|z|] < 0$, is less complicated than the ones analyzed in Proposition 2.1 and $W_3(W_1(h; h_L, u_L))$ can be completely described.

Proposition 2.2. *Suppose that $\theta_R < \theta_L$ and $[|z|] > 0$. There is a critical value $h^\# < h_c \leq h_L$ such that the following properties hold:*

- (1) *The curve 3-wave $W_3(W_1(h; h_L, u_L))$ can be defined for $h_c \leq h \leq h_L$ and it is connected.*
- (2) *Moreover,*

$$\text{Fr}_+(W_3(W_1(h; h_L, u_L))) \leq \text{Fr}_+(W_3(W_1(h_c; h_L, u_L))), \quad h_c \leq h \leq h_L, \quad (26)$$

where $\text{Fr}_+ := \theta^{-1}\beta^{-3/2}\text{Fr}_-$.

- (3) *The curve 3-wave $W_3(W_1(h; h_L, u_L))$ does not exist for $h^\# \leq h < h_c$.*

Proof. (1) In the case $1/\theta > 1 > 1 - [|z|]$, we apply Lemma 2.1-a to show that

$$\text{Fr}_* < 1 < \text{Fr}^*.$$

Let $\eta = 1/\theta + (a(\theta) + 2)(1/\theta - (1 - [|z|]))$ be the value of Fr^2 , where the positive root $\tilde{h}_2(\text{Fr})$ of the derivative of $\partial\psi(h; \theta, [|z|], \text{Fr})/\partial y$ equals to $1/\theta$. From $a(\theta) + 2 > 0$ results that $\eta > 1$, hence $\text{Fr}^* > 1$ and $\tilde{h}_2(1) < 1/\theta$. To show that $\text{Fr}_* < 1$, it is sufficient to prove that the minimum $\psi_2(1)$ of $\psi(y; \theta, [|z|], 1)$ is positive. One has

$$\psi_2(1) = \frac{1}{\theta} - \tilde{h}_2(1) - \tilde{h}_2(1) \left(\tilde{h}_2(1) - (1 - [|z|]) \right) \left(a(\theta) + b(\theta)\tilde{h}_2(1) \right)$$

Since $\tilde{h}_2(1) < 1/\theta$, $a(\theta) < 0$ and $b(\theta) < 0$, it is sufficient that $\tilde{h}_2(1) > (1 - \lfloor |z| \rfloor)$ to prove the inequality $\psi_2(1) > 0$.

Taking into account that $\tilde{h}_2(1)$ is the greatest root of the quadratic equation $\partial_y \psi(y; \theta, \lfloor |z| \rfloor, 1) = 0$, the inequality holds if $\partial_y \psi(1 - \lfloor |z| \rfloor; \theta, \lfloor |z| \rfloor, 1) < 0$. One has

$$\partial_y \psi(1 - \lfloor |z| \rfloor; \theta, \lfloor |z| \rfloor, 1) = -b(\theta)(1 - \lfloor |z| \rfloor)^2 - a(\theta)(1 - \lfloor |z| \rfloor) - 1.$$

The function

$$f(y) := -b(\theta)y^2 - a(\theta)y - 1 = 0$$

is monotone increasing on the interval $(0, \infty)$. Since $0 < 1 - \lfloor |z| \rfloor < 1$, then

$$f(0) < f(1 - \lfloor |z| \rfloor) < f(1).$$

Consequently,

$$-1 < f(1 - \lfloor |z| \rfloor) < -b(\theta) - a(\theta) - 1.$$

As $-b(\theta) - a(\theta) - 1 = -b(\theta) + b(\theta)/\theta = b(\theta)(1/\theta - 1) < 0$, it results that

$$f(1 - \lfloor |z| \rfloor) < 0.$$

So, the root $\tilde{h}_2(1) > 1 - \lfloor |z| \rfloor$.

(2) The inequality (26) can be proved as follows. To obtain a point $W_3(W_1(h; h_L; u_L))$, one must solve the terrain equation when the left state is the point $W_1(h; h_L; u_L)$. Let h_c be the minimum value of h on the interval $(h_L^\#, h_L)$ for which the terrain equation is solvable having the point $W_1(h; h_L; u_L)$ as left state. We show that on the curve $W_3(W_1(h; h_L; u_L))$ the Froude number is a monotone decreasing function of h . Let $W_1(h; h_L; u_L)$ be a left state in the terrain equation, $\text{Fr}_-(h)$ be the Froude number of the left state and let $\beta(h)$ be the solution of the equation. Using $\beta(h)$, we define the 3-wave $W_3(W_1(h; h_L; u_L))$. Let $\text{Fr}_+(h)$ be the Froude number of $W_3(W_1(h; h_L; u_L))$. One has

$$\text{Fr}_+(h) = \frac{\text{Fr}_-(h)}{\theta \beta(h)^{3/2}}$$

and then,

$$\partial_h \text{Fr}_+(h) = \frac{\partial_h \text{Fr}_-(h)}{\theta \beta(h)^{3/2}} - 3/2 \frac{\text{Fr}_-(h)}{\theta \beta(h)^{5/2}} \partial_h \beta(h).$$

It is relatively easy to show that on the curve $W_1(h; h_L; u_L)$, one has $\partial_h \text{Fr}_-(h) < 0$.

We show that $\partial_h \beta(h)$ is a negative function, too. Using a standard implicit function theorem, we can write

$$0 = \psi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) = \partial_y \psi(y; \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) \Big|_{y=\beta(h)} \cdot \partial_h \beta(h) + \phi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)),$$

where we used the notation

$$\begin{aligned} \phi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) := \\ + \partial_{\lfloor |z| \rfloor} \psi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) \cdot \partial_h \lfloor |z| \rfloor(h) + \\ + \partial_{\text{Fr}_-} \psi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) \cdot \partial_h \text{Fr}_-(h). \end{aligned}$$

We have,

$$\begin{aligned} \phi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) = \\ = \frac{\lfloor |z| \rfloor(h)}{h} \beta(h) (a(\theta) + b(\theta) \beta(h)) + 2 \text{Fr}_-(h) (1/\theta - \beta(h)) \partial_h \text{Fr}_-(h) < 0. \end{aligned}$$

For the last inequality, we used that $a(\theta)$ and $b(\theta)$ are negative functions, $1/\theta - \beta(h) > 0$ (see Lemma 2.2-a) and $\partial_h \text{Fr}_-(h) < 0$. Using a standard implicit function theorem, we can write

$$\partial_h \beta(h) = -\partial_h \psi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) / \psi(\beta(h); \theta, \lfloor |z| \rfloor(h), \text{Fr}_-(h)) > 0,$$

consequently,

$$\partial_h \text{Fr}_+(h) < 0.$$

□

In all cases that are not covered by the Proposition 2.1, we can not say in advance if the point $U_L^\#$ is an admissible state for W_3 or if its Froude number is greater or smaller than one.

The general strategy to solve the dam-break problem is to find out if the backward wave $W_2^B(U_R)$ intersects a segment of the composite wave $W_3(W_1(h; U_L))$. In the case of negative answer, we try to interpose a 1-wave between $W_3(W_1)$ and W_2 .

We will show that there are three different structures of the composite waves that can solve the dam-break problem. All three algorithms are effective for the more general Riemann Problem (than the dam-break problem) but restricted to $\text{Fr}_L^2 < 1$ and $\text{Fr}_R^2 < 1$.

2.3.1 Constructive Algorithms

Riemann Constructor. Type I

Step 1 *Build the curve*

$$W_3(W_1(h; h_L, u_L); \theta, [|z|]).$$

Step 2 *Find the intersection point h_* such that*

$$W_2^B(h_*; h_R, u_R) = W_3(W_1(h_*; h_L, u_L); \theta, [|z|]).$$

Step 3 *The composite wave curve of the solution for the Riemann problem is*

$$W_2(h_R; W_3(W_1(h_*; h_L, u_L); \theta, [|z|])).$$

In Step 1, we build the maximal connected curve that includes $h = h_L$. It really exists if $\text{Fr}_L = 0$ or if the jumps in terrain functions are small enough. The algorithm furnishes a solution only if the intersection point h_* searched in Step 2 exists, otherwise this algorithm does not provide a solution. A typical pattern of solution is illustrated in Figure 2.

Riemann Constructor. Type II

Step 1 *Build the curve*

$$W_3(W_1(h; h_L, u_L); \theta, [|z|]).$$

Step 2 *Find the intersection point U_2 and h_1 such that*

$$U_2 = W_3(W_1(h_1; h_L, u_L); \theta, [|z|]), \quad \text{Fr}(U_2) = 1.$$

Step 3 *Find the intersection point $h_* < h_2$ such that*

$$W_2^B(h_*; h_R, u_R) = W_1(h_*; W_3(W_1(h_1; h_L, u_L); \theta, [|z|])).$$

Step 4 *The composite wave curve of the solution for the Riemann Problem is*

$$W_2(h_R; W_1(h_*; W_3(W_1(h_*; h_L, u_L); \theta, [|z|]))).$$

This algorithm works only if there is a point on the curve $W_3(W_1(\cdot; \cdot))$ with Froude number equal to one. A solution of this type exists if the point $W_3(W_1(h_1; U_L); \theta, [|z|])$ is below the curve $W_2^B(h; h_R, u_R)$, for $h < h_R$. A typical pattern of solution is illustrated in Figure 3.

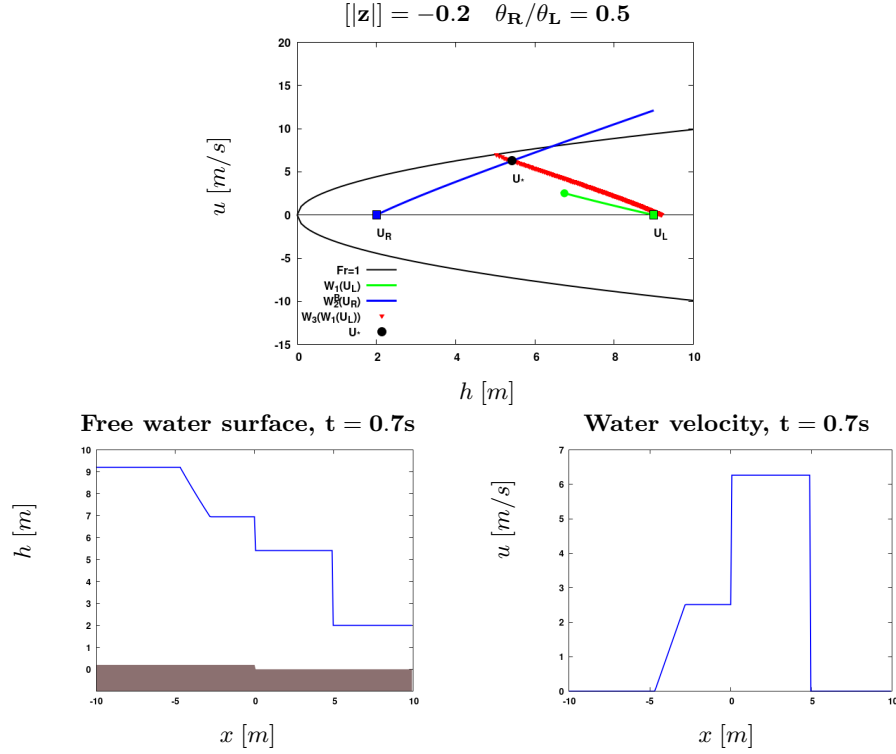


Figure 2: Type I solution of Riemann Problem: the phase portrait (on top), the water level (left bottom) and the water velocity (right bottom).

Riemann Constructor. Type III

Step 1 Find the intersection point h_1 such that

$$\text{Fr}(W_1(h_1; h_L, u_L)) = 1.$$

Step 2 Find the point (h_2, u_2) such that

$$W_3(W_1(h_1; h_L, u_L); \theta, [|z|]) = (h_2, u_2).$$

Step 3 Find the intersection point h_3 such that

$$W_2^B(h_3; h_R, u_R) = W_1(h_3; h_2, u_2).$$

Step 4 The composite wave curve of the solution for the Riemann problem is

$$W_2(h_R; W_1(h_3; W_3(W_1(h_1; h_L, u_L); \theta, [|z|])))$$

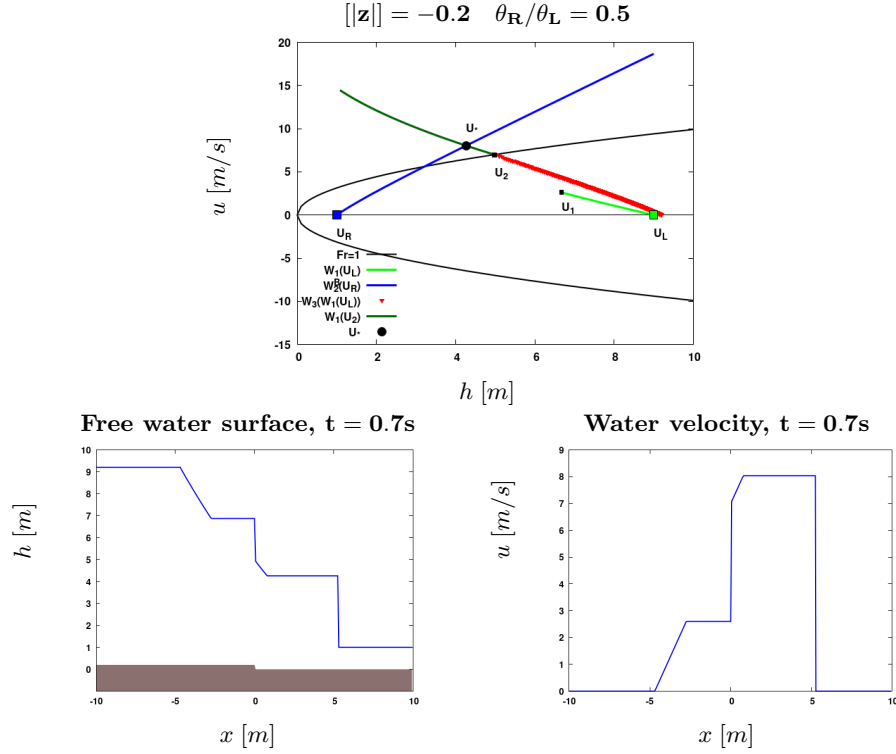


Figure 3: Type II solution of Riemann Problem: the phase portrait (on top), the water level (left bottom) and the water velocity (right bottom).

The algorithm can be used only if U_2 exists and $Fr(U_2) > 1$. The Proposition 2.1 offers results to guaranty that the algorithm of Type III makes sense. To have a solution, one needs to verify that the point h_3 exists and if it is also verified the inequality $h_3 > h_2$, then the shock speed $\sigma_1(h_3; h_2, u_2)$ must be positive. A typical pattern solution is illustrated in Figure 4.

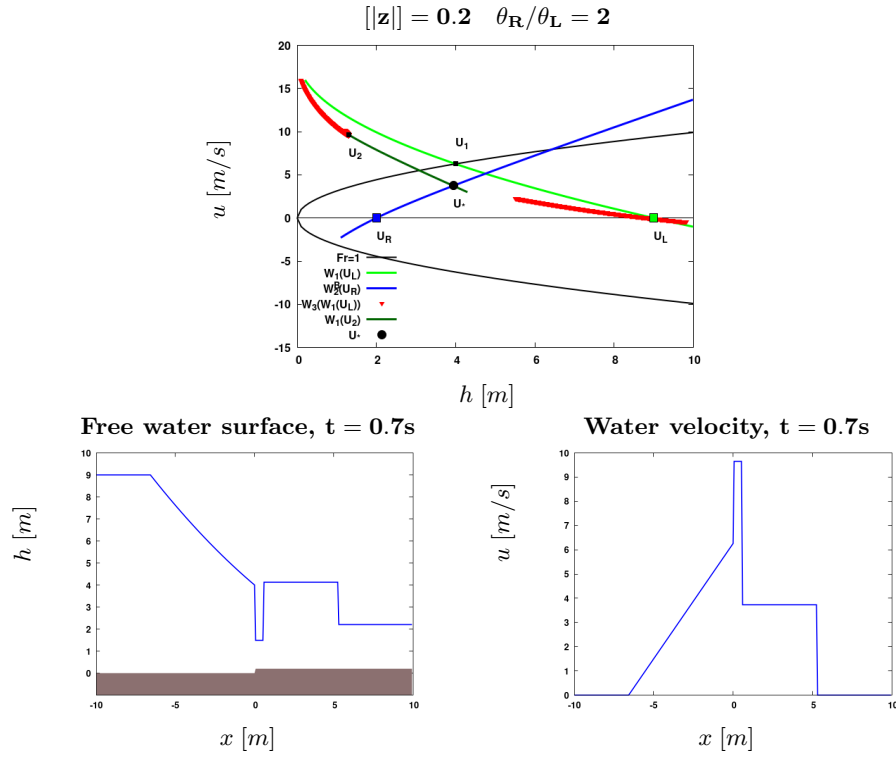


Figure 4: Type III solution of Riemann Problem: the phase portrait (on top), the water level (left bottom) and the water velocity (right bottom).

2.3.2 On the existence of the solution of the dam-break problem

We will provide some sufficient conditions to have a solution by composite waves of the dam-break problem. We will also indicate the cases where the solution by composite wave is not possible. We assume that the terrain data $\{z_L, z_R, \theta_L, \theta_R\}$ and h_L are frozen. We will use the notation $h_{\max} = h_L + z_L - z_R$, $h_L^\#$ and $\tilde{\text{Fr}}(h)$ given by (20) and (21), respectively, the critical number h_c whose existence was proved in proposition 2.2.

The following theorem gathers together the results formulated in Propositions 2.1, 2.2 and in the Constructive Algorithms (of type I, II, III) of the solution for the dam-break problem.

Theorem 2.1. *Let $\{z_L, z_R, \theta_L, \theta_R\}$ and h_L be given. We assume that:*

Case a:

$$\theta_R > \theta_L, \quad z_R > z_L, \quad h_L + z_L > z_R, \quad h_L^\# > (z_R - z_L) \frac{\theta_R}{\theta_R - \theta_L}. \quad (27)$$

There is a value \tilde{h} with $h_L^\# < \tilde{h} < h_L$ such that one can build the curve $W_3(W_1(h; h_L, 0))$ for $\tilde{h} \leq h \leq h_L$, the point $U_2 = W_3(U_L^\#)$ and the 1-wave $W_1(h; U_2)$ for $h < h_0$, $(\sigma_1(h_0; U_2) = 0)$. In this case, there are two values $\xi_1^a < \xi_2^a < h_{\max}$ satisfying the properties:

1. *For $h_R < \xi_1^a$, $W_2^B(h_R, 0)$ intersects $W_1(h; U_2)$ and the solution is given by algorithm III;*
2. *For $\xi_1^a < h_R < \xi_2^a$, there is no solution;*
3. *For $\xi_2^a < h_R < h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I.*

Case b:

$$\theta_R > \theta_L, \quad z_R < z_L. \quad (28)$$

Depending on the value of $\tilde{\text{Fr}}(h_L^\#)$, one has:

Case b1:

$$\tilde{\text{Fr}}(h_L^\#) < 1. \quad (29)$$

There is a value \tilde{h} with $h_L^\# < \tilde{h} < h_L$ such that one can build the curve $W_3(W_1(h; h_L, 0))$ for $\tilde{h} \leq h \leq h_L$, the point $U_2 = W_3(U_L^\#)$ and the 1-wave $W_1(h; U_2)$ for $h < h_0$, $(\sigma_1(h_0; U_2) = 0)$. In this case, there are two values $\xi_1^{b1} < \xi_2^{b1} < h_{\max}$ satisfying the properties:

1. *For $h_R < \xi_1^{b1}$, $W_2^B(h_R, 0)$ intersects $W_1(h; U_2)$ and the solution is given by algorithm III;*

2. For $\xi_1^{b1} < h_R < \xi_2^{b1}$, there is no solution;
3. For $\xi_2^{b1} < h_R < h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I.

Case b2:

$$\tilde{\text{Fr}}(h_L^\#) > 1. \quad (30)$$

One can build the curve $W_3(W_1(h; h_L, 0))$ for $h_L^\# \leq h \leq h_L$. In this case, there is one value $\xi^{b2} < h_{\max}$ with the following properties:

1. For $h_R < \xi^{b2}$ there is no solution;
2. For $\xi^{b2} \leq h_R < h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I.

Case c:

$$\theta_R < \theta_L, \quad z_R < z_L, \quad h_L < (z_R - z_L) \frac{\theta_R}{\theta_R - \theta_L}. \quad (31)$$

One can build the curve $W_3(W_1(h; h_L, 0))$ for $h_L^\# \leq h \leq h_L$. In this case, there is one value $\xi^c < h_{\max}$ with the following properties:

1. For $h_R < \xi^c$ there is no solution;
2. For $\xi^c \leq h_R < h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I.

Case d:

$$\theta_R < \theta_L, \quad z_R > z_L. \quad (32)$$

There is a value h_c with $h_L^\# < h_c < h_L$ such that one can build the curve $W_3(W_1(h; h_L, 0))$ for $h_c \leq h \leq h_L$. Depending of the value of $\text{Fr}_+(W_3(W_1(h_c; h_L, u_L)))$, one has:

Case d1:

$$\text{Fr}_+(W_3(W_1(h_c; h_L, u_L))) < 1 \quad (33)$$

In this case, there is one value $\xi^{d1} < h_{\max}$ with properties:

1. For $h_R < \xi^{d1}$ there is no solution;
2. For $\xi^{d1} \leq h_R < h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I;

Case d2:

$$Fr_+(W_3(W_1(h_c; h_L, u_L))) > 1 \quad (34)$$

There is a value h_c , where $h_c < \tilde{h} < h_L$, with $Fr_+(W_3(W_1(\tilde{h}; h_L, u_L))) = 1$ and one builds the curve $W_3(W_1(h; h_L, 0))$ for $\tilde{h} \leq h \leq h_L$, the point $U_1 = W_3(W_1(\tilde{h}; h_L, 0))$ and the 1-wave $W_1(h; U_1)$ for $h < h_1$. In this case, there is $\xi^{d2} < h_{\max}$ satisfying the properties:

1. For $h_R < \xi^{d2}$, $W_2^B(h_R, 0)$ intersects $W_1(h; U_1)$ and the solution is given by algorithm II;
2. For $\xi^{d2} \leq h \leq h_{\max}$, $W_2^B(h_R, 0)$ intersects $W_3(W_1(h; h_L, 0))$ and the solution is given by algorithm I.

3 Conclusion

In this paper, we provide certain analytical solutions of Riemann Problem for shallow water equations with topography and vegetation when the jump of the initial data is arbitrary large. We introduced three algorithms that allow to solve the dam-break problem. Each algorithm provides a set of elementary waves that are combined to obtain the solution to the dam-break problem. This algorithmic procedure can be extended to solve the general Riemann Problem.

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References

- [1] Luca Cozzolino and Veronica Pepe and Luigi Cimorelli and Andrea D’Aniello and Renata Della Morte and Domenico Pianese, *The solution of the dam-break problem in the Porous Shallow water Equations*, Advances in Water Resources, **114**(2018), 83–101.
- [2] Luca Cozzolino and Luigi Cimorelli and Carmine Covelli and Renata Della Morte and Domenico Pianese, *The analytic solution of the Shallow-Water Equations with partially open sluice-gates: The dam-break problem*, Advances in Water Resources, **80**(2015), 90–102.
- [3] G. Dal Maso, P.G. LeFloch, F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**(1995), 483–548.

- [4] O. Delestre, C. Lucas, P.A. Ksinant, F. Darboux, C. Laguerre, et al. *SWASHES: a compilation of Shallow Water Analytic Solutions for Hydraulic and Environmental Studies*, International Journal for Numerical Methods in Fluids, **72**(3)(2013), 269–300.
- [5] Thierry Gallouet, Jean-Marc Herard, Nicolas Seguin, *Some approximate Godunov schemes to compute shallow-water equations with topography*, Computers&Fluids, **32**(2003) 479–513.
- [6] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Commun. Pure. Appl. Math. **18**(1965), 697–715.
- [7] Vincent Guinot, *A critical assessment of flux and source term closures in shallow water models with porosity for urban flood simulations*, Advances in Water Resources, **109**(2017), 133–157
- [8] S. Ion, D Marinescu and S G Cruceanu, *Riemann Problem for Shallow Water Equation with Vegetation* An. St. Univ. Ovidius, **26**(2)(2018), 145–173.
- [9] S. Ion, D. Marinescu, A. V. Ion, S. G. Cruceanu, V. Iordache *Water ow on vegetated hill. 1D shallow water equation type model*, An. St. Univ. Ovidius, **23**(3)(2015), 83–96.
- [10] P.D. Lax, *Hyperbolic systems of conservation laws. II.*, Commun. Pure Appl. Math. **10**(1957), 537–566.
- [11] P.G. LeFloch, Mai Duc Thanh, *Rieamann problem for the shallow water equations with discountinuous topography*, Commun. Pure Appl. Math. **5**(4)(2007), 865–885.
- [12] P.G. LeFloch, *Hyperbolic systems of conservation laws: the theory of classical and nonclassical shock waves, Lectures in Mathematics*, ETH Zuerich, Birkhauser, 2002.
- [13] P.G. LeFloch, *Graph solutions of nonlinear hyperbolic systems*J. Hyper. Diff. Equa. **1**(2004), 243–289.
- [14] P.G. LeFloch, A.E. Tzavaras, *Representation of weak limits and definition of nonconservative products*, SIAM J. Math. Anal., **30**(6)(1999), 1309–1342.
- [15] H. M. Nepf, *Drag, turbulence, and diffusion in flow through emergent vegetation*, Water Resources Research, **35**(2)(1999), 479–489.

- [16] J.A. Smoller, *Shock Waves and Reaction-Diffusion Equations*, (Second Edition). New York: Springer, 1994.
- [17] E.F. Toro, *Riemann solvers and numerical methods for fluid dynamics*, Berlin: Springer, 1997.
- [18] A.I. Volpert, *The spaces BV and quasilinear equations*, Math. USSR Sbornik, **73**(1967), 255–302.

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