# Modelarea efectului indus de plante asupra proceselor hidrologice
# - Raport tehnic -

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**Instrumente de modelare a proceselor de interfaţă Apă-Sol-Plante pentru administrarea inteligentă şi durabilă a bazinelor hidrografice şi a ecosistemelor dependente de apa subterană**

**Proiect P2 MODSPA-NUTRITOX**

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# 1 Introducere

Covorul vegetal este un factor activ de control al circulației apei în natură. Prin intermediul rădăcinilor, plantele extrag apa din sol și o eliberează în atmosferă la nivelul etajului foliar care la rândul lui reține o parte din apa căzută în timpul precipitațiilor. Prezența plantelor pe terenuri în pantă modifică regimul hidrodinamic al curgerilor de suprafață cu efecte directe asupra proceselor de eroziune și propagării viiturilor în timpul inundațiilor. Scara proceselor este extrem de largă: procesele de absorbţie a apei din sol în plante au loc la scara rădăcinii, procesele de evapotranspiraţie sunt la nivelul frunzei, procesele de eroziune şi inundaţiile sunt semnificative la nivel bazinal.
Studierea acestor procese este de o importanţă vitală pentru climatologie, hidrologie, ecologie şi agricultură întrucât cunoaşterea lor permite elaborarea unor modele de analiză şi prognoză utile în managementul resurselor naturale.
Conceptul de continuum Sol-Plante-Atmosferă permite crearea unui cadru unitar de analiză teoretică a proceselor semnificative atât la scară microlocală cât şi scară macrolocală. Trecerea de la micro la macro poate fi făcută printr-o procedură de mediere a legilor generale de bilanţ.
În capitolul *"Mathematical Models in Hydrology: Shallow Water Type Equations"* descriem în detaliu această procedură aplicătă obţinerii unor modele matematice pentru:

1 curgerea apei în sol;

2 curgerea apei la suprafaţa solului;

3 eroziunea solului produsă de curgerea apei.

Raportul conţine o parte aplicativă şi una teoretică. Partea teoretică este dedicată deducerii modelelor macroscopice.
Ca principiu general, un modelul macroscopic presupune existenţa unui principiu general de bilanţ (conservarea masei, conservarea impulsului sau energiei) formulat în cadrul axiomatic al mecanicii mediilor continue (scara microlocală) şi o formulă de mediere. Prin această tehnică se obţin noi concepte care sunt utilizabile la scara macrolocală. Un exemplu tipic este conceptul de mediu poros cu rădăcini. La nivel micro avem medii distincte: matricea solidă a solului, spaţiul gol dintre particulele solide (porii), rădăcinile plantelor şi mediul fluid (aer, apă, etc.). Mediul fluid circulă prin spaţiu porilor şi poate trece în rădacinile plantelor. La acest nivel, geometria porilor si a rădăcinilor este extrem de complicată şi practic este imposibil să determinăm regimul de viteze al mediului fluid. Prin mediere, mediile îşi pierd identitatea, spaţiul este ocupat de un mediu poros cracterizat de unele proprietăti ca porositate, conductivitate hidraulică, umiditate etc.
În partea aplicativă prezentăm câteva aplicaţii care ilustrează cum poate fi utizat modelul matematic pentru a studia probleme reale. Ne vom restrictiona la trei probleme importante din punct de vedere al aplicaţiilor practice: curgerea apei în sol, curgerea apei pe suprafeţe acoperite cu vegetaţie şi erosiunea solului. Toate rezultatele numerice au fost obţinute cu ajutorul modulelor ASTERIX-CASES şi ASTERIX-CASPA, componente ale softului ASTERIX.

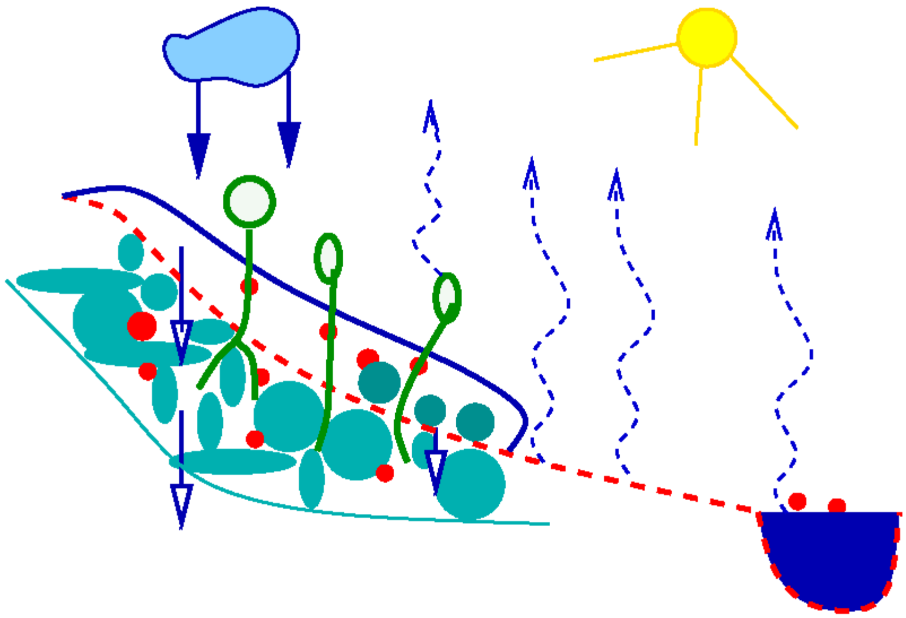

Figure 1: Circulaţia apei în sistemul Sol-Plante-Atmosferă. Procese dominante: precipitaţii, evapotranspiraţie, infiltraţie, curgeri de suprafaţă.

## 2 Curgerea apei în sol

Modelul matematic folosit este un model larg utilizat în științele solului: ecuațiile lui Richards cu surse de masă. Sursele de masă modelează absorbţia apei de către plante prin intermediul rădăcinilor. În acest model, rădăcinile plantelor acționează ca o pompă care extrage apă din sol cu o rată care variază în funcție de densitatea spațială a rădăcinilor şi de potenţialul de evapotransipraţie al plantelor.
Ecuaţia lui Richards este dată de

$$\frac{\partial \theta}{\partial t} - \operatorname{div} K\nabla(\psi + z) = -j_w(t, \boldsymbol{x})$$

unde $\theta$ reprezintă conţinutul de apă din sol, $\psi$ sarcina hidraulică, $K$ conductivitatea hidraulică, iar $z$ coordonata pe verticală. Termenul $j_w(t, \boldsymbol{x})$ modelează rata de absorbţie a apei din sol de către plante. De regulă, în zona rădăcinilor, solul este nesaturat, spaţiul porilor este ocupat parţial cu apă. În această zonă, conductivitatea hidraulică $K$, conţinutul de apă, $\theta$ şi sarcina de presiune $\psi$ sunt legate prin relaţii algebrice neliniare. Relaţiile sunt de natură empirică şi sunt caracteristice unui anumit tip

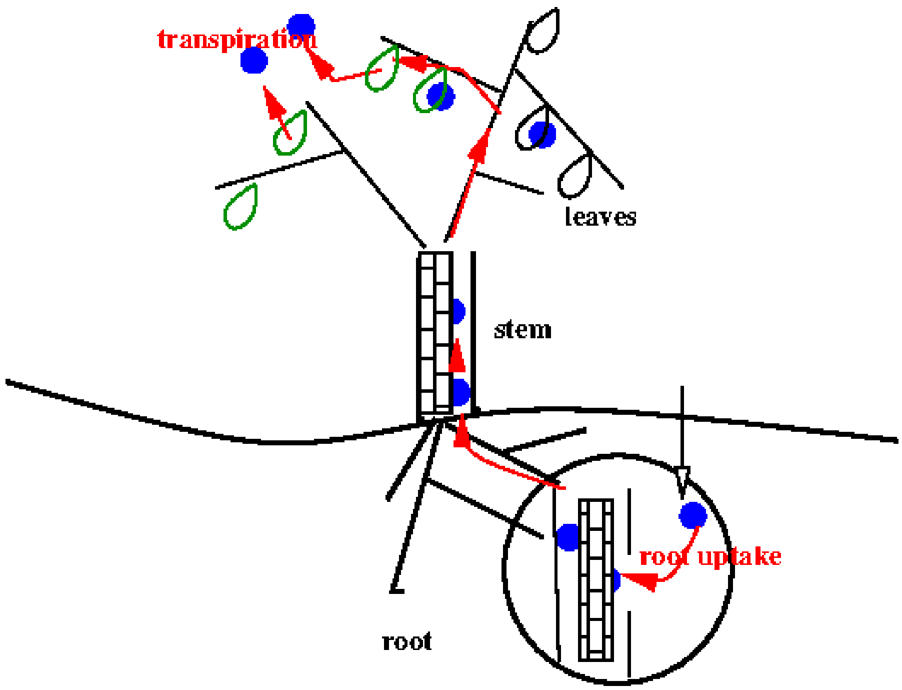


Figure 2: Pentru a trăi şi a se dezvolta, plantele au nevoie de apa din sol

de sol. Un model de relaţii empirice este modelul Mualem-vanGenuchten:

$$S(\psi) = \begin{cases} (1+(\alpha\psi)^n)^{-m}, & \psi < 0, \\ 1, & \psi \geq 0, \end{cases}$$
$$K(S) = \begin{cases} K_s S^l \left(1-\left(1-S^{1/m}\right)^m\right)^2, & 0 < S < 1, \\ K_s, & S \geq 1, \end{cases}$$
$$S = \frac{\theta-\theta_r}{\theta_s-\theta_r}.$$

În acest model intervin următorii parametri:
$K_s$, $\theta_s$ conductiviatea hidraulică şi respectiv conţinutul de apă în regim saturat,
$\theta_r$ conţinutul de apă rezidual;
$n, l$ parametrii specifici modelului Mualem-vanGenuchten, $m = 1 - 1/n$.
Pentru determinarea ratei de absorţie, în literatură au fost propuse mai multe formule. În acest raport folosim o relaţie de tipul:

$$j_w = \gamma(\psi)\Pi_m(t, \boldsymbol{x})$$

în care $\Pi_m(t, \boldsymbol{x})$ este rata normalizată de absorţie a apei, $\gamma(\psi)$ este o funcţie care modelează stresul hidric,

$$\Pi_m(t, \boldsymbol{x}) = T_p(t)\frac{\beta(\boldsymbol{x})}{\int_{\Omega_r}\beta(\boldsymbol{x})\mathrm{d}x},$$

$\beta(\boldsymbol{x})$ modelează densitatea de distribuţie a rădăcinilor. $T_p$ este potenţialul de transpiraţie al plantelor. Funcţiile $\gamma(\psi)$ şi $\beta(\boldsymbol{x})$ considerate au expresiile:

$$\beta(z) = \left(1 - \frac{z}{z_m}\right) \mathrm{e}^{-\frac{p_z}{z_m}|z^* - z|}, \quad \gamma(\psi) = \frac{1}{1 + \left(\dfrac{\psi}{\psi_{50}}\right)^p},$$

unde $z^*$, $p_z$, $p$ sunt parametri. $\psi_{50}$ este sarcina hidraulică la care rata de evapotranspiraţie se reduce la jumătate.
Provocarea principală a unui model matematic este să măsoare variaţia variabilelor procesele hidrodinamice în raport cu variaţia caracteristicilor bio-fizice ale plantelor. În modelul propus de noi, caracteristicile bio-fizice ale plantelor sunt cuantificate prin intermediul a şase parametri. Trei dintre aceştia caracterizează geometria spaţială a sistemului radicular, iar ceilalţi trei parametri ţin de metabolismul intern al plantelor.

**Aplicaţie.** *Infiltraţia apei de ploaie intr-o coloană de sol acoperit cu vegetaţie.*
**Obiective.** Influenţa parametrilor asociaţi plantelor asupra proceselor de evapotranspiratie şi drenaj.

Pentru simularea numerică am considerat o coloană de sol cu grosimea de 1 m compusă din două straturi, stratul 1 fiind la suprafaţa solului. Datele pentru ploie provin din înregistrări zilnice, cantitea de apă cazută intr-o zi, pe un interval de 30 de zile. Calculul soluţiei numerice a fost efectuat cu următoarele condiţii la limită: flux $q_r(t)$ impus la suprafaţa solului. La baza solului am considerat drenaj liber.
Fluxul $q_r(t)$ a fost calculat prin distribuirea în mod uniform a precipitaţiilor dintr-o zi pe perioada întregii zile:

$$q_r(24 * t_d + t^*) = r(t_d)/24,$$

unde $t_d$ este timpul măsurat în zile, $r(t_d)$ cantitatea de apă căzută în ziua $t_d$, iar $t^*$ timpul zilnic măsurat în ore, $t^* \in [0, 24]$.
Pentru analiza variaţiei regimului de curgere în sol în funcţie de parametrii asociaţi plantelor am analizat patru cazuri distincte. Tabelul 1 conţine valorile parametrilor utilizaţi pentru fiecare caz în parte. Timpul de simulare 30 de zile.
Experimentul numeric arată ca modelul matematic este suficient de sensibil la variaţia parametrilor caracteristici plantelor. El poate fi utilizat pentru elaborarea unor scenarii privind protecţia pânzei freatice împotriva contaminării cu ape de suprafaţă contaminate.

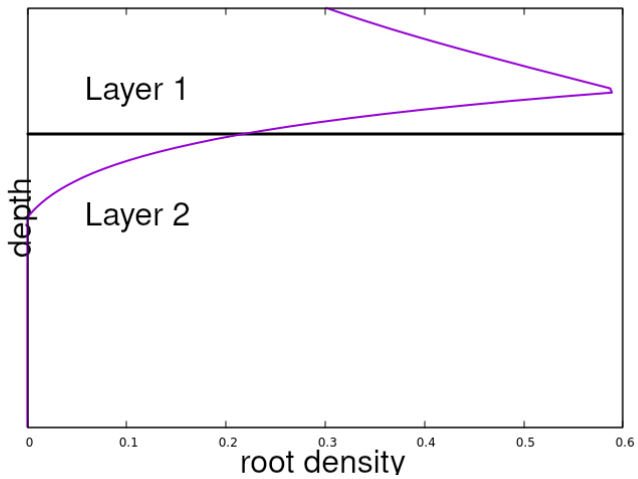


Figure 3: Configuraţia spaţială a coloanei de sol şi distribuţia rădăcinilor.

| Parametrii asociaţi solului | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Grosime | $n$ | $\alpha$ | $l$ | $K_s$ | $\theta_s$ | $\theta_r$ |
| Stratul 1 | $[0, 0.3]$ | 2 | 3.35 | 0.5 | 0.3318 | 0.368 | 0.102 |
| Stratul 2 | $[0.3, 1]$ | 2 | 3.35 | 0.5 | 0.118 | 0.409 | 0.082 |

| Experiment | Parametrii asociaţi plantelor | | | | | |
|---|---|---|---|---|---|---|
| | Geometria rădacinilor | | | Fiziologia plantelor | | |
| | $z_m$ | $p_z$ | $z^*$ | $p$ | $\psi_{50}$ | $T_p$ |
| 1 | 0.5 | 3 | 0.1 | 3 | $-6$ | $2.06e-4$ |
| 2 | 0.5 | 3 | 0.2 | 3 | $-6$ | $2.06e-4$ |
| 3 | 0.5 | 3 | 0.2 | 3 | $-6$ | $1.06e-4$ |
| 4 | 0.5 | 3 | 0.2 | 3 | $-6$ | 0 |

Table 1: Parametrii asociaţi solului şi plantelor. Unităţi de măsură: L[m], T[h].

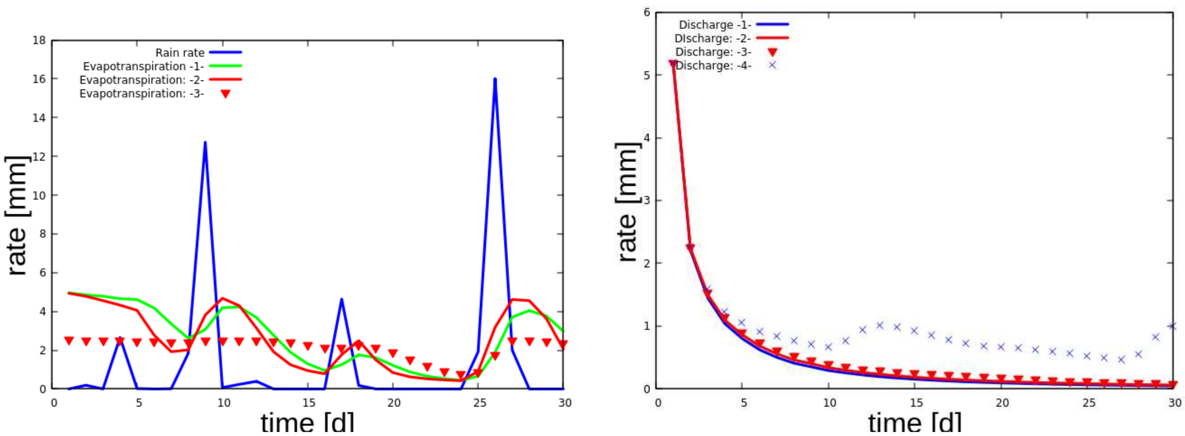


Figure 4: Distribuţia ratei de evapotranspiraţie (imaginea din stânga) şi a ratei de infiltraţie (imaginea din dreapta). Datele privind caracteristicile solului şi plantelor sunt cele din Tabelul 1.

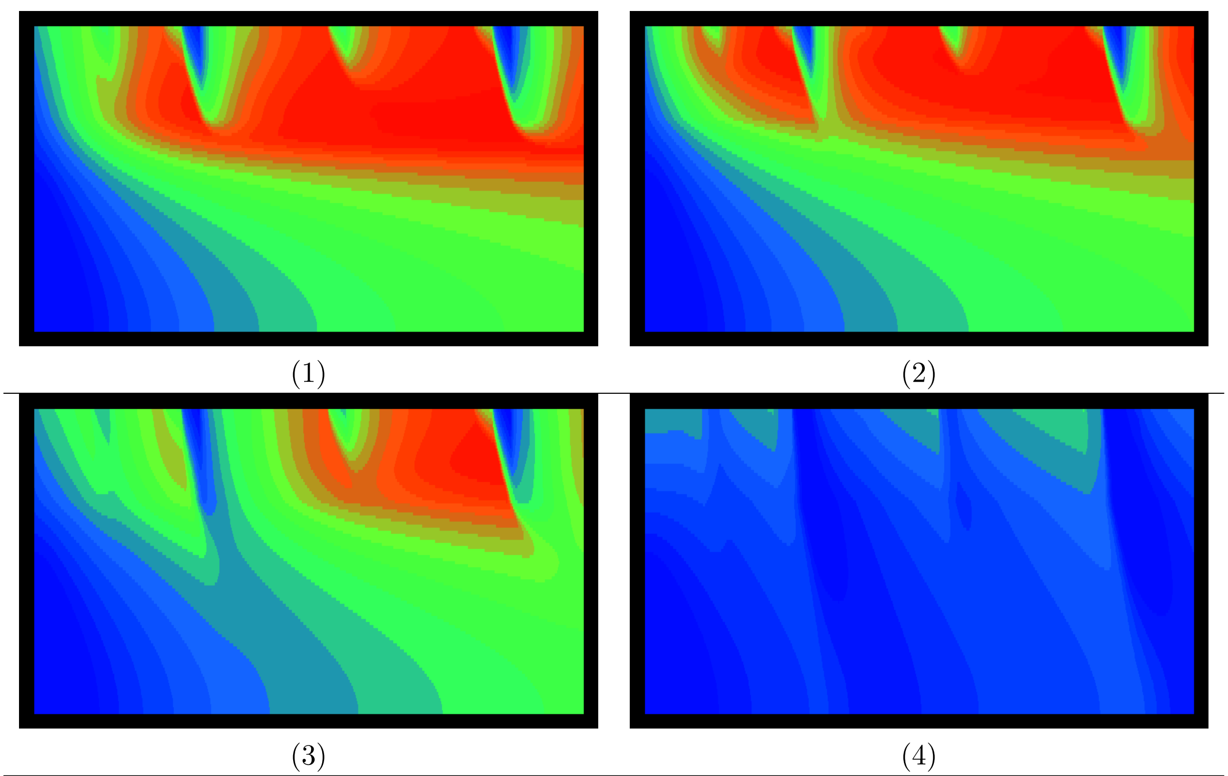


Figure 5: Distribuţia conţinutului de apă în sol

# 3 Curgerea apei pe suprafeţe în pantă şi acoperite cu vegetaţie

Modelul folosit este dat de ecuațiile Saint-Venant cu porozitate și frecare. Caracteristicile solului sunt luate în considerare prin intermediul coeficientului de frecare apă-sol, iar cele ale covorului vegetal prin intermediul porozității covorului vegetal și al coeficientului de frecare apă-plante. Din păcate, există puține date experimentale pentru a putea valida modelul numeric pe baza lor. Dispunem de un singur set date măsurate într-un experiment de laborator și făcute disponibile de autorii experimentului, [2]. Pe lângă aceasta confruntare cu datele experimentale, au fost efectuate și validări teoretice pe baza unor soluții analitice. Determinarea soluțiilor analitice este o problemă dificilă și a necesitat un efort deosebit. Simulările au fost făcute cu programul ASTERIX- CASES.

**Aplicaţie.** *Curgerea pe terenuri acoperite cu vegetaţie.*
**Obiective.** Influenţa parametrilor asociaţi plantelor asupra regimului de curgere.

În cadrul acestei aplicaţii au fost efectuate două simulari numerice. O simulare a fost făcută în scopul validarii modelului teoretic şi a schemei numerice. Datele de intrare au fost cele corespunzătoare experimentului prezentat în figura 6.

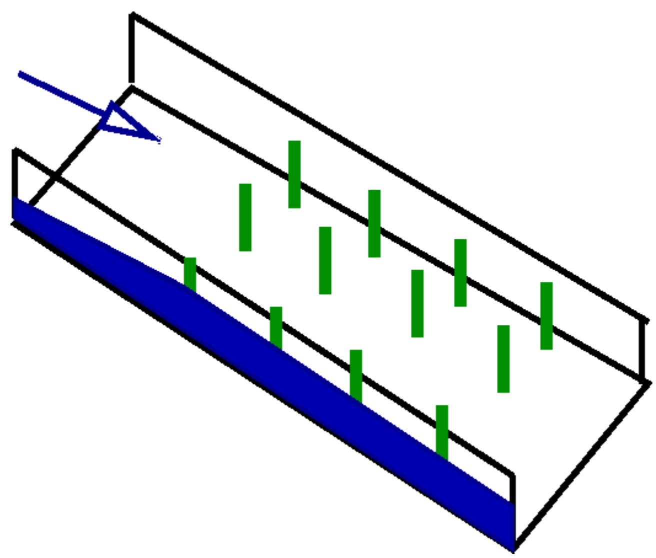

Figure 6: Instalaţie experimentală pentru studiul curgerii apei pe suprafeţe acoperite cu vegetaţie. Jgeabul prin care curge apa are 18 m lungime, 1 m lăţime şi o înclinare de $S = 1.05$mm/m. Covorul vegetal este modelat printr-un şir de cilindri verticali cu raza de 5 mm. Porositatea covorului vegetal este $\theta = 0.99336$. La partea superioară jgeabul este alimentat cu un debit constant, iar partea inferioară este liberă. Alţi parametri: coeficientul Manning $n^2 = 3.005785$, coeficientul de frecare apă-plante $\alpha_p = 74.7643$ m$^{-1}$.

O comparaţie a datelor experimentale cu datele numerice este prezentată în figura 7.
În al cel de al doilea experiment numeric dorim să punem în evidenţă implicaţiile despăduririlor asupra propagării viiturilor de apă în zonele montane şi de deal. Pentru aceasta am considerat o suprafaţă

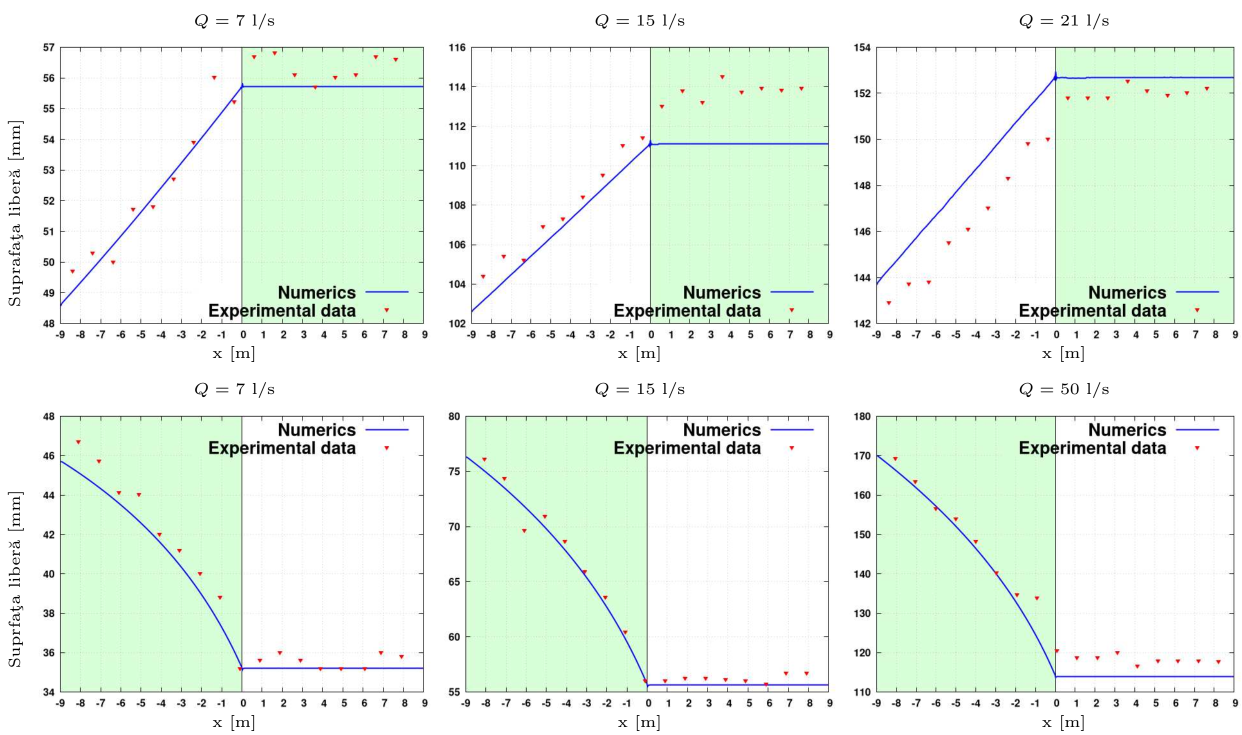


Figure 7: Curgere pe o suprafaţă cu vegetaţie: comparaţie experiment - numeric. $Q$ reprezintă debitul de alimentare.

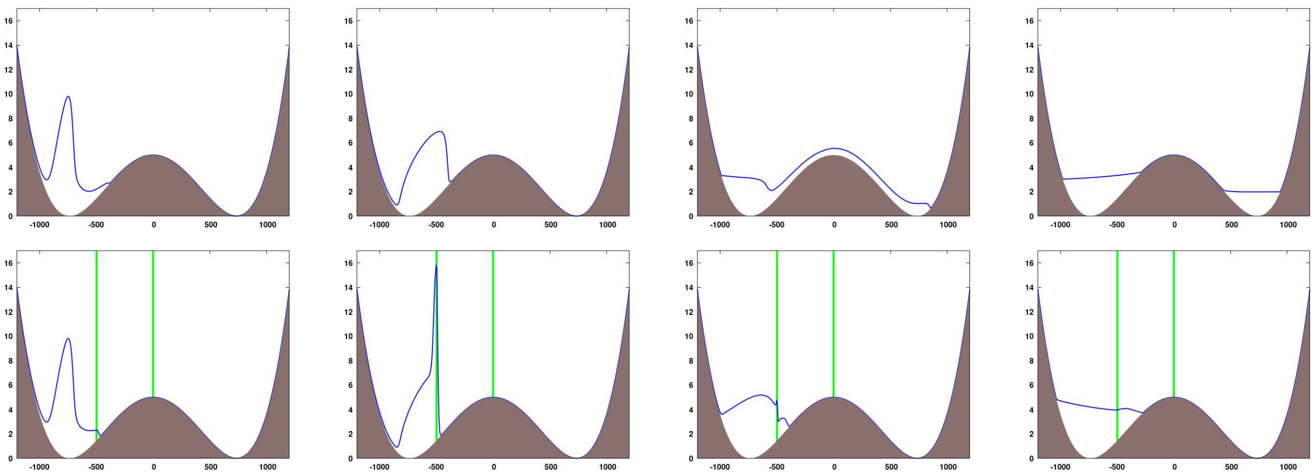


Figure 8: Imaginile reprezintă nivelul apei la momentele de timp $t = 40,\ 60,\ 200,\ 400$ s în două configuraţii distincte: suprafaţă netedă peste tot (prima linie) şi suprafaţă acoperită parţial cu un filtru vegetal (linia a doua).

convex-concavă. Iniţial, suprafaţa este parţial acoperită (zona cu panta în cădere) cu un strat uniform de apă şi la momentul de timp $t = 0$ apa este eliberată provocând o viitură. În funcţie de energia curgerii, apa poate urca sau nu obstacolul prezent în centrul terenului. Rezulatele numerice, vezi figura 8, arată că prezenţa covorului vegetal provoacă o disiparea a energiei cinetice astfel încât zona din spatele obstacolului este protejată la inundaţii. Acest lucru nu se mai întâmplă însă în absenţa filtrului vegetal.

# 4 Eroziunea solului acoperit cu vegetaţie

Modelul combină ecuaţiile Saint-Venant cu vegetaţie cu modelul Hairsine-Rose de eroziune. În acest caz, procedeul de mediere include şi fracţiile de sediment aflate în suspensie.

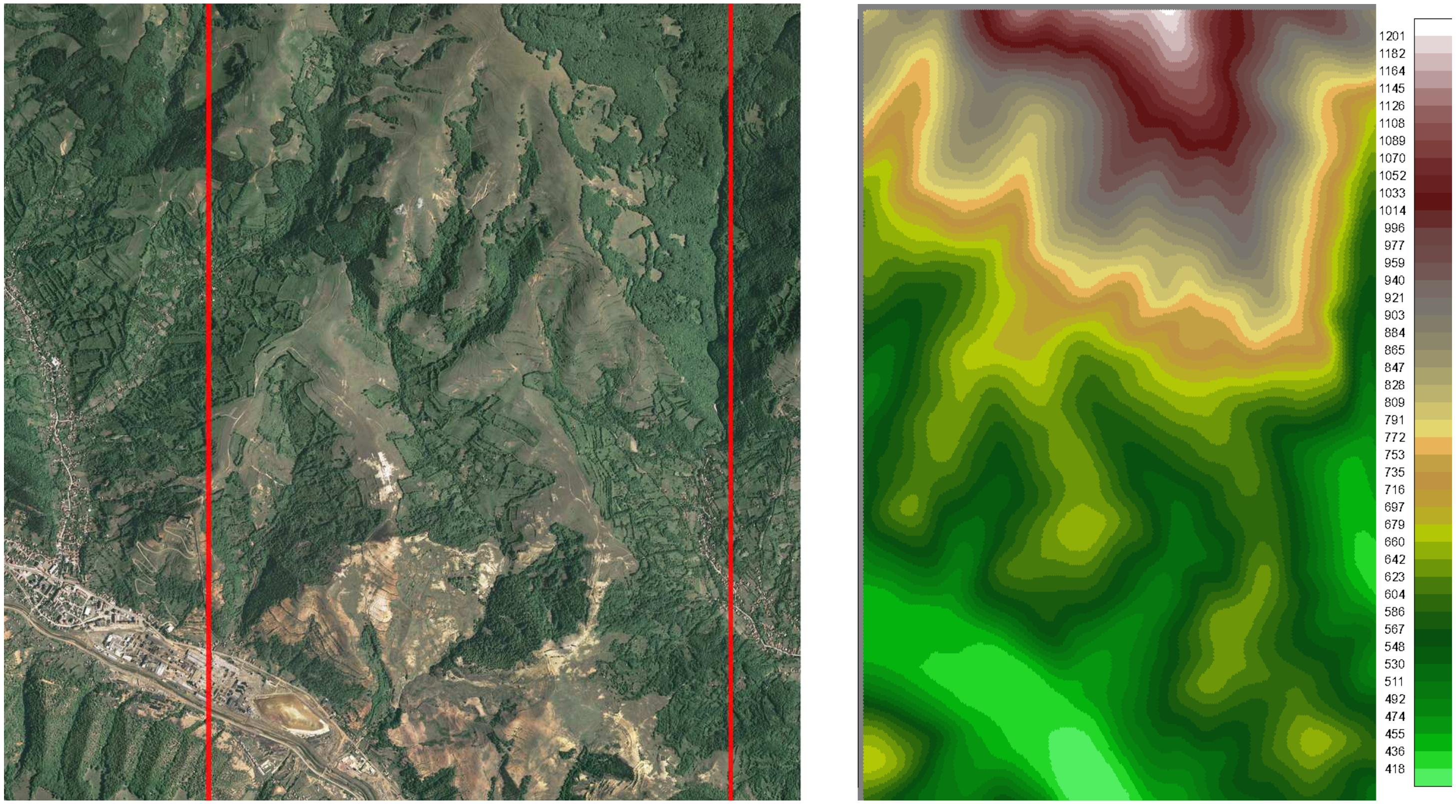


Figure 9: Bazinul hidrografic din Valea lui Paul. Poză aeriană (stânga) şi terenul reconstruit (din date GIS) pe o reţea hexagonală (dreapta).

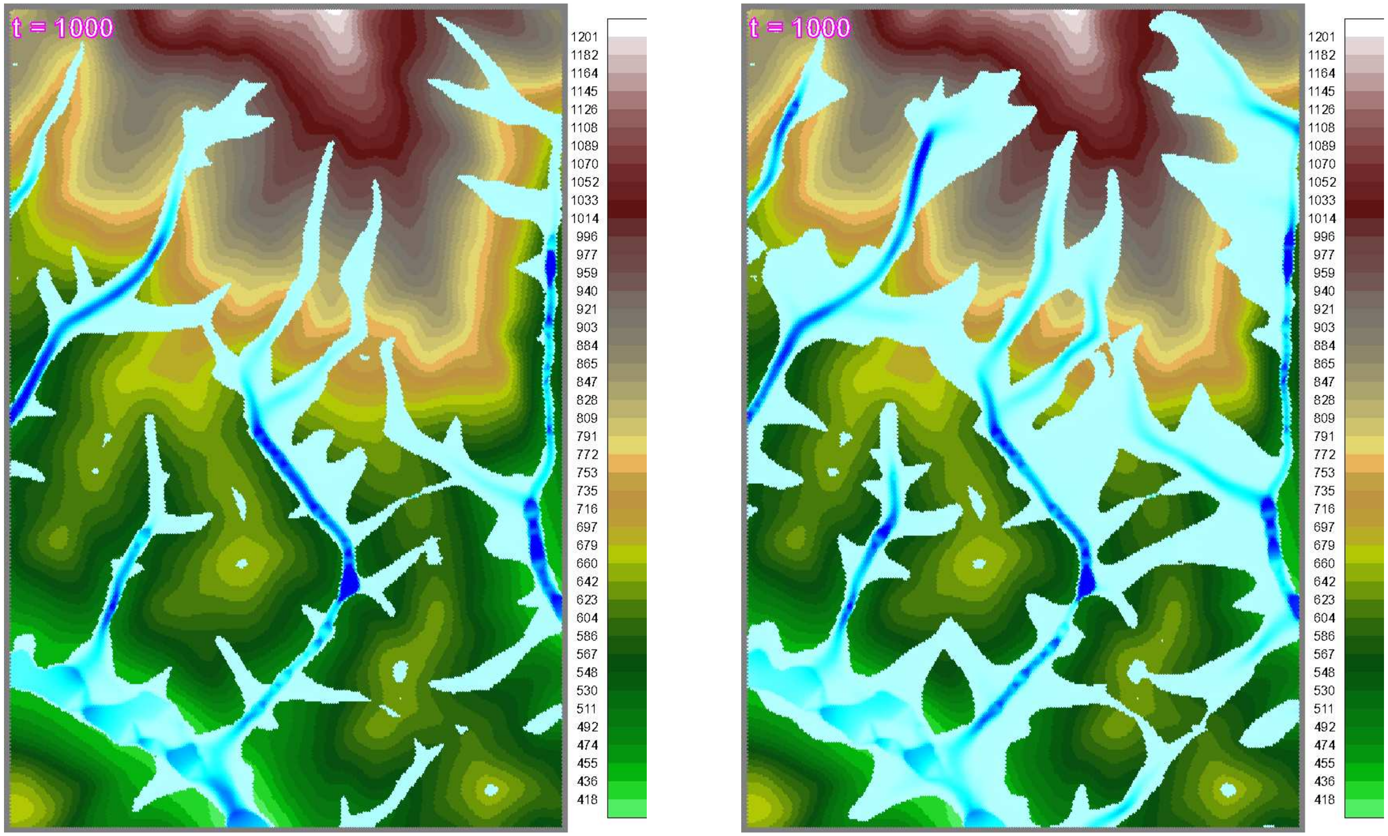


Figure 10: Stop cadru cu distribuţia apei în bazinul hidrografic din Valea lui Paul. Observaţii directe indică faptul că timpul de rezindenţă al apei din bazin depinde de densitatea de vegetaţie. Datele numerice obţinute de noi sunt consistente cu observaţiile din teren: timpul de scurgere al apei din bazin este mai mare în cazul unei densităţi mai mari de vegetaţie. În această simulare s-a considerat terenul acoperit cu o vegetaţie uniformă: $\theta = 0.993$ pentru poza din stânga şi $\theta = 0.974$ pentru poza din dreapta.

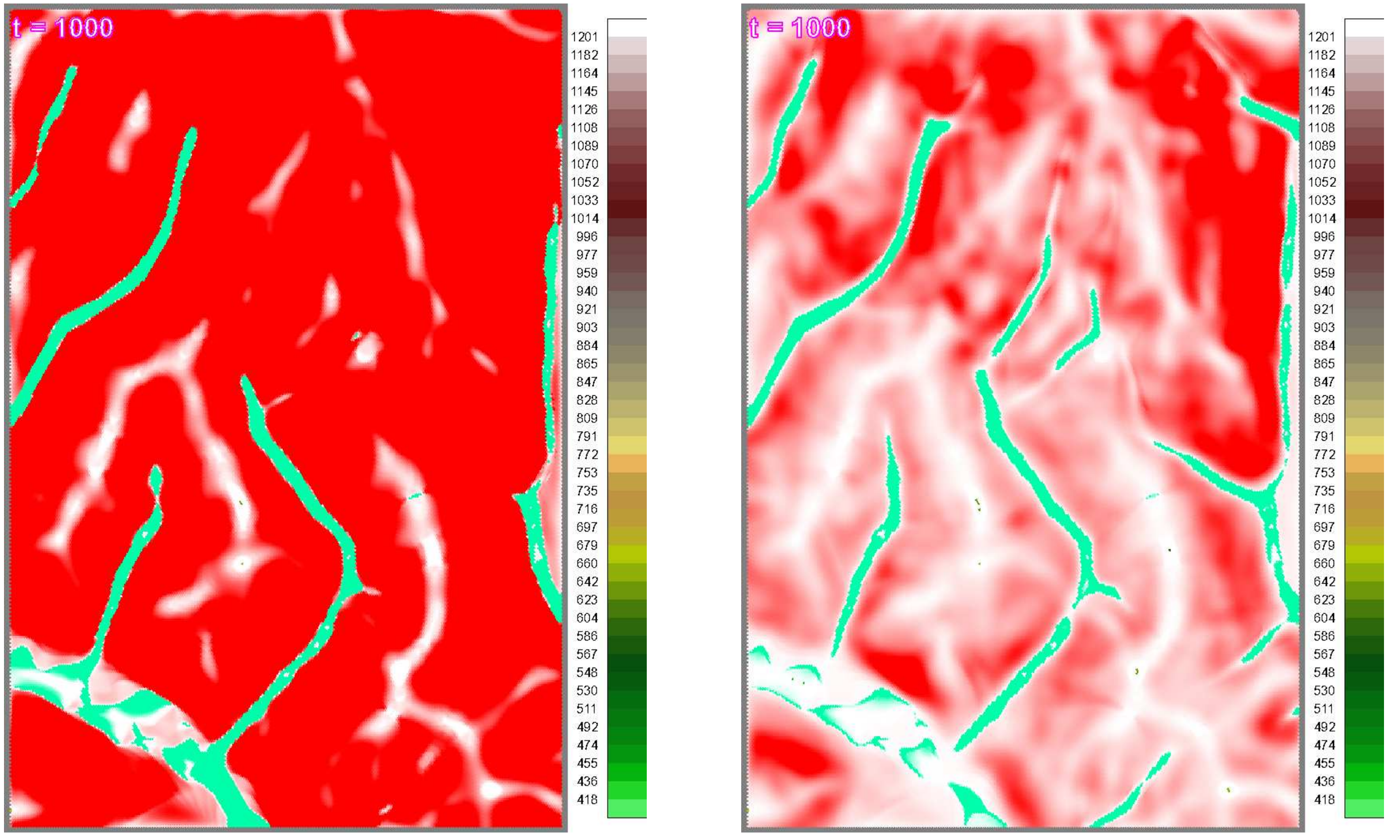


Figure 11: Distribuţia sedimentului erodat la $t = 1000$ s în bazinul hidrografic din Valea lui Paul pentru două distribuţii uniforme de vegetaţie diferite: $\theta = 0.993$ pentru poza din stânga şi $\theta = 0.974$ pentru poza din dreapta. Procesul de eroziune implică două componente: eroziune netă (pierdere de masă) şi depunere netă (câştig de masă). Nuanţele de roşu indică eroziune netă, iar nuanţele de verde indică depunere netă.

## 5 Concluzii

Rezultatele numerice arată că modelele dezvoltate sunt suficient de versatile pentru a prinde heterogenitatea proceselor de mediu. De asemenea, ele ne permit formularea unor noi ipoteze privind rolul plantelor în cadrul proceselor complexe din continuumul Sol-Plante-Apă. Apreciem câ pachetele de programe dezvoltate în cadrul proiectul INTER-ASPA pot fi extrem de utile în proiectarea unor schem de monitoring integrat al bazinelor hidrografice în vederea unei bune administrări a resurselor de apă.

# Mathematical Models in Hydrology: Shallow Water type Equations

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# Contents

# Chapter 1

# General Balance Equations for Fluids

The main subject of our work is the flow a fluid on the soil surface. We assume that thermal effects has a small influence of the mechanical characteristics of the fluid, consequently we deal only with the two balance equations, namely, mass balance equation and momentum balance equation. A general balance equation can be formulate as integral form or as local form. Usually the later is obtained from the first by using several variant of the flux divergence formula. We will present a general form of the balance equation and then we will show how one can obtain different variants of the local form. An instructive exercise is to obtain the differential form in a general curvilinear coordinate system from integral form.

In all our consideration one assume that there exist a Cartesian coordinate system $Ox$ in the reference Euclidean space $E^3$. Let $V$ be a volume in the domain of the flow and let $\psi$ be the mass density of a mechanical quantity. The balance equation of a mechanical quantity defined by the mass density $\psi$ read as

$$\partial_t \int\limits_V \rho\psi \mathrm{d}x + \int\limits_{\partial V} \rho\psi(v_n - u_n)\mathrm{d}\sigma = \int\limits_{\partial V} \mathbf{\Phi}_\psi \boldsymbol{n} \mathrm{d}\sigma + \int\limits_V \rho\boldsymbol{\phi}_\psi \mathrm{d}x \tag{1.1}$$

where $\rho$ is mass density of the fluid, $\mathbf{n}$ is the unitary normal to the boundary $\partial V$ of $V$ outward orientated, $\mathbf{v}$ is the velocity of the fluid. The terms $u_n$ stands for the velocity displacement of a point of boundary $\partial V$. $\mathbf{\Phi}_\psi$ is the flux density of $\psi$ and $\phi$ is mass density supply of $\psi$. For more details the reader is referred to [1] and [2]. The general balance equation (1.1) must be red as

*For any domain $V$ in the domain fluid flow the time variation of the mechanical quantity with the mass density $\psi$ obey the law* (1.1).

If all fields that appear in the balance law are smooth enough one obtain local form or differential form of the balance law.

Table 1.1: The entry fields in the mass and linear momentum balance equation for viscous fluid.

| Mechanical | $\psi$ | $\mathbf{\Phi}_\psi$ | $\phi$ | Comment |
|---|---|---|---|---|
| Mass | 1 | 0 | 0 | For reactive fluids one must specify the mass production $\phi$ |
| Linear momentum | $\boldsymbol{v}$ | $\mathbf{t}$, $\mathrm{t}^{ij} = -p\delta^{ij} + \tau^{ij}$ | $\boldsymbol{f}$ | $\mathbf{t}$ is the stress tensor, $p$ is the pressure field and $\boldsymbol{\tau}$ is viscous stress tensor, $f$ is the body force |

If the integral form can be written only in Cartesian coordinate the differential form can be written as well as in any curvilinear system,$\{x^I\})I = 1, 3$. The invariant form read as

$$\partial_t(\rho\psi^I) + \left(\rho\psi^I v^J\right)_{;J} = \Phi_{\psi}{}^{IJ}_{;J} + \rho\phi^I \tag{1.2}$$

where subscript ; stands for covariant derivative. For a vectorial quantity $\psi$ an alternative form is

$$\partial_t(\sqrt{g}\rho\psi^i)+\partial_j\left(\sqrt{g}\rho\psi^i v^j\right)+\sqrt{g}\psi^k v^j\Gamma^i_{kj} = \partial_j\left(\sqrt{g}\Phi^{ij}_\psi\right)+\sqrt{g}\Phi^{kj}_\psi\Gamma^i_{kj}+\rho\phi^i_\psi \tag{1.3}$$

where $g = \det g_{\cdot,\cdot}$ is the determinant of the metric tensor $g$ and $\Gamma^k_{ij}$ are the Christoffel symbols.

# Chapter 2

# Fluid Flow on Unvegetated Hillslope

The accumulation of water on the surface of the soil is a process that implies rain and infiltration into soil, the rain drops produce a layer of water if rain rate is greater than infiltration rate. This stratum of exceeding water moves on the soil surface down the hill. This flow was modeled in [3] and the flow described by this model is usually named hortonian flow.

There exists different models of water flow on hill slope and each of them is process oriented. As a consequence it is very hard to extrapolate an existent model to a new context or to establish what is generally common for all flow process on hill slope. We find that the Saint Venant equations can be considered as common ground for most models used for hill slope flow. The Saint Venant equations in turn are obtained from Navier Stokes equations by using an asymptotic analysis and a space averaged technique, [8], [9]. Here we present a variant of Saint Venant equations for overland flow obtained from Navier Stokes equations.

## 2.1 Depth average form of the balance of laws of mass and momentum

The physical problem considered here is the motion of a film of fluid along a soil surface. Generally in such problems the topographical characteristics of soil exhibits variation and the depth of fluid is small comparative with the soil area occupied by the fluid. In the water domain the velocity of the fluid is almost parallel with the soil surface and their variation along the depth of the fluid is a small quantity as compared with its averaged.

That two characteristics namely small aspect ratio of depth versus soil surface area and small normal components of the velocity field allows one to use model equations of water as simplified form of full Navier Stokes equations.

Shallow water type equations (SWE) are a typical approximation model used in many applications.

A common way to obtain a SWE model is firstly *depth average differential form of the balance equations* and then obtain a set of equations for the *depth averaged mechanical quantities*. To obtain these equations one essential ingredient is to assume that interface air-water or interface soil-water are material surface with respect to water motion.

The interface water-air is a mobile surface that occupies different space positions at different moments of time. Its mobility is mainly due to the motion of the fluid, but there is another process that can affect its space position: the mass transfer between atmosphere and water body. The rain raises the water level and the evaporation decreases the water level. Both processes do not affect the motion of the water body material.

As regarding the interface water-soil if one take into account the erosion of soil by water moving also give rise to surface that is not a material surface with respect to fluid motion. As conclusion we admit that the water-air interface and soil-water interface are not necessary material surfaces with respect to the fluid motion.

In sequel we present a new method to obtain a shallow water type equation that avoid the assumption that the interfaces that separate the fluid domain from the external media are material surface. In our approach we start with the integral form of the balance laws.

Let $\boldsymbol{Ox^1x^2x^3}$ be a Cartesian coordinate system in the reference Euclidean space $E^3$ and let $V$ be an arbitrary domain in the domain flow. The integral form of the balance laws of momentum and mass are given by

$$\partial_t \int\limits_V \rho \boldsymbol{v} \mathrm{d}x + \int\limits_{\partial V} \rho \boldsymbol{v}(v_n - u_n)\mathrm{d}\sigma = \int\limits_{\partial V} \boldsymbol{t_n} \mathrm{d}\sigma + \int\limits_V \rho \boldsymbol{f} \mathrm{d}x \tag{2.1}$$

$$\partial_t \int\limits_V \rho \mathrm{d}x + \int\limits_{\partial V} \rho(v_n - u_n)\mathrm{d}\sigma = 0 \tag{2.2}$$

respectively. $\rho$ is the mass density of the fluid, $\boldsymbol{v}$ the velocity field of fluid, $\boldsymbol{f}$ is the density of the applied force, $\boldsymbol{t_n}$ is the stress vector and $u_n$ is the normal component of the velocity of a point of the boundary $\partial V$ of $V$.

To perform the depth average we choose a cylindrical volume control $V$ with the top face on the free surface and the bottom face on the soil surface. To take advantage from that the velocity field is almost parallel with soil surface one need to introduce a soil surface based coordinate system. If the soil surface is a mobile surface this give rise to a new difficulty. We assume that the basal surface deviate in a small amount from a fix surface and we choose that surface as a referential surface. Let $\mathcal{B}$ be the referential surface, $\mathcal{S}$ be the soil surface and $\mathcal{U}$ be the free surface.

Let $\mathcal{B}$ be represented by

$$x^i = b^i(y^1, y^2),\ (y^1, y^2) \in \widetilde{D} \subset \mathbb{R}^2. \tag{2.3}$$

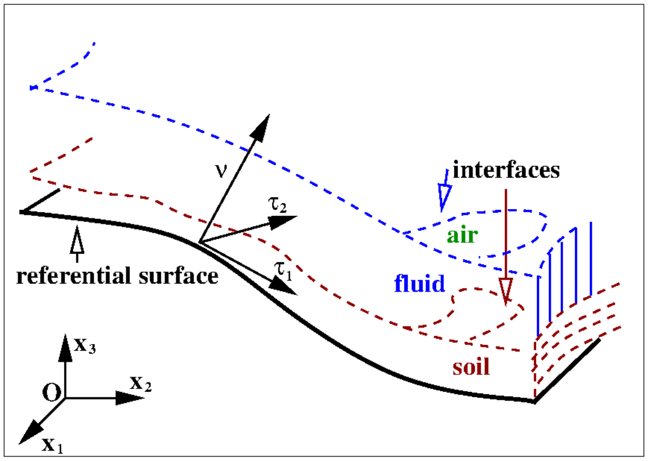


Figure 2.1: Flow domain of the watersheet flow

Given (2.3) one introduces a new coordinate system in euclidean space $E^3$ by

$$x^i = b^i(y^1, y^2) + y^3\nu^i(y^1, y^2),\ (y^1, y^2) \in \widetilde{D} \in \mathbb{R}^2,\ y^3 \in I \in \mathbb{R}, \tag{2.4}$$

where $\boldsymbol{\nu}$ is the unitary normal to the reference surface. It is supposed that the application

$$\boldsymbol{x} : D \times I \to \mathbb{R}^3$$

define a coordinate transformation and its image cover the entire domain of flow

Assume also that there exists the functions

$$\eta : \widetilde{D} \times (0, \infty) \to [0, \infty),$$
$$s : \widetilde{D} \times (0, \infty) \to \mathbb{R}$$

such that the upper surface $\mathcal{U}$ can be parameterized by

$$x^i = b^i(y^1, y^2) + \eta(y^1, y^2, t)\nu^i(y^1, y^2) \tag{2.5}$$

and the bottom surface $\mathcal{S}$ can be parameterized by

$$x^i = b^i(y^1, y^2) + s(y^1, y^2, t)\nu^i(y^1, y^2). \tag{2.6}$$

In the new coordinate system the domain flow is defined by

$$\{(y^1, y^2, y^3) | (y^1, y^2) \in \widetilde{D}, s(y^1, y^2, t) < y^3 < \eta(y^1, y^2, t)\}$$

The function $h(y^1, y^2, t) := \eta(y^1, y^2, t) - s(y^1, y^2, t)$ measures the depth of the fluid along the normal to the reference surface.

Let $D$ be an arbitrary domain in $\widetilde{D}$, $u(y_1, y_2)$ and $w(y_1, y_2)$ two functions defined on $\widetilde{D}$ such that

$$s(y_1, y_2, t) \le u(y_1, y_2, t) < w(y_1, y_2, t) \le \eta(y_1, y_2, t)$$

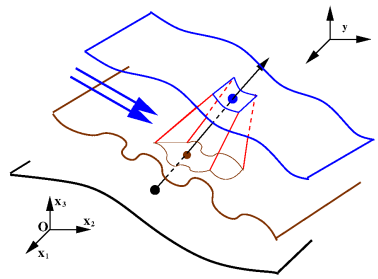


Figure 2.2: Configuration of a volume control.

one defines the cylinder $V$ by

$$V = \left\{ \boldsymbol{x} | \boldsymbol{x} = \boldsymbol{b}(y^1, y^2) + y^3 \boldsymbol{\nu}, u(y^1, y^2) < y^3 < w(y^1, y^2), (y^1, y^2) \in D \right\} \quad (2.7)$$

For cylinder $V$ given by (2.7) one introduce the notations:

$$\begin{aligned} \dot{\mathcal{H}}^i_{\boldsymbol{v}}(D; (u, w)) &:= \quad \partial_t \int_V \rho \boldsymbol{v} \mathrm{d}x + \int_{\partial V} \rho \boldsymbol{v}(v_n - u_n) \mathrm{d}\sigma, \\ \dot{\mathcal{H}}^i_m(D; (u, w)) &:= \quad \partial_t \int_V \rho \mathrm{d}x + \int_{\partial V} \rho(v_n - u_n) \mathrm{d}\sigma, \\ \mathcal{F}^i_{\text{stress}}(D; (u, w)) &:= \int_{\partial V} \boldsymbol{t}_{\boldsymbol{n}} \mathrm{d}\sigma, \\ \mathcal{F}^i_{\text{ext}}(D; (u, w)) &:= \int_V \rho \boldsymbol{f} \mathrm{d}x. \end{aligned} \quad (2.8)$$

By using the above notation one rewrite the balance of the momentum of $V$ as

$$\dot{\mathcal{H}}^i_{\boldsymbol{v}}(D; (u, w)) = \mathcal{F}^i_{\text{stress}}(D; (u, w)) + \mathcal{F}^i_{\text{ext}}(D; (u, w)) \quad (2.9)$$

and balance of mass of $V$ as

$$\dot{\mathcal{H}}^i_m(D; (u, w)) = 0. \quad (2.10)$$

To obtain the averaged form of balance of equations in the based surface coordinate $\{y_1, y_2, y_3\}$ one rewrites the equations (2.9), (2.10) as function of curvilinear components of the velocity and tensor fields.

By using the stress tensor $\mathbf{t}$ one can evaluate the stress vector on the surface $\Sigma_D$ by

$$t^i_n = t^{ij} n_j$$

The stress tensor for an incompressible can be written as

$$t^{ij} = -\rho p \delta^{ij} + \rho \sigma^{ij}$$

where $p$ is the pressure field and $\sigma$ is the viscous part of the stress tensor.

On an interface one write the stress vector as

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = -\sigma^{\perp}\boldsymbol{n} + \sigma^{\parallel a}\boldsymbol{\zeta}_a \tag{2.11}$$

where the where $\boldsymbol{\zeta}_{1,2}$ are two directions in the tangent plan at the interface and $\boldsymbol{n}$ is a unitary normal to the interface.

In the new coordinate system one has

$$\begin{aligned} v^i &= v^a q_a^c \tau_c^i + v^3 \nu^i, \\ f^i &= f^a q_a^c \tau_c^i + f^3 \nu^i, \\ t^{ij} &= t^{ab} q_a^c q_b^e \tau_c^i \tau_e^j + t^{a3} q_a^c \tau_c^i \nu^j + t^{3b} q_b^e \nu^i \tau_e^j + t^{33}\nu^i\nu^j, \end{aligned} \tag{2.12}$$

(see appendix for notations).

**Lemma 1** *Let* $\boldsymbol{y}$ *be the surface based coordinates introduced as in (2.4). Then*

$$\begin{aligned} \mathcal{F}^i_{-\mathrm{p}}(D;(u,w)) = & -\iint_D \int_u^w \partial_a\left(p g^{ab}\zeta_b^i \vartheta\right) \mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 + \\ & -\iint_D \nu^i\, p\vartheta|_{y^3=w}\,\mathrm{d}y^1\mathrm{d}y^2 + \iint_D \nu^i\, p\vartheta|_{y^3=u}\,\mathrm{d}y^1\mathrm{d}y^2 \\ = & -\iint_D \int_u^w \left(\partial_a p g^{ab}\zeta_b^i + p\nu^i g^{ab}\Gamma^3_{ba}\vartheta\right)\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 \\ & -\iint_D \nu^i\, p\vartheta|_{y^3=w}\,\mathrm{d}y^1\mathrm{d}y^2 + \iint_D \nu^i\, p\vartheta|_{y^3=u}\,\mathrm{d}y^1\mathrm{d}y^2 \\ = & -\iint_D \int_u^w \left(\partial_a p g^{ab}\zeta_b^i + \nu^i\partial_3 p\right)\vartheta \mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathcal{F}^i_{\mathrm{visc}}(D;(u,w)) = & \iint_D \tau_c^i \left(\frac{\partial}{\partial y^a}\int_u^w q_b^c\sigma^{ba}\vartheta(y)\mathrm{d}y^3 + \gamma_{ae}^c\int_u^w q_b^e\sigma^{ba}\vartheta(y)\mathrm{d}y^3 - \kappa_a^c\int_s^w \sigma^{a3}\vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 + \\ & + \iint_D \nu^i\left(\kappa_{ca}\int_u^w q_b^c\sigma^{ba}\vartheta(y)\mathrm{d}y^3 + \frac{\partial}{\partial y^a}\int_u^w q_b^c\sigma^{3a}\vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 + \\ & + \iint_D \tau_c^i q_a^c\left(\sigma^{\parallel a}\varsigma(w) - \sigma^{\perp}g^{ab}\partial_b w\right) + \nu^i\left(\partial_a w\ \sigma^{\parallel a}\varsigma(w) + \sigma^{\perp}\right)\Big|_{y^3=w}\vartheta(w)\mathrm{d}y^1\mathrm{d}y^2 \\ & - \iint_D \tau_c^i q_a^c\left(\sigma^{\parallel a}\varsigma(u) - \sigma^{\perp}g^{ab}\partial_b u\right) + \nu^i\left(\partial_a u\ \sigma^{\parallel a}\varsigma(u) + \sigma^{\perp}\right)\Big|_{y^3=u}\vartheta(u)\mathrm{d}y^1\mathrm{d}y^2 \end{aligned} \tag{2.14}$$

$$\dot{\mathcal{H}}^i_{\boldsymbol{v}}(D;(u,w)) = \iint\limits_D \frac{\partial}{\partial t}\left(\tau^i_c \int\limits_u^w \rho q^c_b v^b \vartheta(y)\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 + \nu^i \int\limits_u^w \rho v^3 \vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 +$$
$$+ \iint\limits_D \tau^i_c \left(\frac{\partial}{\partial y^a}\int\limits_u^w \rho q^c_b v^b v^a \vartheta(y)\mathrm{d}y^3 + \gamma^c_{ae}\int\limits_u^w q^e_b v^b v^a \vartheta(y)\mathrm{d}y^3 - \kappa^c_a \int\limits_u^w \rho v^a v^3 \vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 +$$
$$+ \iint\limits_D \nu^i \left(\kappa_{ca}\int\limits_u^w \rho q^c_b v^b v^a \vartheta(y)\mathrm{d}y^3 + \frac{\partial}{\partial y^a}\int\limits_u^w \rho v^3 v^a \vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 +$$
$$+ \iint\limits_D \rho\tau^i_c q^c_a \left(v^{\|a}\varsigma(w) - v^{\perp} g^{ab}\partial_b w\right) + \nu^i\left(\partial_a w\ v^{\|a}\varsigma(w) + v^{\perp}\right)\Big|_{y^3=w}\vartheta(w)(v_n - u_n)\mathrm{d}y^1\mathrm{d}y^2$$
$$- \iint\limits_D \rho\tau^i_c q^c_a \left(v^{\|a}\varsigma(u) - v^{\perp} g^{ab}\partial_b u\right) + \nu^i\left(\partial_a u\ v^{\|a}\varsigma(u) + v^{\perp}\right)\Big|_{y^3=u}\vartheta(u)(v_n - u_n)\mathrm{d}y^1\mathrm{d}y^2 \tag{2.15}$$

$$\dot{\mathcal{H}}^i_m(D;(u,w)) = \iint\limits_D \frac{\partial}{\partial t}\int\limits_u^w \rho\vartheta(y)\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 + \iint\limits_D \frac{\partial}{\partial y^a}\int\limits_u^w \rho v^a \vartheta(y)\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 +$$
$$+ \iint\limits_D \rho\vartheta(\eta)(v_n - u_n)\big|_{y^3=w}\mathrm{d}y^1\mathrm{d}y^2 - \iint\limits_D \rho\vartheta(\eta)(v_n - u_n)\big|_{y^3=u}\mathrm{d}y^1\mathrm{d}y^2 \tag{2.16}$$

$$\mathcal{F}^i_{\mathrm{ext}}(D;(s,\eta)) = \iint\limits_D \left(\tau^i_c \int\limits_u^w \rho q^c_b f^b \vartheta(y)\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2 + \nu^i \int\limits_u^w \rho f^3 \vartheta(y)\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 \tag{2.17}$$

*where*

$$\varsigma(r) = \sqrt{1 + g^{ab}\frac{\partial r}{\partial y^a}\frac{\partial r}{\partial y^b}}$$

*Proof* See appendix.

Let us introduce the averaged quantities

$$
\begin{aligned}
&h\widetilde{h} = \int_s^\eta \Delta(y, y^3)\mathrm{d}y^3 \\
&h\widetilde{v}^{ca} = \int_s^\eta q_b^c v^b v^a \Delta(y, y^3)\mathrm{d}y^3,\ h\widetilde{v}^{3a} = \int_s^\eta v^3 v^a \Delta(y, y^3)\mathrm{d}y^3, \\
&h\widetilde{\tau}^{ca} = \int_s^\eta q_b^c \tau^{ba} \Delta(y, y^3)\mathrm{d}y^3,\ h\widetilde{\tau}^{3a} = \int_s^\eta \tau^{3a} \Delta(y, y^3)\mathrm{d}y^3, \\
&h\widetilde{v}^{c} = \int_s^\eta q_b^c v^b \Delta(y, y^3)\mathrm{d}y^3, h\widetilde{v}^{3} = \int_s^\eta v^3 \Delta(y, y^3)\mathrm{d}y^3, \\
&h\overline{v}^{c} = \int_s^\eta v^c \Delta(y, y^3)\mathrm{d}y^3, \\
&h\widetilde{p}^{ca} = \int_s^\eta p q_b^c g^{ab} \Delta(y, y^3)\mathrm{d}y^3, \\
&h\widetilde{p}^{c} = \int_s^\eta \partial_a p q_b^c g^{ab} \Delta(y, y^3)\mathrm{d}y^3, \\
&h\widetilde{f}^{c} = \int_s^\eta q_b^c f^b \Delta(y, y^3)\mathrm{d}y^3,\ h\widetilde{f}^{3} = \int_s^\eta f^3 \Delta(y, y^3)\mathrm{d}y^3,
\end{aligned}
\tag{2.18}
$$

The governing equations of the averaged fields result from lemma 1 by taking into account that the domain $D$ is an arbitrary domain in $\widetilde{D}$.

**Proposition 1 (Depth Averaged form of mass balance equation)** *Assume that all integrands in the integrals appearing in the Lemma 1 are continuous functions. Then*

$$
\frac{\partial}{\partial t} h\widetilde{h} + \frac{1}{\beta}\frac{\partial}{\partial y^a}\beta h\overline{v^a} = \Delta(y,\eta)\ (v_n - u_n)|_{\mathcal{U}} - \Delta(y,s)\ (v_n - u_n)|_{\mathcal{S}} \tag{2.19}
$$

By using the linear independence of the vectors $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\nu})$, in the whole domain flow the equality (2.2) can be written component-wise. The components in the tangent plan read as

**Proposition 2 (Depth Averaged form of momentum equation)** *The projection in the tangent plan of the reference surface of the mediate momentum equations are given by*

$$
\begin{aligned}
&\frac{\partial}{\partial t} h\widetilde{v}^c + \frac{1}{\beta}\frac{\partial}{\partial y^a}\beta\, h\left(\widetilde{v}^{ca} - \widetilde{\sigma}^{ca}\right) + h\gamma_{ab}^c\left(\widetilde{v}^{ab} - \widetilde{\sigma}^{ab}\right) - h\kappa_a^c(\widetilde{v}^{a3} - \widetilde{\sigma}^{a3}) + h\widetilde{p}^c = \\
&h\widetilde{f}^c + \mathcal{E}^c(\eta)\Delta(y)|_{y^3=\eta} - \mathcal{E}(s)^c\Delta(y)|_{y^3=s},\ c = \overline{1,2}.
\end{aligned}
\tag{2.20}
$$

*where*

$$
\mathcal{E}^c(\eta) = q_a^c\left[\left(\sigma^{\parallel a}\varsigma(\eta) - \sigma^{\perp} g^{ab}\partial_b\eta\right) - \left(v^{\parallel a}\varsigma(\eta) - v^{\perp} g^{ab}\partial_b\eta\right)(v_n - u_n)\right]
$$

*and*

$$
\mathcal{E}^c(s) = q_a^c\left[\left(\sigma^{\parallel a}\varsigma(s) - t^{\perp} g^{ab}\partial_b s\right) - \left(v^{\parallel a}\varsigma(s) - v^{\perp} g^{ab}\partial_b s\right)(v_n - u_n)\right]
$$

We follow an idea of [10] to use the projection in the normal direction to obtain the pressure field. On write again the integral form but this time we chose a volume $V$ given by

$$\{(y^1,y^2,y^3)|(y^1,y^2)\in D, \xi<y^3<\chi\}$$

where $\xi,\chi$ is such that $s(y^1,y^2,t)<\xi<\chi<\eta(y^1,y^2,t),(y^1,y^2)\in D$. By writing the integral form as

$$\dot{\mathcal{H}}^i(D;(\xi,\chi))=\mathcal{F}^i_{\text{stress}}(D;(\xi,\chi))+\mathcal{F}^i_{\text{ext}}(D;(\xi,\chi)) \tag{2.21}$$

one can obtain the normal projection that read as

$$\begin{aligned}&\frac{\partial}{\partial t}\int\limits_\xi^\chi v^3\vartheta(y)\mathrm{d}y^3+\kappa_{ca}\int\limits_\xi^\chi q^c_b v^b v^a\vartheta(y)\mathrm{d}y^3+\frac{\partial}{\partial y^a}\int\limits_\xi^\chi\left(v^3v^a-\sigma^{3a}\right)\vartheta(y)\mathrm{d}y^3+\\&+(v^3v^3-\sigma^{33})\vartheta(y,\chi)-(v^3v^3-\sigma^{33})\vartheta(y,\xi)+\int\limits_\xi^\chi\partial_3p\vartheta(y)\mathrm{d}y^3=\int\limits_\xi^\chi\vartheta(y)f^3\mathrm{d}y^3\end{aligned} \tag{2.22}$$

The equations (2.19), (2.20) and (2.22) result from the general balance equations and are general true. In the next section we obtain simplified variants of it by using several constitutive hypothesis concerning the fluid and its interaction with the external media and by performing an asymptotic analysis with respect to a small parameter.

## 2.2 Hydrostatic approximation

To perform of an asymptotic analysis of the equation (**??**) one assume that the new coordinate system have the dimension of the length. One introduce a characteristic length $L$ of the domain $\widetilde{D}$, and a characteristic velocity $V_0$.

**Proposition 3 (Pressure field)** *Let the external force be the gravitational force*

$$\boldsymbol{f}=-g_{\text{gravit}}\boldsymbol{i}_3.$$

*On assume that*

$$\begin{aligned}&||\boldsymbol{t}_a||=\mathcal{O}(1),\frac{\eta}{L}=\mathcal{O}(\epsilon),\frac{v^3}{V_0}=\mathcal{O}(\epsilon),\frac{v^a}{V_0}=\mathcal{O}(1),\\&\frac{\tau^{a3}}{\rho V_0^2}=\mathcal{O}(\epsilon),\frac{\tau^{33}}{\rho V_0^2}=\mathcal{O}(\epsilon),\end{aligned} \tag{2.23}$$

*Then:*
(a) *up to $\mathcal{O}(\epsilon)$ the pressure field and velocity field satisfy*

$$\kappa_{ba}v^bv^a+\partial_3p=f^3+\mathcal{O}(\epsilon), \tag{2.24}$$

(b) *if the curvatures of the support surface is also a small quantities $\kappa=\mathcal{O}(\epsilon)$ one has*

$$p(y,\xi)=-f^3(\eta-\xi)+\mathcal{O}(\epsilon^2). \tag{2.25}$$

*Proof* By using the equation (2.22) and the assumptions (2.23) one obtains

$$\kappa_{ca} \int_{\xi}^{\chi} q_b^c v^b v^a \Delta(y) \mathrm{d}y^3 + \int_{\xi}^{\chi} \partial_3 p \Delta(y, \xi) \mathrm{d}y^3 = \int_{\xi}^{\chi} \Delta(y, y^3) f^3 \mathrm{d}y^3 + \mathcal{O}(\epsilon) \tag{2.26}$$

and then by taken into account that the $\chi$ and $\xi$ are arbitrary numbers results (2.24).

The most important consequence of the assumptions (2.23) and the assumption regarding the curvature of the support surface is that one can decouple the calculation of the velocity field from calculation of the pressure field. The relation (2.25) provide the pressure field distribution. Similarly to the case of flat surface the pressure field is linear distributed along the fluid depth.

Let us now analyze the consequences of the hydrostatic approximation of the pressure by (2.25) on the averaged balance momentum (2.20).

One introduce the vector

$$\widetilde{\mathfrak{p}}^c := h\widetilde{p}^c - h\widetilde{f}^c = \int_s^{\eta} \left(\partial_a p g^{ab} - f^b\right) q_a^c \Delta(y, y^3) \mathrm{d}y^3$$

**Lemma 2** *For a pressure field given by (2.25) and for the force field $\boldsymbol{f}$ given by gravitational force one has*

$$\widetilde{\mathfrak{p}}^a = h g_{\text{gravit}} \beta^{ac} \partial_c \left(x^3(y^1, y^2) + \eta(y^1, y^2)\nu^3(y^1, y^2)\right) + h\mathcal{O}(\epsilon\kappa) \tag{2.27}$$

*Proof* To prove the formula (2.27) one needs to express the component

$$f^3 = -g_{\text{gravit}}\nu^3,\ f^a = -g_{\text{gravit}} g^{ab} e_b^3; e_b^3 := \partial_b(x^3 + y^3\nu^3)$$

of the gravitational force and to introduce the metric tensor $g$ by

$$g^{ab} = Q_e^a Q_f^b \beta^{ef}, Q_c^a q_b^c = \delta_b^a.$$

One obtains

$$\widetilde{\mathfrak{p}}^a = g_{\text{gravit}} \partial_a (x^3 + \eta\nu^3) \int_s^{\eta} g^{ab} q_b^c \Delta \mathrm{d}y^3$$

and then one perform the integrals.

**Remarks** *One note that*

$$x^3(y^1, y^2) + \eta(y^1, y^2)\nu^3(y^1, y^2)$$

*is the $x^3$ components of the water surface in the surface base coordinate system.*

By introducing the potential force

$$w = g_{\text{gravit}} \left(x^3(y^1, y^2) + \eta(y^1, y^2)\nu^3(y^1, y^2)\right)$$

we can write the integral form of momentum balance equation up to $\mathcal{O}(\epsilon\kappa)$ as follows

$$\begin{aligned} &\frac{\partial}{\partial t} h\widetilde{v}^c + \frac{1}{\beta}\frac{\partial}{\partial y^a} \beta\, h\left(\widetilde{v}^{ca} - \widetilde{\sigma}^{ca}\right) + h\gamma_{ab}^c \left(\widetilde{v}^{ab} - \widetilde{\sigma}^{ab}\right) + h\beta^{ca}\partial_a w = \\ &\mathcal{E}^c(\eta)\Delta(y)\big|_{y^3=\eta} - \mathcal{E}(s)^c \Delta(y)\big|_{y^3=s} + h\mathcal{O}(\epsilon\kappa),\ c = \overline{1,2}. \end{aligned} \tag{2.28}$$

## 2.3 Closure relations

In order to be solved, the equations obtained in the hydrostatic approximation require some closure relations concerning the viscosity, frictional force and mixed averaged quantities concerning quadratic velocity terms. We analyze here a model that can be relevant in the case of water moving on hill slope with small curvature and thin water film.

### 2.3.1 Saint Venant's equations with curvature

The models in this class are simplified variant of the equations (2.19) and (2.28).

**Proposition 4 (Saint Venant)** *In addition to the assumptions (2.23) of the proposition (3) one consider that*

$$\begin{aligned} &\kappa_{ab} = \mathcal{O}(\epsilon) \\ &v^a(y^1, y^2, y^3, t) = \overline{v}^a(y^1, y^2, t) + \mathcal{O}(\epsilon) \\ &\sigma^{ab} = \mathcal{O}(\epsilon). \end{aligned} \tag{2.29}$$

*Assume that the friction vector obey the Darcy-Weisbach law*

$$t^a_{(s)} = f|v|v^a \tag{2.30}$$

*Then the model equations of water flow on an unvegetated hillslope is given by*

$$\begin{aligned} \frac{\partial}{\partial t}\beta h + \frac{\partial}{\partial y^a}\beta h v^a = &\quad \beta(\mathfrak{m}_r - \mathfrak{m}_i) \\ \frac{\partial}{\partial t}h\beta v^c + \frac{\partial}{\partial y^a}\beta\, h v^c v^a + h\beta\gamma^c_{ab}v^a v^b + h\beta\beta^{ca}\partial_a w = &\quad -\beta f|v|v^c. \end{aligned} \tag{2.31}$$

**Proposition 5** *The system (2.31) has the properties:*
*(a) it preserve the steady state of a lake*

$$x^3 + h\nu^3 = \text{constant}$$

*(b) There exists a conservative equation of the energy*

$$\frac{\partial}{\partial t}h\beta\mathcal{E} + \frac{\partial}{\partial y^a}h\beta v^a\left(\mathcal{E} + g_{\text{gravit}}\frac{h}{2}\nu^3\right) = \beta\mathfrak{M}(-\frac{1}{2}|v|^2 + w) - f\beta|v|^3 \tag{2.32}$$

*where*

$$\mathcal{E} := \frac{1}{2}|v|^2 + g_{\text{gravit}}(x^3 + \frac{h}{2}\nu^3), \mathfrak{M} = \mathfrak{m}_r - \mathfrak{m}_i$$

*(c) Bernoulli law. In a steady state in the absence of the mass source and without friction force the total energy, i.e*

$$\mathcal{E}^{\text{t}} = \frac{1}{2}|v|^2 + g_{\text{gravit}}x^3 + p(y, h)$$

*is constant along of a current line*

$$v^a\partial_a\mathcal{E}^{\text{t}} = 0. \tag{2.33}$$

*Prof.* The affirmation (a) is evident. To prove (b) one rewrite momentum balance equation as

$$\frac{\partial}{\partial t}v^c + v^a\frac{\partial}{\partial y^a}v^c + \gamma^c_{ab}v^av^b + \beta^{ca}\partial_a w = -\frac{v^c}{h}\left(\mathfrak{M} + f|v|\right) \tag{2.34}$$

Then one multiplies the (2.34) by $v_c$ and using that

$$v_cv^a\partial_a v^c + \gamma^c_{ab}v^av^bv_c = \frac{1}{2}v^a\partial_a v_bv^b$$

one can write

$$\frac{1}{2}\frac{\partial}{\partial t}v_cv^c + v^a\frac{\partial}{\partial y^a}\left(\frac{1}{2}v_bv^b + w\right) = -\frac{v_cv^c}{h}\left(\mathfrak{M} + f|v|\right)$$

Then one multiplies by $h\beta$ and one obtains

$$\frac{1}{2}\frac{\partial}{\partial t}h\beta|v|^2 + w\frac{\partial}{\partial t}h\beta + \frac{\partial}{\partial y^a}h\beta v^a\left(\frac{1}{2}|v|^2 + w\right) = \beta\left(\left(-\frac{1}{2}|v|^2 + w\right)\mathfrak{M} - f|v|^3\right)$$

that is the relation (b) The affirmation (c) results from the observation

$$w = g_{\text{gravit}}x^3 + p(y,h)$$

# Chapter 3

# Fluid Flow on Vegetated Hillslope

The presence of plants on the hill creates a resistance force to the water flow and influences the process of water accumulation on the soil surface. The large diversity of plants growing on a hill makes the elaboration of an unitary model of the water flow over a soil covered by vegetation very difficult. Here, we present a model based on water mass and momentum balance equations that takes into account the presence of certain type of plants.

More precisely, the plants form a dense net of rigid vertical tubes and the water fills the "voided" space up to a level not higher than these plant tubes, see Figure 3.1.

## 3.1 Space Averaging Models

Space averaging is a method to define a unique continuous model associated to a heterogeneous fluid-solid mechanical system. The method is largely used in porous soil media models [4], [5]. For the fluid-plants physical system, the porous analogy was also used in [6], [7] especially in the case of submerged vegetation.

At a hydrographic basin scale, there are variations in the geometrical properties of the terrain (curvature, orientation, slope) and vegetation density or vegetation type etc. Assume there is a map that models the terrain surface

$$x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2, \quad i = 1, 2, 3. \tag{3.1}$$

Denote the tangent vectors to the coordinate curves on this surface by

$$\varsigma_a = \partial_a \boldsymbol{b} := \frac{\partial \boldsymbol{b}}{\partial \xi^a}, \quad a = 1, 2. \tag{3.2}$$

Using this fixed surface, one introduces a new coordinate $y^3$ along the normal direction $\boldsymbol{\nu}$ to the surface. A point in the neighborhood of this surface is defined

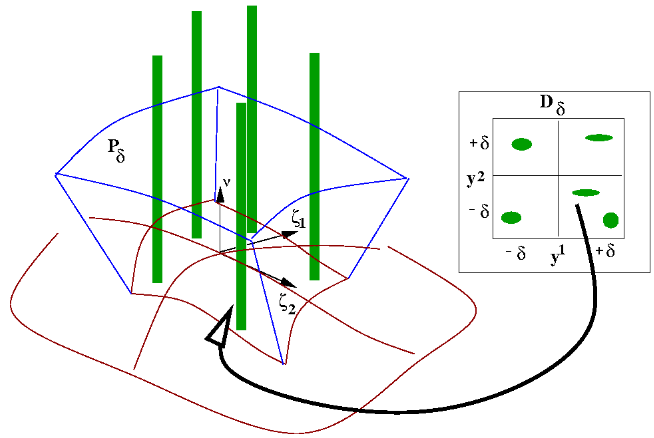


Figure 3.1: The representative element of the volume $P_\delta$ used for mediation. The bottom surface of $P_\delta$ has a representative width $\delta$ along two orthogonal directions on this surface. The water depth $h$ associated to $P_\delta$ is the averaged value of the physical water depth $\widetilde{h}$ inside $P_\delta$.

in this new system of coordinates $Y = (\xi^1, \xi^2, y^3)$ by

$$x^i = b^i(\xi^1, \xi^2) + y^3\nu^i, \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2, \quad y^3 \in J \in \mathbb{R}, \quad i = 1, 2, 3, \tag{3.3}$$

where $\boldsymbol{\nu} = (\nu^1, \nu^2, \nu^3)$ represents the unit normal to the surface.

We introduce the tangent vectors to the coordinate curves defined by $Y$

$$\boldsymbol{\zeta}_I := \partial_I \boldsymbol{x}, \quad I = 1, 2, 3. \tag{3.4}$$

One has

$$\boldsymbol{\zeta}_3 = \boldsymbol{\nu}, \quad \boldsymbol{\zeta}_a = (\delta_a^b - y^3\kappa_a^b)\boldsymbol{\varsigma}_b, \quad a = 1, 2, \tag{3.5}$$

where $\boldsymbol{\kappa}$ is the curvature tensor of the terrain surface.

In the presence of vegetation on the hill slope, the fluid occupies the free space between plant bodies and the mechanical characteristics of the fluid flow are defined only in the domain occupied by the fluid.

We adopt the following
**General convention:** *any variable bearing a tilde over it designates a micro-local physical quantity, while the absence of tilde indicates the corresponding averaged quantity. When the micro-local quantity does not differ from the corresponding averaged quantity we denote the micro-local quantity without tilde.*

Denote by $\Omega_f$ and $\Omega_p$ the spatial domain occupied by fluid and plants, respectively. Consider $\widetilde{\psi}$ to be some microscopic quantity that refers to the fluid. Let $\boldsymbol{y} = (y^1, y^2)$ be a point in $D$. One introduces the rectangular domain

$$D_\delta = D_\delta(\boldsymbol{y}) := [y^1 - \delta, y^1 + \delta] \times [y^2 - \delta, y^2 + \delta]. \tag{3.6}$$

Define the spatial averaging volume

$$P = P(\boldsymbol{y}) = \big\{(x^1, x^2, x^3) \,|\, x^i = b^i(\xi^1, \xi^2) + y^3\nu^i, \\ 0 < y^3 < \bar{h}(\xi^1, \xi^2),\ (\xi^1, \xi^2) \in D_\delta(\boldsymbol{y}),\ i = 1, 2, 3\big\}.$$

Here, $\bar{h}$ is some extension of $\widetilde{h}$ to the domain $D$, where $\widetilde{h}$ is the function describing the free water surface outside the domain occupied by plants.

Denote by $P^f$ the fluid domain inside $P$,

$$P^f := P \cap \Omega^f.$$

The boundary of $P^f$ can be partitioned as

$$\partial P^f = \Sigma^{fp} \cap \Sigma^{ff} \cap \Sigma^{fa} \cap \Sigma^{fs},$$

where $\Sigma^{fp}$ is the fluid-plant contact surface inside $P^f$, $\Sigma^{fa}$ is the free surface of the fluid inside $P^f$, $\Sigma^{fs}$ is the fluid-soil contact surface inside $P^f$, and $\Sigma^{ff}$ is the boundary surface separating the fluid inside and outside $P^f$.

The general form of a balance equation is [**?** ]

$$\partial_t \int\limits_{P^f} \widetilde{\rho}\,\widetilde{\psi}\mathrm{d}V + \int\limits_{\partial P^f} \widetilde{\rho}\,\widetilde{\psi}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\partial P^f} \widetilde{\boldsymbol{\Phi}}_\psi \cdot \boldsymbol{n}\mathrm{d}\sigma + \int\limits_{P^f} \widetilde{\rho}\,\widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V. \tag{3.7}$$

Here, the significance of the above quantities are:
- $\widetilde{\rho}$ – the micro-local mass density of the fluid;
- $\widetilde{\boldsymbol{v}}$ – the micro-local velocity of the fluid;
- $\boldsymbol{n}$ – the exterior unit normal on $\partial P^f$;
- $\widetilde{\boldsymbol{\Phi}}_\psi$ – the micro-local flux density of $\widetilde{\psi}$;
- $\widetilde{\boldsymbol{\phi}}_\psi$ – the micro-local mass density of supply $\widetilde{\psi}$;
- $u_n$ – the normal surface velocity;
- $\mathrm{d}V$ – the volume element;
- $\mathrm{d}\sigma$ – the surface element.

To obtain a mathematical treatable model, one needs to make some assumptions concerning the complex fluid-plant-soil system. The first assumption refers to the plant cover.

**Asummption 1 (Vegetation structure)** *The plant cover satisfies:*
A1. *The plants are almost normal to the terrain surface and they behave like rigid sticks.*
A2. *The water depth is smaller than the height of the plants.*

Assumption A1 is often used in the porous model of the vegetation and assumption A2 is proper to the overland flow.

The soil-fluid $\mathcal{I}_{fs}$ and fluid-air $\mathcal{I}_{fa}$ interfaces can be represented as

$$\mathcal{I}_{fs} := \{\boldsymbol{x} \,|x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D^f,\ i = 1, 2, 3\}$$

and

$$\mathcal{I}_{fa} := \{\boldsymbol{x}\left| x^i = b^i(\xi^1,\xi^2) + \widetilde{h}(\xi^1,\xi^2)\delta_3^i, \quad (\xi^1,\xi^2) \in D^f,\ i = 1,2,3\right.\},$$

respectively, where $D^f := \left\{(\xi^1,\xi^2) \in D \left| \boldsymbol{b}(\xi^1,\xi^2) \in \Omega_f\right.\right\}$.

Define the averaged water depth by

$$h(y^1,y^2,t) := \frac{1}{\omega_f}\int\limits_{D_\delta^f} \widetilde{h}(\xi^1,\xi^2,t)\beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{3.8}$$

where $\omega_f$ measures the area of $\Sigma^{fs}$,

$$\omega_f := \int\limits_{D_\delta^f} \beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2. \tag{3.9}$$

The volume of the fluid inside the elementary domain $P$ is given by

$$\mathrm{vol}(P^f) = \omega_f h. \tag{3.10}$$

A pure geometrical result which refers to the flux of $\widetilde{\psi}$ through the boundary $\Sigma^{ff}$ is formulated as:

**Lemma 3**

$$\int\limits_{\Sigma^{ff}} \widetilde{\rho}\widetilde{\psi}\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma = \partial_a \int\limits_{D^f}\int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\widetilde{\psi}\widetilde{v}^a\Delta\mathrm{d}y^3\beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{3.11}$$

where $\Delta = 1 - y^3K_M + (y^3)^2K_G$, with $K_M$ and $K_G$ the mean and Gauss curvature respectively, and $\beta\mathrm{d}\xi\mathrm{d}\eta$ is the area element of the terrain surface. The quantities $\widetilde{v}^a$, with $a = 1,2$ stand for the contravariant components of the velocity fields in the local basis $\{\boldsymbol{\zeta}_I\}_{I=\overline{1,3}}$

$$\widetilde{\boldsymbol{v}} = \widetilde{v}^a\boldsymbol{\zeta}_a + \widetilde{v}^3\boldsymbol{\nu}.$$

In Lemma 3, the partial differentiation $\partial_a$ stands for

$$\partial_a := \frac{\partial}{y^a}.$$

### 3.1.1 Averaged mass balance equation

Although the water density is considered to be a constant function, we keep it in the mass balance formulation for emphasizing the physical meaning of the equations. Define the averaged water flux by

$$\rho v^a(\boldsymbol{x},t) := \frac{1}{\mathrm{vol}(P^f)}\int\limits_{D_\delta^f}\int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\,\widetilde{v}^a\Delta\mathrm{d}y^3\beta\mathrm{d}\xi^1\mathrm{d}\xi^2. \tag{3.12}$$

The mass balance equation results from (3.7) by taking $\widetilde{\psi} = 1$, $\widetilde{\boldsymbol{\Phi}}_\psi = 0$ and $\widetilde{\phi}_\psi = 0$. Since the plants are treated as solid bodies and the water does not penetrate the plant bodies, the water flux through the boundary of the elementary volume $P^f$ reduces to

$$\int\limits_{\partial P^f} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\Sigma^{ff}} \widetilde{\rho}\, \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma + \int\limits_{\Sigma^{fa}} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \widetilde{\rho}\, \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma.$$

The second integral in the r.h.s. of the above relation represents the water flux due to the rain which leads to the water mass gain inside $P^f$. The third term corresponds to the water flux due to the infiltration which contributes to the water loss inside $P^f$. Using Lemma 3 and the definition of the averaged quantities, one can write the mass balance:

$$\frac{\partial}{\partial t}(\omega_f h) + \partial_a(\omega_f h v^a) = \omega r - \omega_f i, \tag{3.13}$$

with

$$\int\limits_{\Sigma^{fa}} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = -\rho\omega r \quad \text{and} \quad \int\limits_{\Sigma^{fs}} \widetilde{\rho}\, \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma = \rho\omega_f i \tag{3.14}$$

representing the rain and the infiltration rates, respectively. Here, as in (3.9), $\omega$ is defined as

$$\omega := \int_{D_\delta} \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2.$$

### 3.1.2 Averaged Momentum Balance Equations

The momentum balance equation results from (3.7) with $\widetilde{\psi} = \widetilde{\boldsymbol{v}}$, $\widetilde{\boldsymbol{\Phi}}_\psi = \widetilde{\boldsymbol{T}}$, where $\widetilde{\boldsymbol{T}}$ is the stress tensor and $\widetilde{\phi}_\psi = \widetilde{\boldsymbol{f}}$, with $\widetilde{\boldsymbol{f}}$ denoting the body forces. Here, we only consider the gravitational force.

In contrast to the planar case, there are some difficulties in writing component-wise the space averaging balance momentum equations. These difficulties appear due to the point dependence of the local basis. In the euclidean basis of $X$, the momentum of the elementary volume $P^f$ is given by

$$\mathcal{H}^i(P^f) = \int\limits_{P^f} \widetilde{\rho}\, \widetilde{v}^i \mathrm{d}V.$$

Using the components of $\widetilde{\boldsymbol{v}}$ in the basis of $Y$ coordinates, we obtain

$$\mathcal{H}^i(P^f) = \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\, \zeta_a^i\, \widetilde{v}^a \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\, \nu^i\, \widetilde{v}^3 \Delta \mathrm{d}y^3 \mathrm{d}\sigma, \tag{3.15}$$

which can be rewritten as

$$\mathcal{H}^i(P^f) = \varsigma_a^i \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\, \widetilde{v}^a \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \nu^i \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\, \widetilde{v}^3 \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \mathcal{E}_1^i(\widetilde{\boldsymbol{v}}, P^f). \tag{3.16}$$

Here and in what follows, we make the following convention: $\boldsymbol{\varsigma}_a = \boldsymbol{\varsigma}_a(\boldsymbol{y})$, where $\boldsymbol{y} = (y^1, y^2)$ is the point defining the domain $D_\delta(\boldsymbol{y})$ from (3.6). When it appears inside the integral, the unit normal $\boldsymbol{\nu}$ is a variable quantity depending on the current point from the domain $D_\delta$, but when it appears outside the integral, it is the unit normal defined by the same $\boldsymbol{y}$ as $\boldsymbol{\varsigma}_a$.

The term

$$\mathcal{E}_1^i(\widetilde{\boldsymbol{v}}, P^f) := \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}(\zeta_a^i - \varsigma_a^i) \widetilde{v}^a \Delta \mathrm{d}y^3 \mathrm{d}\sigma$$

represents an error introduced by neglecting the variation of the basis $\boldsymbol{\zeta}_I$ along the domain $P^f$.

By averaging, from (3.16) one has

$$\mathcal{H}(P^f) = \rho h \omega_f v^a \boldsymbol{\varsigma}_a + \rho h \omega_f v^3 \boldsymbol{\nu} + \mathcal{E}_1(\widetilde{\boldsymbol{v}}, P^f). \tag{3.17}$$

If one neglects the momentum transfer on the fluid-air and fluid-soil interfaces, then the flux of the momentum through the boundary $\partial P^f$ can be reduced to

$$\mathcal{F}(\widetilde{\rho}\, \widetilde{\boldsymbol{v}}, \partial P^f) := \int\limits_{\partial P^f} \widetilde{\rho}\, \widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n) \mathrm{d}\sigma = \int\limits_{\Sigma^{ff}} \widetilde{\rho}\, \widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}) \mathrm{d}\sigma.$$

Using Lemma 3, one has

$$\mathcal{F}(\widetilde{\rho}\, \widetilde{\boldsymbol{v}}, \partial P^f) = \partial_a \int\limits_{D^f} \int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\, \widetilde{\boldsymbol{v}}\, \widetilde{v}^a \Delta \mathrm{d}y^3 \beta(\xi^1, \xi^2) \mathrm{d}\xi^1 \mathrm{d}\xi^2,$$

and then,

$$\begin{aligned} \mathcal{F}(\widetilde{\rho}\, \widetilde{\boldsymbol{v}}, \partial P^f) = & \\ & \partial_a(\rho \omega_f h v^b v^a \boldsymbol{\varsigma}_b) + \partial_a(\rho \omega_f h w^{ba} \boldsymbol{\varsigma}_b) + \partial_a(\rho \omega_f h v^3 v^a \boldsymbol{\nu}) + \\ & \mathcal{E}_2(\widetilde{v}^2, P^f), \end{aligned} \tag{3.18}$$

where the fluctuation

$$\rho w^{ab} := \frac{1}{\omega_f h} \int_{\Sigma^f} \int_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}(\widetilde{v}^b - v^b) \widetilde{v}^a y^3 \beta(\xi^1, \xi^2) \mathrm{d}\xi^1 \mathrm{d}\xi^2.$$

The quantity $\mathcal{E}_2(\widetilde{v}^2, P^f)$ (as $\mathcal{E}_1(\widetilde{v}, P^f)$ appearing above), represents the error introduced by approximating the variable local basis $(\boldsymbol{\zeta}_1(\xi^1, \xi^2, y^3)$, $\boldsymbol{\zeta}_2(\xi^1, \xi^2, y^3)$,

$\boldsymbol{\nu}(\xi^1, \xi^2, 0))$ with the fixed local basis $(\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2, \boldsymbol{\nu})$ at $(y^1, y^2, 0)$. The quantities $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ introduced in what follows are errors of the same nature.

Rel. (3.18) can be rewritten as

$$\begin{aligned}
&\mathcal{F}(\widetilde{\rho}\,\widetilde{\boldsymbol{v}}, \partial P^f) = \\
=&\partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f h v^b v^a \partial_a \boldsymbol{\varsigma}_b + \partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \rho\omega_f h w^{ba}\partial_a \boldsymbol{\varsigma}_b + \\
&\partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \rho\omega_f h v^3 v^a \partial_a \boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f) \\
=&\partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f(h v^b v^a + w^{ba})(\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \kappa_{ab}\boldsymbol{\nu}) + \\
&\partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} - \rho\omega_f h v^3 v^a \kappa^b_a \boldsymbol{\varsigma}_b + \mathcal{E}_2(\widetilde{v}^2, P^f) \\
=&\partial_a(\rho\omega_f h(v^b v^a + w^{ba}))\boldsymbol{\varsigma}_b - \rho\omega_f h v^3 v^a \kappa^b_a \boldsymbol{\varsigma}_b + \rho\omega_f(h v^b v^a + w^{ba})\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \\
&\rho\omega_f(h v^b v^a + w^{ba})\kappa_{ab}\boldsymbol{\nu} + \partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f),
\end{aligned} \tag{3.19}$$

where $\gamma^c_{ab}$ are the Christoffel symbols.

To express the contribution of the stress forces to the momentum balance we decompose the stress tensor field $\widetilde{\boldsymbol{T}}$ in two components: the pressure field $\widetilde{p}$ and the viscous part of the stress tensor field $\widetilde{\boldsymbol{\tau}}$

$$\widetilde{\boldsymbol{T}} = -\widetilde{p}\boldsymbol{I} + \widetilde{\boldsymbol{\tau}}.$$

The flux of the stress vector can now be written as

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \partial P_f) = \mathcal{F}(-p\boldsymbol{I}, \partial P_f) + \mathcal{F}(\widetilde{\boldsymbol{\tau}}, \partial P_f).$$

An elementary calculation show that

$$\mathcal{F}(-p\boldsymbol{I}, \partial P_f) = -\int_{D^f} \int_0^{\widetilde{h}(\xi^1,\xi^2,t)} \left(\partial_a p g^{ab}\boldsymbol{\zeta}_b + \partial_3 p\boldsymbol{\nu}\right) \Delta \mathrm{d}y^3 \beta \mathrm{d}\xi^1 \mathrm{d}\xi^2 \tag{3.20}$$

The pressure field is determined up to a constant value. If we subtract the atmospheric pressure from the water pressure, on the interface fluid-air the pressure must be zero. We assume the pressure field to be hydrostatically distributed.

Let $\boldsymbol{g} = -g\boldsymbol{i}_3$ be the gravitational force acting on the mass unit. In the local frame of coordinates related to the free surface of the fluid this force has the representation

$$\boldsymbol{g} = \widetilde{f}^a\boldsymbol{\zeta}_a - \widetilde{f}^3\boldsymbol{\nu}.$$

**Asummption 2 (Hydrostatic approximation)** *One assume that,*
A3. *The hydrostatic pressure field has the form,*

$$\widetilde{p}(\xi^1, \xi^2, y^3) = \widetilde{\rho}\,\widetilde{f}^3(\widetilde{h}(\xi^1, \xi^2) - y^3).$$

We neglect the shear forces on the fluid-air interface, i.e.

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fa}) = 0.$$

On the fluid-soil interface the stress vector $\widetilde{\boldsymbol{t}} := \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n}$ can be written as

$$\widetilde{\boldsymbol{t}} = \widetilde{t}^a \boldsymbol{\zeta}_a + \widetilde{t}^3 \boldsymbol{\nu}.$$

On the interface soil-water we can write

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fs}) = \varsigma_a \int\limits_{\Sigma^{fs}} \widetilde{t}^a \mathrm{d}\sigma + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{3.21}$$

Introducing the shear force at the fluid-soil interface

$$\sigma_s^a = \frac{1}{\rho \omega_f} \int\limits_{\Sigma^{fs}} \widetilde{t}^a \mathrm{d}\sigma,$$

(3.21) takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fs}) = \varsigma_a \rho \omega_f \sigma_s^a + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{3.22}$$

On the fluid-plant interface

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \int\limits_{\Sigma^{fp}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma = \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma, \tag{3.23}$$

where $\Sigma_l^{fp}$ is the fluid-plant surface corresponding to the plant $l$. Obviously, $\bigcup\limits_l \Sigma_l^{fp} = \Sigma^{fp}$. Since the plant stems are supposed to be perpendicular to the ground surface, (3.23) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \varsigma_a \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{t}^a \mathrm{d}\sigma + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) \tag{3.24}$$

and introducing the plant resistance force

$$\sigma_p^a = \frac{1}{\rho \omega} \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{t}^a \mathrm{d}\sigma,$$

(3.24) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \varsigma_a \rho \omega \sigma_p^a + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}). \tag{3.25}$$

On the fluid interface of $P^f$ invoking again Lemma 3, the contribution of the viscous part of the stress tensor on the interface fluid-fluid takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\tau}^{ba} \boldsymbol{\zeta}_b \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \partial_a \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\tau}^{3a} \boldsymbol{\nu} \Delta \mathrm{d}y^3 \mathrm{d}\sigma.$$

Then, we write the above quantity as,

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a(\omega_f h \tau^{ba} \boldsymbol{\varsigma}_b) + \partial_a(\omega_f h \tau^{3a} \boldsymbol{\nu}) + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f). \tag{3.26}$$

Rel. (3.26) implies,

$$\begin{aligned} &\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \\ =&\partial_a(\omega_f h \tau^{ba})\boldsymbol{\varsigma}_b + \omega_f h \tau^{ba} \partial_a \boldsymbol{\varsigma}_b + \partial_a(\omega_f h \tau^{3a})\boldsymbol{\nu} + \omega_f h \tau^{3a} \partial_a \boldsymbol{\nu} + \\ &\mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f) \\ =&\partial_a(\omega_f h \tau^{ba})\boldsymbol{\varsigma}_b + \omega_f h \tau^{ba}(\gamma^c_{ab}\boldsymbol{\varsigma}_c + \kappa_{ab}\boldsymbol{\nu}) + \partial_a(\omega_f h \tau^{3a})\boldsymbol{\nu} - \\ &\omega_f h \tau^{3a} \kappa^b_a \boldsymbol{\varsigma}_b + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f) = \\ =&\partial_a(\omega_f h \tau^{ba})\boldsymbol{\varsigma}_b - \omega_f h \tau^{3a} \kappa^b_a \boldsymbol{\varsigma}_b + \omega_f h \tau^{ba} \gamma^c_{ab} \boldsymbol{\varsigma}_c + \omega_f h \tau^{ba} \kappa_{ab} \boldsymbol{\nu} + \\ &\partial_a(\omega_f h \tau^{3a})\boldsymbol{\nu} + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f). \end{aligned} \tag{3.27}$$

For the supply $\widetilde{\Phi}_\psi$ we only consider the contribution of the gravitational force. Proceeding by components as in (3.16), the second term in the r.h.s. of (3.7) is finally expressed as

$$\int\limits_{P^f} \widetilde{\rho}\widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V = \int\limits_{D^f} \int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \left(\widetilde{f}^a \boldsymbol{\zeta}_a - \widetilde{f}^3 \boldsymbol{\nu}\right) \Delta \mathrm{d}y^3 \beta \mathrm{d}\xi^1 \mathrm{d}\xi^2 \tag{3.28}$$

The relations (3.17, 3.19, 3.20, 3.22, 3.25, 3.27) and some order assumptions are the basis for averaged momentum equations.

The porosity of the plant cover $\theta$ and is defined by

$$\theta = \frac{\omega_f}{\omega}.$$

Let $\beta_0 = \beta(y_1, y_2)$, where $\boldsymbol{y} = (y^1, y^2)$ is the point defining the domain $D_\delta(\boldsymbol{y})$ from (3.6).

Let $\epsilon$ be a small parameter.

**Asummption 3 (Kinematical and topographical assumptions)** *Suppose that the physical processes satisfy the following properties:*
A4. The water depth. $\widetilde{h} = O(\epsilon)$.
A5. The velocity. $v^3 = O(\epsilon)$.
A6. Geometric assumptions:
A6.1. Curvature. *The terrain surface curvatures and the curvature of the coordinate curves are of order of $\epsilon$. This means that locally the surface is almost planar.*
A6.2. Metric tensor. $\beta = \beta_0 + O(\epsilon)$.
A7. The averaged dimension $\delta$. $d_p \ll \delta \ll L$ *and* $\delta K_M = O(\epsilon)$.

In what follows, by abuse of notations, we denote $\beta_0$ by $\beta$.

The shallow water type approximation of the averaged momentum balance for an incompressible fluid results by an asymptotic analysis.

**Theorem 1 (Averaged momentum equations)** *Under assumptions* A1–A7, *the first order approximation for the momentum equations are given by*

$$\partial_t(h\beta\theta v^a) + \partial_b \mathfrak{F}^{ab}(h,v) + h\beta\theta\beta^{ab}\partial_a w = \mathfrak{G}^a(h,v), \quad a = 1,2, \tag{3.29}$$

*where*

$$w = g_{\text{gravit}}(b^3 + h\nu^3),$$

$$\mathfrak{F}^{ab}(h,v) = h\beta\theta\left(v^a v^b + w^{ab} - \frac{1}{\rho}\tau^{ab}\right),$$

$$\mathfrak{G}^a(h,v) = \beta\sigma_p^a + \beta\theta\sigma_s^a - \gamma_{bc}^a\eta^{bc}$$

*and*

$$\eta^{ac} = h\beta\theta\left(v^a v^b + w^{ab} - \frac{1}{\rho}\tau^{ab}\right).$$

*Sketch of proof.* Using Assumption 3 and relations (3.17, 3.19, 3.22, 3.25, 3.27) one can prove that the terms $\mathcal{E}_1, \ldots, \mathcal{E}_5$ are of order $\epsilon^2$. For $\epsilon \ll 1$ these terms as well as the terms containing the factors $v^3h$, $h\kappa$ or $h^2$ (which are of same order $\epsilon^2$) can be neglected.

The equations (3.29) must be supplemented by empirical laws concerning the *averaged stress tensor* $\boldsymbol{\tau}$, the *averaged vegetation force resistance* $\boldsymbol{\sigma}_p$, the *averaged shear fluid-soil force* $\boldsymbol{\sigma}_s$ and the *averaged fluctuation* $w^{ab}$. These empirical laws are expressed by functions depending on the averaged velocity $\boldsymbol{v}$, the averaged water depth $h$ and a set of parameters $\boldsymbol{\lambda}$ defined by the characteristics of the plant cover.

$$\begin{cases} \tau^{ab} = \mathfrak{T}^{ab}(\nabla\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_p^b = \mathfrak{S}_p^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_s^b = \mathfrak{S}_s^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ w^{ab} = \mathfrak{W}^{ab}(\boldsymbol{v}, h, \boldsymbol{\lambda}). \end{cases} \tag{3.30}$$

## 3.2 Closure Relations

The averaged models of water flow on a vegetated hillslope consists in mass balance equation (3.13), momentum balance equations (3.29) and a set of empirical relations (3.30).

### 3.2.1 The averaged vegetation force resistance

The most used empirical relations that relates the vegetation resistance and fluid velocity has the form [7], [6]

$$\sigma_p^a = -\frac{1}{2} C_d m d |\boldsymbol{v}| v^a, \tag{3.31}$$

where $m$ is the number of stems on the surface $\omega$ and $d$ is the averaged diameters of the stems. The bed shear stress

$$\sigma_b^a = -\frac{g_{\text{gravit}}}{C_b^2} |\boldsymbol{v}| v^a, \tag{3.32}$$

$|\boldsymbol{v}|$ being the magnitude of the averaged velocity *i.e.*

$$|\boldsymbol{v}|^2 = \beta_{ab} v^a v^b.$$

One assumes that the viscosity of fluid and the fluctuation of the velocity field have a small effect as compared with the bed friction and plant resistance. Therefore the base model is given by

$$\begin{aligned} \frac{\partial}{\partial t}\left(h\beta\theta\right) + \partial_a\left(h\beta\theta v^a\right) &= \beta(\mathfrak{m}_r - \theta\mathfrak{m}_i), \\ \frac{\partial}{\partial t} h\theta\beta v^c + \frac{\partial}{\partial y^a}\theta\beta\, h v^c v^a + h\theta\beta\gamma_{ab}^c v^a v^b + h\beta\theta\beta^{ca}\partial_a w &= -\,\beta\mathcal{K}(h,\theta)|\boldsymbol{v}|v^c. \end{aligned} \tag{3.33}$$

The parameter function $\mathcal{K}(h,\theta)$ is given by

$$\mathcal{K}(h,\theta) = \frac{1}{2} C_d m(\boldsymbol{y}) h d + \frac{g\theta}{C_b^2}$$

here $m$ stands for the density number of the stems on surface area. In our model of plant the porosity $\theta$ and the density number $m$ are related by

$$\theta = 1 - m\frac{\pi d^2}{4}.$$

such that one can write

$$\mathcal{K}(h,\theta) = \alpha_p h(1-\theta) + \alpha_s\theta,$$

where the new parameters are given by

$$\alpha_p = \frac{2C_d}{\pi d}, \quad \alpha_s = \frac{g}{C_b^2}.$$

Note that the system equations modeling the water flow on an unvegetated hill can be obtained from the model (3.33) by simply considering the porosity $\theta = 1$.

# Chapter 4

# SWE models

The full PDE model for the water flow on vegetated hill is given by (3.33). The system is hyperbolic with source terms and there is an energy function that is a conserved quantity in the absence of plants and water-soil friction. Also, the model preserves the steady state of the lake.

**Proposition 6** *The model* (3.33) *is of hyperbolic type with source terms.*
(a) *The conservative form of the system is given by*

$$\partial_t \mathcal{H}^i(\boldsymbol{y}, t, \boldsymbol{u}) + \partial_a \mathcal{F}^{ia}(\boldsymbol{y}, t, \boldsymbol{u}) = \mathcal{P}^i(\boldsymbol{y}, t, \boldsymbol{u}), \tag{4.1}$$

*where*

$$\boldsymbol{u} = \begin{pmatrix} h \\ v^1 \\ v^2 \end{pmatrix}, \quad \mathcal{H}(\boldsymbol{y}, t, \boldsymbol{u}) = \begin{pmatrix} \beta\theta h \\ \beta\theta h v^1 \\ \beta\theta h v^2 \end{pmatrix},$$

$$\mathcal{F}(\boldsymbol{y}, t, \boldsymbol{u}) = \begin{pmatrix} \beta\theta h v^1 & \beta\theta h v^2 \\ \beta\theta(hv^1v^1 + g_{\text{gravit}}\nu^3\beta^{11}h^2/2) & \beta\theta(hv^1v^2 + g_{\text{gravit}}\nu^3\beta^{12}h^2/2) \\ \beta\theta(hv^2v^1 + g_{\text{gravit}}\nu^3\beta^{21}h^2/2) & \beta\theta(hv^2v^2 + g_{\text{gravit}}\nu^3\beta^{22}h^2/2) \end{pmatrix},$$

*and*

$$\mathcal{P}(\boldsymbol{y}, t, \boldsymbol{u}) = \begin{pmatrix} \beta(\mathfrak{m}_r - \theta(y)\mathfrak{m}_i) \\ -\beta\theta h\gamma^1_{ab}v^av^b - g_{\text{gravit}}h\left[\beta\theta\beta^{1a}\left(\partial_a x^3 + \frac{h}{2}\partial_a\nu^3\right) - \frac{h}{2}\nu^3\partial_a\beta\theta\beta^{1a}\right] - \beta\mathcal{K}|v|v^1 \\ -\beta\theta h\gamma^1_{ab}v^av^b - g_{\text{gravit}}h\left[\beta\theta\beta^{2a}\left(\partial_a x^3 + \frac{h}{2}\partial_a\nu^3\right) - \frac{h}{2}\nu^3\partial_a\beta\theta\beta^{2a}\right] - \beta\mathcal{K}|v|v^2 \end{pmatrix}.$$

(b) *For any unitary vector $\boldsymbol{n} \in \mathbb{R}^3$, the eigenvalue problem [11]*

$$\left(\frac{\partial}{\partial u^i}\mathcal{F}^{ja}n_a - \lambda\frac{\partial}{\partial u^i}\mathcal{H}^j\right) r^i = 0 \tag{4.2}$$

*has three solutions:*

$$\lambda_- = v^a n_a - \sqrt{g_{\text{gravit}}\nu^3 h}, \quad \lambda_0 = v^a n_a, \quad \lambda_+ = v^a n_a + \sqrt{g_{\text{gravit}}\nu^3 h}. \tag{4.3}$$

*Proof.* In order to prove the existence of the solution for (4.2), it is sufficient to show that

$$\frac{\partial}{\partial u^i}\mathcal{F}^{ja}n_a - \lambda\frac{\partial}{\partial u^i}\mathcal{H}^j = \beta\theta\begin{pmatrix} \delta & hn_1 & hn_2 \\ v^1\delta + g_{\text{gravit}}\nu^3 h\beta^{1a}n_a & h\delta + hv^1 n_1 & hv^1 n_2 \\ v^2\delta + g_{\text{gravit}}\nu^3 h\beta^{2a}n_a & hv^2 n_1 & h\delta + hv^2 n_2 \end{pmatrix},$$

where $\delta = v^a n_a - \lambda$. The solutions (4.3) results then from straightforward calculations.

**Proposition 7** *The following properties hold for system* (3.33)*:*
*(a) it preserve the steady state of a lake*

$$x^3 + h\nu^3 = \text{constant}$$

*(b) There is a conservative equation for the energy*

$$\frac{\partial}{\partial t}h\beta\theta\mathcal{E} + \frac{\partial}{\partial y^a}h\beta\theta v^a\left(\mathcal{E} + g_{\text{gravit}}\frac{h}{2}\nu^3\right) = \beta\left(\left(\mathfrak{M}(-\frac{1}{2}|v|^2 + w\right) - \mathcal{K}|v|^3\right) \tag{4.4}$$

*where*

$$\mathcal{E} := \frac{1}{2}|v|^2 + g_{\text{gravit}}(x^3 + \frac{h}{2}\nu^3), \quad \mathfrak{M} = \mathfrak{m}_r - \theta\mathfrak{m}_i$$

*(c) Bernoulli's law. At a steady state, in the absence of mass source and friction force, the total energy*

$$\mathcal{E}^{\text{t}} = \frac{1}{2}|v|^2 + g_{\text{gravit}}x^3 + p(y,h)$$

*is constant along a current line*

$$v^a\partial_a\mathcal{E}^{\text{t}} = 0. \tag{4.5}$$

## 4.1 Simplified models

The mathematical model (3.33) is too complicated for many practical applications, but it represents a great start to generate simplified models of certain realistic problems. A simplified version of the full model corresponds to a given soil surface topography and a given structure of the plant cover. In what follows, we introduce a simplified variant of (3.33) that allows variations in the soil topography and plant porosity, but for which one must consider small departures from some constant states.

Assume that the soil surface is represented by

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = z(y^1, y^2) \tag{4.6}$$

and the surface is such that the first derivatives of the function $z(y^1, y^2)$ are small quantities.

**Assumptions**:
(a) *Geometrical assumptions:*

$$|\nabla z|^2 \approx 0, \quad \nabla^2 z \approx 0.$$

One these grounds, equations (3.33) can be approximated as

$$\begin{aligned} \frac{\partial}{\partial t}\theta h + \partial_a\left(\theta h v^a\right) = \mathfrak{M}, \\ \frac{\partial}{\partial t}\theta h v_a + \partial_b \theta h v_a v^b + \theta h \partial_a w = -\mathcal{K}(h,\theta)|v|v_a, \end{aligned} \tag{4.7}$$

where

$$\mathcal{K}(h,\theta) = \alpha_p h(1-\theta) + \theta\alpha_s, \quad \mathfrak{M} = \mathfrak{m}_r - \mathfrak{m}_i\theta, \quad w = g(z(y^1,y^2)+h). \tag{4.8}$$

The simplified model (4.7) preserves the main properties of the full model.

**Proposition 8** *The reduce model* (4.7) *of equations for the water flow on vegetated hill is of hyperbolic type with source terms.*
(a) *The conservative form of the system is given by*

$$\begin{aligned} \frac{\partial}{\partial t}\theta h + \partial_a\left(\theta h v^a\right) = \mathfrak{M}, \\ \frac{\partial}{\partial t}\theta h v_a + \partial_b\left(\theta h v_a v^b + \delta_a^b \theta g\frac{h^2}{2}\right) = -hg\partial_a z - g\frac{h^2}{2}\partial_a\theta - \mathcal{K}(h,\theta)|v|v_a. \end{aligned} \tag{4.9}$$

(b) *For any unitary vector $\boldsymbol{n} \in \mathbb{R}^3$, the solutions of the eigenvalue problem are given by*

$$\lambda_- = v^a n_a - \sqrt{gh}, \quad \lambda_0 = v^a n_a, \quad \lambda_+ = v^a n_a + \sqrt{gh}. \tag{4.10}$$

**Proposition 9** *The system* (4.7) *has the following properties:*

(a) *it preserves the steady state of a lake*

$$x^3 + h = \text{constant},$$

(b) *there is a conservative form of the equation for the energy dissipation*

$$\frac{\partial}{\partial t}\theta h\mathcal{E} + \frac{\partial}{\partial y^a}\theta h v^a\left(\mathcal{E} + g_{\text{gravit}}\frac{h}{2}\right) = \left(\left(\mathfrak{M}(-\frac{1}{2}|v|^2 + w\right) - \mathcal{K}|v|^3\right), \tag{4.11}$$

*where*

$$\mathcal{E} := \frac{1}{2}|v|^2 + g(x^3 + \frac{h}{2})$$

,

(c) *Bernoulli's law. At a steady state, in the absence of mass source and friction force, the total energy*

$$\mathcal{E}^{\text{t}} = \frac{1}{2}|v|^2 + gx^3 + p(y,h)$$

*is constant along of a current line*

$$v^a\partial_a\mathcal{E}^{\text{t}} = 0. \tag{4.12}$$

The presence of the plants and the existence of the frictional interaction between water and soil induce and energetic lost. To put in evidence such phenomenon, let us consider a domain $\Omega$ and $\boldsymbol{n}$ the unitary normal to the $\partial\Omega$ outward orientated. One assumes that the $\partial\Omega$ consists in an impermeable portion and an exit portion $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\boldsymbol{n}\cdot\boldsymbol{v} = 0$ on $\Gamma_1$ and $\boldsymbol{n}\cdot\boldsymbol{v} > 0$ on $\Gamma_2$. One of the two portions can be a void set.

**Proposition 10 (Energy disipation)** *Assume that there is no mass production. Then the energy of $\Omega$ is a decreasing function with respect to time*

$$\partial_t \int\limits_{\Omega} h\beta\theta\mathcal{E}\mathrm{d}x < 0 \tag{4.13}$$

To prove the assertion, one integrates the energy dissipation equation (4.11)

$$\partial_t \int\limits_{\Omega} h\beta\theta\mathcal{E}\mathrm{d}x + \int\limits_{\partial\Omega} h\beta\theta\boldsymbol{v}\cdot\boldsymbol{n}\mathcal{E}^t\mathrm{d}s = -\int\limits_{\Omega} \mathcal{K}|v|^3\mathrm{d}x$$

and observes that the second integral from the l.h.s. is a positive quantity.

## 4.2 Mathematical model of soil erosion in the presence of vegetation

Soil erosion is a complex and not yet very well understood process. To fill this gap in the mathematical modeling of this process, there are several empirical relations that relate the soil production with some soil properties and water motion characteristics. The concept of "sediment transport" refers to the transport of the eroded as suspended sediment in the water. One assumes that the soil particle velocity components in the tangent plane at soil surface are approximately equal to the velocity of the mixture - one ignores the diffusion processes of the sediment. Also, one assumes that the surface is an almost planar surface.

The erosion model we consider here couples the shallow water equations with the Hairsin-Rose model for soil erosion and takes into account the presence of the plants on the soil surface, [15]

$$\begin{gathered}\partial_t\theta h + \partial_a(\theta h v^a) = 0,\\ \partial_t(\theta h v^a) + \partial_b(\theta h v^a v^b) + \theta h g\delta^{ab}\partial_b(z+h) = \tau_v^a + \tau_s^a, \quad a = 1,2\end{gathered} \tag{4.14}$$

$$\partial_t(\theta h\rho_\alpha) + \partial_a(\theta\rho_\alpha h v^a) = \theta(e_\alpha + e_\alpha^r - d_\alpha), \quad \alpha = \overline{1,N}, \tag{4.15}$$

$$\partial_t m_\alpha = \theta(d_\alpha - e_\alpha^r), \quad \alpha = \overline{1,N}. \tag{4.16}$$

The unknown variables are $h(t,x)$ - water depth, $v^a(t,x)$ - components of the water speed, $\rho_\alpha(t,x)$ - mass density of the suspended sediment of the size class $\alpha$ and $m_\alpha(t,x)$ - mass density of the deposited sediment of the size class $\alpha$.

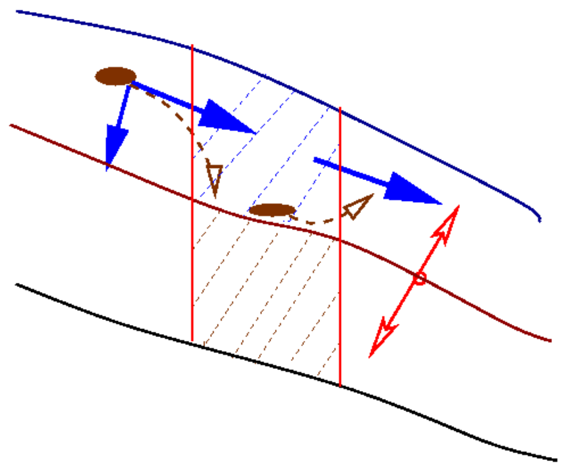

Figure 4.1: Representative elementary volume in the sediment mass balance equation

The sediment is partioned in $N$ size classes. The soil surface is modeled by the altitude function $z(x)$ and the vegetation is quantified by the porosity function $\theta(x)$. The terms $\tau_v^a$ and $\tau_s^a$ quantify the interaction water-plant and water-soil, respectively. The erosion and sedimentation processes are modeled by the terms $e_\alpha$ - entrainment rate, $e_\alpha^r$ - re-entrainment rate and $d_\alpha$ - deposition rate of the sediment from the size class $\alpha$, respectively.

Equations (4.14-4.16) need empirical relations to relate the erosion rates, the deposition rate, and the flow resistance to the unknown functions. One assumes that the flow resistance exercised by plants and soil obey laws (4.17) and (4.18), respectively

$$\tau_v^a = -\alpha_v h \,(1-\theta)\,|v| v^a, \tag{4.17}$$

$$\tau_s^a = -\theta \alpha_s |v| v^a, \tag{4.18}$$

where $\alpha_v$ and $\alpha_s$ are material parameters. The coefficient $\alpha_v$ depends on the geometry of the plants from the vegetation cover, while $\alpha_s$ depends on the soil roughness.

The Hairsine-Rose model [13, 14, 16], uses a set of empirical relationships based on the "power stream" concept, originally introduced by Bagnold [12] for determining the sediment transport in rivers, and then extended to flows on sloping surfaces

$$\begin{aligned} d_\alpha &= \nu_{s\,\alpha} \rho_\alpha, \\ e_\alpha &= p_\alpha (1-H) \frac{F\,(\Omega - \Omega_{cr})}{J}, \\ e_\alpha^r &= H \frac{m_\alpha}{m_t} \frac{\gamma_s}{\gamma_s - 1} \frac{F\,(\Omega - \Omega_{cr})}{gh}, \end{aligned} \tag{4.19}$$

where $p_\alpha$ is the proportion of the sediment in the original soil, $\nu_{s,\alpha}$ is the settling velocity of the sediment in the size class $\alpha$, and $\gamma_s$ is specific weight of sediment.

The parameters $F$ - effective fraction of power stream, $J$ - energy of soil particles detachment and $\Omega_{cr}$ - critical power stream are specific to a given type of soil. The erosion processes are controlled by the water flow through the stream power $\Omega$. In the present paper we use the law

$$\Omega = \theta \rho_{\mathrm{w}} |\tau_s||v|. \tag{4.20}$$

The function

$$H = \min\left\{\frac{m_t}{m_t^\star}, 1\right\} \tag{4.21}$$

plays the role of a protecting factor of the original soil to the erosion process. The terms

$$m_t = \sum_{a=1}^{N} m_a$$

and $m_t^\star$ from (4.21) are the total mass of sediment deposited on the soil and the mass required to protect the original soil from erosion, respectively.

# Appendix A

# Basics of differential geometry in $\mathbb{E}^3$

## A.1 Curvilinear coordinate

Let $Ox$ be a Cartesian coordinate system $Ox$ in the reference Euclidean space $\mathbb{E}^3$. Let $\{y^I\}_{I=\overline{1,3}}$ be another coordinate system and let

$$x^i = x^i(y^1, y^2, y^3), \quad \boldsymbol{y} \in D \tag{A.1}$$

be the transformation rule. By coordinate line, one understands the curves generated by the variation of a single variable $y^I$, while the rest are kept constants. The tangent vector to the coordinate lines are defined by

$$\boldsymbol{e}_I = \partial_i \boldsymbol{x}. \tag{A.2}$$

The set of vectors $\{\boldsymbol{e}_I\}_{I=\overline{1,3}}$ give rise to a new base of tensor fields. For the vectors and tensor of rank 2, one writes

$$\boldsymbol{v} = v^I \boldsymbol{e}_I, \quad \mathbf{t} = \mathrm{t}^{IJ} \boldsymbol{e}_I \boldsymbol{e}_J.$$

In the new coordinate system, the components of the metric tensor $\boldsymbol{g}$ is given by

$$g_{IJ} = \delta_{ij} e^i_I e^j_J \tag{A.3}$$

and

$$g^{IJ} = \delta^{ij} h^I_i h^J_j, \tag{A.4}$$

where

$$h^I_j = \partial_j y^I. \tag{A.5}$$

One has

$$e^j_I h^I_i = \delta^j_i, \quad e^j_I h^J_j = \delta^J_I \tag{A.6}$$

and then

$$g^{IK}g_{KJ} = \delta^I_J.$$

The volume element is

$$J = \varepsilon_{ijk}e^i_1 e^j_2 e^k_3. \tag{A.7}$$

From (A.7) and (A.3), one obtains

$$\det g_{..} = J^2. \tag{A.8}$$

The variation of the basis $\{\boldsymbol{e}_I\}_I$ with respect to the $y$ coordinate is stored inside Christoffel's symbols $\Gamma$

$$\partial_I \boldsymbol{e}_J = \Gamma^L_{IJ}\boldsymbol{e}_L. \tag{A.9}$$

Alternatively, one can calculate the $\Gamma$ coefficients by

$$\begin{aligned}
\Gamma^L_{IJ} &= h^L_i \partial_J e^i_I, \\
\Gamma^L_{IJ} &= -e^i_I e^j_J \partial_i h^L_j, \\
\Gamma^L_{IJ} &= \frac{1}{2}g^{LK}\left(\partial_I g_{KJ} + \partial_J g_{KI} - \partial_K g_{IJ}\right).
\end{aligned} \tag{A.10}$$

Define the covariant derivative of a vector by

$$v^I_{;L} = \partial_L v^I + v^K\Gamma^I_{LK} \tag{A.11}$$

and the covariant derivative of tensor by

$$t^{IJ}_{;L} = \partial_L t^{IJ} + t^{KJ}\Gamma^I_{LK} + +t^{IK}\Gamma^J_{LK}. \tag{A.12}$$

An elementary way to introduce the covariant derivative is to estimate the difference of vector fields between two neighbor points

$$\boldsymbol{v}(\boldsymbol{x}+\triangle\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{x}) = v^I\boldsymbol{e}_I(\boldsymbol{y}+\triangle\boldsymbol{y})-v^I\boldsymbol{e}_I(\boldsymbol{y}) = \left(\partial_L v^I + v^K\Gamma^I_{LK}\right)\boldsymbol{e}_I\triangle y^L+O(\triangle\boldsymbol{y}^2)$$

## A.2 Basic notions of differential geometry on a surface in $\mathbb{E}^3$

For completeness, we present here the essential facts about the differential geometry of the surface in the euclidean space $E^3$; as a reference, one can consults the classical books [**?** ]. Let $O\boldsymbol{x}$ be a Cartesian coordinate system in the reference Euclidean space $\mathbb{E}^3$. Let $\mathcal{S}$ be a surface in $E^3$ and let

$$x^i = b^i(y^1, y^2), \quad (y^1, y^2) \in D \in \mathbb{R}^2 \tag{A.13}$$

be a parameterization of $\mathcal{S}$. One defines the tangent vectors to the surface by

$$\tau^i_a = \frac{\partial b^i}{\partial y^a} \tag{A.14}$$

and the oriented normal direction to the surface by

$$\mathcal{N}_i = \epsilon_{j\,k\,i}\tau_1^j\tau_2^k. \tag{A.15}$$

The unitary normal $\boldsymbol{\nu}$ to the surface is given by

$$\nu_i = \frac{\mathcal{N}_i}{||\boldsymbol{\mathcal{N}}||}. \tag{A.16}$$

**Metric tensor $\boldsymbol{\beta}$ of the surface**. The covariant components of $\boldsymbol{\beta}$ are given by

$$\beta_{ab} = \delta_{ij}\tau_a^i\tau_b^j \tag{A.17}$$

and the contravariant components $\beta^{ab}$ of it are defined by the relations

$$\delta_b^a = \beta^{ac}\beta_{cb} = \beta_{bc}\beta^{ca}. \tag{A.18}$$

The area element of the surface is defined by

$$\mathrm{d}\sigma = \beta \mathrm{d}y^1 \mathrm{d}y^2, \tag{A.19}$$

where

$$\beta = \sqrt{\epsilon^{ab}\beta_{a1}\beta_{b2}}. \tag{A.20}$$

Note that

$$||\boldsymbol{\mathcal{N}}|| = \beta.$$

**The curvature tensor $\boldsymbol{\kappa}$**. The curvature tensor (introduced by Wiengartern) and the affine connection $\gamma$ are defined by

$$\begin{aligned}\frac{\partial \boldsymbol{\tau}_a}{\partial y^b} &= \gamma_{a\,b}^c\boldsymbol{\tau}_c + \kappa_{a\,b}\boldsymbol{\nu}\\ \frac{\partial \boldsymbol{\nu}}{\partial y^a} &= -\kappa_a^b\boldsymbol{\tau}_b.\end{aligned} \tag{A.21}$$

## A.3 Surface Based Curvilinear Coordinate System

A surface $\mathcal{S}$ based coordinate system in the space $\mathbb{E}^3$ is introduced as follows. Given a parameterization (A.13) of the surface, one defines the applications

$$x^i = b^i(y^1, y^2) + y^3\nu^j, \quad (y^1, y^2) \in \widetilde{D} \subset \mathbb{R}^2, \quad y^3 \in \widetilde{I} \in \mathbb{R}, \tag{A.22}$$

where $\widetilde{I}$ is an open neighborhood of zero. Assume that (A.22) defines a coordinate transformation from $\widetilde{D} \times \widetilde{I}$ to a space neighborhood $\Omega$ of the surface $\mathcal{S}$. The surface $\mathcal{S}$ in the new coordinate system is given by $y^3 = 0$.
Furthermore, we have:

- the vectors tangent to the coordinate lines

$$\boldsymbol{e}_I = \frac{\partial \boldsymbol{x}}{\partial y^I} \Longrightarrow \begin{cases} \boldsymbol{e}_a = q_a^b\boldsymbol{\tau}_b, \quad q_a^b := \delta_a^b - y^3\kappa_a^b, \quad a = \overline{1,2} \\ \boldsymbol{e}_3 = \boldsymbol{\nu} \end{cases}. \tag{A.23}$$

- the coefficients of the metric tensor

$$g_{IJ} = \delta_{ij} e^i_I e^j_J \Longrightarrow \begin{cases} g_{ab} = q^c_a q^d_b \beta_{cd}, & g_{a3} = 0, \\ g_{3a} = 0, & g_{33} = 1, \end{cases} \tag{A.24}$$

with

$$\sqrt{\det g_{..}} = \beta \Delta, \quad \Delta := 1 - 2y^3 K_M + (y^3)^2 K_G \tag{A.25}$$

- the affine connection

$$\frac{\partial \boldsymbol{e}_I}{\partial y^J} = \Gamma^L_{IJ} \boldsymbol{e}_L \Longrightarrow \begin{cases} \Gamma^c_{ab} = \left(\gamma^e_{ab} - y^3 \left(\partial_a \kappa^d_b + \kappa^f_b \gamma^d_{af}\right)\right) Q^c_d, & \Gamma^c_{a3} = -\kappa^e_a Q^c_e, \\ \Gamma^3_{ab} = (\delta^e_a - y^3 \kappa^e_a) \kappa_{cb}, & \Gamma^3_{a3} = 0, \end{cases} \tag{A.26}$$

where $Q$ is defined by

$$\boldsymbol{\tau}_a = Q^b_a \boldsymbol{e}_b \Longrightarrow \begin{cases} Q^1_1 = \dfrac{1 - y^3 \kappa^2_2}{\Delta(y)}, & Q^2_1 = \dfrac{y^3 \kappa^2_1}{\Delta(y)}, \\ Q^1_2 = \dfrac{y^3 \kappa^1_2}{\Delta(y)}, & Q^2_2 = \dfrac{1 - y^3 \kappa^1_1}{\Delta(y)}. \end{cases} \tag{A.27}$$

**Obs.** For any $y^3 \in I$, the tangent vectors $\boldsymbol{t}_a$ belong to the tangent plane at the surface $y^3 = \text{const}$ and they are orthogonal to the normal $\boldsymbol{\nu}$. In the new coordinate system, the elementary element volume is given by

$$\vartheta(y) = \epsilon_{i\,j\,k} t^i_1 t^j_2 t^k_3 = \sqrt{\det g_{..}} = \left(1 - 2y^3 K_M + (y^3)^2 K_G\right) \beta, \tag{A.28}$$

where $K_M = 1/2 \kappa^a_a$ and $K_G = \epsilon_{a,b} \kappa^a_1 \kappa^b_2$ are the mean curvature and the Gauss curvature of the surface, respectively.

### A.3.1 Integrals of vectors and second order tensors

Let $V$ be a domain in $\mathbb{E}^3$ defined by

$$\boldsymbol{x} = \boldsymbol{b}(y^1, y^2) + y^3 \boldsymbol{\nu}, \quad (y^1, y^2) \in D, \quad u(y^1, y^2) < y^3 < w(y^1, y^2)$$

where $D$ is a open closed domain with boundary $\partial D$, $u(y^1, y^2)$ and $w(y^1, y^2)$ are two functions that define some surfaces in $\mathbb{E}^3$. We are interested in calculating the flux of vectors or tensors through the boundary of $V$, to evaluate integral of vectors in $V$ or to calculate integrals of vectors on surfaces. In $\mathbb{E}^3$, such integrals define global quantities of the same kind as the integrands: scalar defines scalars, vector defines vectors and second tensors define second tensors. If one uses curvilinear coordinate, such invariant properties are lost.

Let $\mathcal{S}$ and $V$ be a surface and a domain in $\mathbb{E}^3$, respectively. Define the flux of $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ through a surface by

$$\mathcal{F}_{\boldsymbol{f}}(S) := \int_S f^i \, n_i \mathrm{d}\sigma,$$

$$\mathcal{F}^i_{\boldsymbol{\Phi}}(S) := \int_S \Phi^{ij} \, n_j \mathrm{d}\sigma,$$

and the integrals of a vector field by

$$\mathcal{I}^j_{\boldsymbol{f}}(V) = \int\limits_V f^j \mathrm{d}x,$$
$$\mathcal{I}^j_{\boldsymbol{f}}(S) = \int\limits_S f^j \mathrm{d}\sigma,$$

where $\boldsymbol{n}$ stands for outward oriented unitary normal to the surface.

Let $\mathcal{S}_r$ be the surface defined by some function $r(y^1, y^2)$

$$\boldsymbol{x} = \boldsymbol{b}(y^1, y^2) + r(y^1, y^2)\boldsymbol{\nu}, \quad (y^1, y^2) \in D.$$

One denotes the "vertical" boundary of $V$ by

$$\Sigma = \big\{\boldsymbol{x} \in \mathbb{E}^3 | \quad \begin{array}{l} \boldsymbol{x} = \boldsymbol{b}(y^1(s), y^2(s)) + y^3\nu(y^1(s), y^2(s)), \\ s \in (0, L), \quad u(y^1(s), y^2(s)) < y^3 < w(y^1(s), y^2(s))\big\} \end{array} \tag{A.29}$$

where $\big(y^1(s), y^2(s)\big)$, $s \in (0, L)$ is a parameterization of $\partial D$.

Let $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ be a vector field and a second order tensor field in $\mathbb{E}^3$, respectively. Using the law of transformation of coordinate system of a tensorial field under coordinate transformation, one can write

$$f^i = f^I t^i_I, \quad \Phi^{ij} = t^i_I t^j_J \Phi^{IJ}.$$

Next lemma refers to various integrals.

**Lemma 4** *Let $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ be some smooth fields on a domain $\Omega \subset \mathbb{E}^3$. Let $S_r$, $V$ and $\Sigma$ be a surface, domain and portion of $\partial V$ as the ones previously defined, respectively. Then:*

$$\begin{aligned}
\mathcal{I}^i_f(V) &= \iint\limits_D \left( \tau^i_a \int\limits_u^w q^a_b f^b \vartheta \mathrm{d}y^3 + \nu^i \int\limits_u^w f^3 \vartheta \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}_f(S_r) &= \iint\limits_D \vartheta(y) \left( f^3 - f^a \frac{\partial r}{\partial y^a} \right)\Bigg|_{y^3=r} \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}_f(\Sigma) &= \iint\limits_D \frac{\partial}{\partial y^a} \int\limits_u^w \vartheta f^a \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}^i_\Phi(S_r) &= \iint\limits_D \left[ \left( \tau^i_c q^c_b \left( \Phi^{b3} - \frac{\partial w}{\partial y^a} \Phi^{ba} \right) + \nu^i \left( \Phi^{33} - \frac{\partial w}{\partial y^a} \Phi^{3a} \right) \right) \vartheta(y) \right]\Bigg|_{y^3=r} \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}^i_\Phi(\Sigma) &= \iint\limits_D \tau^i_c \left( \frac{\partial}{\partial y^a} \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 + \gamma^c_{ae} \int\limits_u^w q^e_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 - \kappa^c_a \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2 + \\
&\quad + \iint\limits_D \nu^i \left( \kappa_{ca} \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 + \frac{\partial}{\partial y^a} \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2.
\end{aligned} \tag{A.30}$$

*Proof.* Let $(y^1(s), y^2(s))$, $s \in (0, L)$ be a parameterization of the boundary $\partial D$. On $\Sigma$, the tangent directions are given by

$$\begin{aligned} \boldsymbol{t}_s &= \boldsymbol{t}_a w^a, \\ \boldsymbol{t}_3 &= \boldsymbol{\nu}, \end{aligned}$$

where $w^a = \dfrac{\mathrm{d}y^a}{\mathrm{d}s}$ and the outward normal direction is given by

$$N_i := \epsilon_{jki} t_3^j t_s^k = \epsilon_{jki} \nu^j t_a^k w^a.$$

Thus, one can evaluate the flux as

$$\mathcal{F}_f(\Sigma) := \int_\Sigma f^i n_i \mathrm{d}\sigma = \int_0^L \int_{u(s)}^{w(s)} f^i N_i \mathrm{d}y^3 \mathrm{d}s.$$

Then, one writes $\boldsymbol{f}$ on the local basis $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ and obtains

$$f^i N_i = (f^b t_b^i + f^3 \nu^i) N_i = \epsilon_{jki} \nu^j t_a^k t_b^i w^a f^b = \vartheta(y) \epsilon_{ab} w^a f^b$$

and

$$\mathcal{F}_f(\Sigma) = \int_0^L \int_{u(s)}^{w(s)} \vartheta(y) \epsilon_{ab} w^a f^b \mathrm{d}y^3 \mathrm{d}s \Longrightarrow \int_0^L \epsilon_{ab} w^a \int_{u(s)}^{h(s)} \vartheta(y) f^b \mathrm{d}y^3 \mathrm{d}s.$$

Observe that $\epsilon_{ab} w^a = \epsilon_{ab} \dfrac{\partial y^a}{\partial s}$ is the normal direction to the boundary $\partial D$ and use the flux-divergence theorem and to obtain

$$\mathcal{F}_f(\Sigma) = \iint_D \frac{\partial}{\partial y^a} \int_u^w \vartheta(y) f^a \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2. \tag{A.31}$$

On $S_r$, one has tangent vectors

$$\boldsymbol{\zeta}_a = \frac{\partial \boldsymbol{x}}{\partial y^a} = \boldsymbol{t}_a + \frac{\partial r}{\partial y^a} \boldsymbol{\nu} \tag{A.32}$$

and normal direction

$$N_i = \epsilon_{jki} (t_1^j + \frac{\partial r}{\partial y^1} \nu^j)(t_1^k + \frac{\partial r}{\partial y^2} \nu^k) \tag{A.33}$$

that implies

$$f^i N_i = \vartheta(y)(f^3 - \frac{\partial r}{\partial y^a} f^a).$$

Then

$$\mathcal{F}_f(S_r) = \iint_D \vartheta(y)(f^3 - \frac{\partial r}{\partial y^a} f^a)\bigg|_{y^3 = r} \mathrm{d}y^1 \mathrm{d}y^2 \tag{A.34}$$

Consider now a second order tensor $\Phi$. The coordinate transformation (**??**) implies that the contravariant components of the tensor in the two coordinate system are related by

$$\Phi^{ij} = t^i_I t^j_J \Phi^{IJ}.$$

The main difficulty in this case is that the vectors of the basis depend on the variables $(y^1, y^2, y^3)$ and there is no sense to find the components of the vector global quantity $\mathcal{F}_\Phi$ in the new coordinate system. We proceed to find the Cartesian components of $\mathcal{F}_\Phi$, but calculated as functions of the contravariant components $\Phi^{IJ}$.

On the surface $\Sigma$, one has

$$\Phi^{ij} N_j = t^i_I t^j_J T^{IJ} N_j = \vartheta(y) \epsilon_{ab} w^a t^i_I \Phi^{Ib}$$

and the flux is given by

$$\mathcal{F}^i_\Phi(\Sigma) = \iint\limits_D \frac{\partial}{\partial y^a} \int\limits_u^w \vartheta(y) t^i_I \Phi^{Ia} \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2$$

Next, one uses the relations (A.23) to get

$$\mathcal{F}^i_\Phi(\Sigma) = \iint\limits_D \frac{\partial}{\partial y^a} \left( \tau^i_c \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 + \nu^i \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2.$$

Using Weigartern formula, we can write

$$\mathcal{F}^i_\Phi(\Sigma) = \iint\limits_D \left( \tau^i_c \frac{\partial}{\partial y^a} \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 + \nu^i \frac{\partial}{\partial y^a} \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2 +$$
$$+ \iint\limits_D \tau^i_c \left( \gamma^c_{ae} \int\limits_u^w q^e_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 - \kappa^c_a \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2 +$$
$$+ \iint\limits_D \nu^i \kappa_{ea} \int\limits_u^w q^e_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2.$$

Regrouping terms, one obtain the result for $\mathcal{F}^i_\Phi(\Sigma)$.

**Lemma 5** *Let the stress tensor of the fluid be given as*

$$t^{ij} = -p\delta^{ij} + \tau^{ij}$$

*and set*

$$\mathcal{F}^i_{\text{stress}}(S_r) = \iint\limits_{S_r} t^{ij} n_j \mathrm{d}\sigma.$$

*Then*

$$\mathcal{F}^i_{\text{stress}}(S_r) = \iint\limits_D \left[ \tau^i_c q^c_a \left( (p - \widetilde{\tau}^{33}) g^{ab} \frac{\partial r}{\partial y^b} + \widetilde{\tau}^{a3} \sqrt{1 + g^{ab} \frac{\partial r}{\partial y^a} \frac{\partial r}{\partial y^b}} \right) \vartheta(y) \right] \Bigg|_{y^3 = r(y^1, y^2)} \mathrm{d}y^1 \mathrm{d}y^2 +$$
$$+ \iint\limits_D \left[ \nu^i \left( -p + \widetilde{\tau}^{33} + \frac{\partial r}{\partial y^a} \widetilde{\tau}^{a3} \sqrt{1 + g^{ab} \frac{\partial r}{\partial y^a} \frac{\partial r}{\partial y^b}} \right) \vartheta(y) \right] \Bigg|_{y^3 = r(y^1, y^2)} \mathrm{d}y^1 \mathrm{d}y^2. \tag{A.35}$$

*Proof* Let $r(y^1, y^2)$ a parameterization of the surface $S_r$ and let $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$, $\boldsymbol{n}$ be the tangent vectors and the unit normal given by (A.32) and (A.33), respectively. One can write

$$t^{ij} n_j = -pn^i + \tau^{ij} n_j = -pn^i + \widetilde{\tau}^{a3} \zeta^i_a + \widetilde{\tau}^{33} n^i. \tag{A.36}$$

Rewrite the unit normal and the tangent vectors on the basis $\{\boldsymbol{t}_I\}$ as

$$\boldsymbol{n} = n^a \boldsymbol{t}_a + n^3 \boldsymbol{\nu}, \quad n^a = -g^{ab} \frac{\partial r}{\partial y^a} \frac{\vartheta(y)}{||\boldsymbol{N}||}, \quad n^3 = \frac{\vartheta(y)}{||\boldsymbol{N}||},$$
$$||\boldsymbol{N}|| = \vartheta(y) \sqrt{1 + g^{ab} \frac{\partial r}{\partial y^a} \frac{\partial r}{\partial y^b}}, \quad y^3 = r(y^1, y^2)$$

and

$$\boldsymbol{\zeta}_a = \boldsymbol{t}_a + \frac{\partial r}{\partial y^a} \boldsymbol{n} \boldsymbol{u},$$

respectively. Taking into account that the area element is given by

$$\mathrm{d}\sigma = ||\boldsymbol{N}|| \mathrm{d}y^1 \mathrm{d}y^2,$$

then, the result of the lemma will immediately follow.

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