THE COMPLEX ANGLE IN NORMED SPACES

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We consider a concept of a generalized angle in complex normed vector spaces. Its definition corresponds to the definition of the well known Euclidean angle in real inner product spaces. Not surprisingly, it yields complex values as 'angles'. This 'angle' has some simple properties, which are known from the usual angle in real inner product spaces. To do ordinary Euclidean geometry, real angles are necessary. We show that in a complex normed space there are many pure real valued 'angles'. In an inner product space we have even better conditions. There we can use the known theory of orthogonal systems to find many pairs of vectors with real angles, and it is possible to do geometry, which is based on principles already known by the Greeks 2000 years ago.

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1. INTRODUCTION

There are some attempts already to generalize the well known Euclidean angle of real inner product spaces to real normed spaces; see the references [2, 3, 6, 8, 9, 15–18]. Also angles in complex inner product spaces were considered, see [4, 5, 12]. In this paper a complex-valued 'angle' is defined for the first time for all complex normed spaces, and we investigate its properties. We believe that this concept of a complex-valued angle is superior to the enforcement of pure real angles, since real angles may suppress some true properties of a complex normed vector space.

To initiate the constructions that follow we begin with the special case of an inner product space $(X, < . | . >)$ over the complex field $\mathbb{C}$. It is well known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$ we can write $< \vec{x} \mid \vec{y} > =$

$$(1) \qquad \frac{1}{4} \cdot \left[\, \|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 \; + \mathbf{i} \cdot \left(\, \|\vec{x}+\mathbf{i}\cdot\vec{y}\|^2 - \|\vec{x}-\mathbf{i}\cdot\vec{y}\|^2 \,\right) \right],$$

where the symbol '**i**' means the imaginary unit.

For two vectors $\vec{x}, \vec{y} \neq \vec{0}$ it holds $< \vec{x} \mid \vec{y} > = \|\vec{x}\| \cdot \|\vec{y}\| \cdot < \frac{\vec{x}}{\|\vec{x}\|} \mid \frac{\vec{y}}{\|\vec{y}\|} >$. We use these two facts and ideas from [13] to generate a continuous product in all complex normed vector spaces $(X, \|\cdot\|)$, which is just the inner product in the special case of a complex inner product space.

Definition 1.1. Let $\vec{x}, \vec{y}$ be two arbitrary elements of a complex normed space $(X, \|\cdot\|)$. In the case of $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ we set $< \vec{x} \,|\, \vec{y} > := 0$, and if $\vec{x}, \vec{y} \neq \vec{0}$, i.e. $\|\vec{x}\| \cdot \|\vec{y}\| > 0$, we define the complex number $< \vec{x} \,|\, \vec{y} > :=$

$$\|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right.$$
$$\left. + \mathbf{i} \cdot \left(\left\| \frac{\vec{x}}{\|\vec{x}\|} + \mathbf{i} \cdot \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \mathbf{i} \cdot \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \right].$$

It is easy to show that the product fulfils the conjugate symmetry $(< \vec{x} \,|\, \vec{y} > \; = \; \overline{< \vec{y} \,|\, \vec{x} >})$, where $\overline{< \vec{y} \,|\, \vec{x} >}$ is the complex conjugate of $< \vec{y} \,|\, \vec{x} >$, the positive definiteness $(< \vec{x} \,|\, \vec{x} > \geq 0$, and $< \vec{x} \,|\, \vec{x} >= 0$ only for $\vec{x} = \vec{0})$, the homogeneity for real numbers, $(< r \cdot \vec{x} \,|\, \vec{y} > \; = \; r \cdot < \vec{x} \,|\, \vec{y} >)$, and the homogeneity for pure imaginary numbers, $(< r \cdot \mathbf{i} \cdot \vec{x} \,|\, \vec{y} > \; = \; r \cdot \mathbf{i} \cdot < \vec{x} \,|\, \vec{y} > \; = \; < \vec{x} \,|\, -r \cdot \mathbf{i} \cdot \vec{y} >)$, for $\vec{x}, \vec{y} \in X$, $r \in \mathbb{R}$. Further, for $\vec{x} \in X$ it holds $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >}$.

If we switch for a moment to real inner product spaces $(X, < . \,|\, . >_{real})$ we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the usual Euclidean angle $\angle_{Euclid}$. It can be defined also in terms of the norm by

$$\angle_{Euclid}(\vec{x}, \vec{y})$$
$$= \arccos\left(\frac{< \vec{x} \,|\, \vec{y} >_{real}}{\|\vec{x}\| \cdot \|\vec{y}\|} \right) = \arccos\left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right).$$

For two vectors $\vec{x}, \vec{y} \neq \vec{0}$ from a complex normed vector space $(X, \|\cdot\|)$ we use the defined product $< \cdot \,|\, \cdot >$, and we are able to define an 'angle', which coincides with the definition of the Euclidean angle in real inner product spaces.

Definition 1.2. Let $\vec{x}, \vec{y}$ be two elements of $X \backslash \{\vec{0}\}$. We define the complex number

$$\angle(\vec{x}, \vec{y}) \;:=\; \arccos\left(\frac{< \vec{x} \,|\, \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} \right).$$

This number $\angle(\vec{x}, \vec{y}) \in \mathbb{C}$ is called the *angle of the pair* $(\vec{x}, \vec{y})$.

We state that the angle $\angle(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$. Since we deal with complex vector spaces, it is not surprising that we get complex numbers as 'angles'. For the definition we need the extension of the cosine and arccosine functions on complex numbers. We use two subsets $\mathcal{A}$ and $\mathcal{B}$ of the complex plane $\mathbb{C}$, where

$$\begin{aligned}\mathcal{A} &:= \{a+\mathbf{i}\cdot b \in \mathbb{C} \mid 0<a<\pi,\ b\in\mathbb{R}\} \cup \{0,\pi\}, \quad \text{and}\\ \mathcal{B} &:= \mathbb{C}\backslash\{r\in\mathbb{R} \mid r<-1 \ \text{ or } \ r>+1\}.\end{aligned}$$

We have two known homeomorphisms $\cos : \mathcal{A} \xrightarrow{\cong} \mathcal{B}$ and $\arccos : \mathcal{B} \xrightarrow{\cong} \mathcal{A}$. The cosines of the 'angles' are in the *complex unit square* $\mathcal{CUS} := \{r+\mathbf{i}\cdot s \in \mathbb{C} \mid -1 \le r,s \le +1\} \subset \mathcal{B}$. The values of the 'angles' are from its image $\arccos(\mathcal{CUS})$, which forms a symmetric hexagon (with concave sides) in $\mathcal{A}$. Its center is $\pi/2$, two corners are 0 and π. We get a third corner if we use one of four possibilities of $\pm 1 \pm \mathbf{i}$, for instance $\arccos(1+\mathbf{i}) \approx 0.90 - \mathbf{i}\cdot 1.06$.

The 'angle' in Definition 1.2 has eight simple properties (`An 1`)–(`An 8`), which are known from the Euclidean angle $\angle_{Euclid}$ of real inner product spaces.

THEOREM 1.3. *In a complex normed space $(X,\|\cdot\|)$ the angle $\angle$ has the following properties. We assume that $\vec{x},\vec{y}$ are non-zero elements of X.*

- (`An 1`) *$\angle$ is a continuous map from $\left(X\backslash\{\vec{0}\}\right)^2$ onto a subset of $\arccos(\mathcal{CUS}) \subset \mathcal{A}$. The image of $\angle$ is symmetric to $\pi/2$.*
- (`An 2`) $\angle(\vec{x},\vec{x}) = 0$,
- (`An 3`) $\angle(-\vec{x},\vec{x}) = \pi$,
- (`An 4`) $\angle(\vec{x},\vec{y}) = \overline{\angle(\vec{y},\vec{x})}$,
- (`An 5`) *for real numbers $r,s>0$ we have $\angle(r\cdot\vec{x}, s\cdot\vec{y}) = \angle(\vec{x},\vec{y})$,*
- (`An 6`) $\angle(-\vec{x},-\vec{y}) = \angle(\vec{x},\vec{y})$,
- (`An 7`) $\angle(\vec{x},\vec{y}) + \angle(-\vec{x},\vec{y}) = \pi$.
- (`An 8`) *If $\vec{x},\vec{y}$ are two linear independent vectors of $(X,\|\cdot\|)$ there is a continuous injective map $\Theta : \mathbb{R} \longrightarrow \mathcal{A},\ t \mapsto \angle(\vec{x},\vec{y}+t\cdot\vec{x})$. The limits are $\lim_{t\to-\infty}\Theta(t) = \pi$ and $\lim_{t\to\infty}\Theta(t) = 0$.*

In Table 1 we list some angles and their cosines. We take arbitrary two elements $\vec{x},\vec{y} \neq \vec{0}$ of a complex normed space $(X,\|\cdot\|)$, and six suitable real numbers a,b,r,s,v,w with $-\frac{\pi}{2} \le a,v \le \frac{\pi}{2}$ and $-1 \le r,s \le 1$, such that $\angle(\vec{x},\vec{y}) = \frac{\pi}{2}+a+\mathbf{i}\cdot b \in \mathcal{A}$, $\cos(\angle(\vec{x},\vec{y})) = \cos\left(\frac{\pi}{2}+a+\mathbf{i}\cdot b\right) = r+\mathbf{i}\cdot s \in \mathcal{B}$, and $\angle(\mathbf{i}\cdot\vec{x},\vec{y}) = \frac{\pi}{2}+v+\mathbf{i}\cdot w \in \mathcal{A}$.

Note that the cosines of all angles in the table (third column) have the same modulus $\sqrt{r^2+s^2}$.

Table 1

pair of vectors	their angle $\angle$	the cosine of $\angle$	the angle for $\vec{x}=\vec{y}$	its cosine for $\vec{x}=\vec{y}$
$(\vec{x},\vec{y})$	$\frac{\pi}{2}+a+\mathbf{i}\cdot b$	$r+\mathbf{i}\cdot s$	0	1
$(-\vec{x},\vec{y})$	$\frac{\pi}{2}-a-\mathbf{i}\cdot b$	$-r-\mathbf{i}\cdot s$	π	-1
$(\vec{y},\vec{x})$	$\frac{\pi}{2}+a-\mathbf{i}\cdot b$	$r-\mathbf{i}\cdot s$	0	1
$(-\vec{y},\vec{x})$	$\frac{\pi}{2}-a+\mathbf{i}\cdot b$	$-r+\mathbf{i}\cdot s$	π	-1
$(\mathbf{i}\cdot\vec{x},\vec{y})$	$\frac{\pi}{2}+v+\mathbf{i}\cdot w$	$-s+\mathbf{i}\cdot r$	$\frac{\pi}{2}-\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$\mathbf{i}$
$(\vec{y},\mathbf{i}\cdot\vec{x})$	$\frac{\pi}{2}+v-\mathbf{i}\cdot w$	$-s-\mathbf{i}\cdot r$	$\frac{\pi}{2}+\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$-\mathbf{i}$
$(\vec{x},\mathbf{i}\cdot\vec{y})$	$\frac{\pi}{2}-v-\mathbf{i}\cdot w$	$s-\mathbf{i}\cdot r$	$\frac{\pi}{2}+\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$-\mathbf{i}$
$(\mathbf{i}\cdot\vec{y},\vec{x})$	$\frac{\pi}{2}-v+\mathbf{i}\cdot w$	$s+\mathbf{i}\cdot r$	$\frac{\pi}{2}-\mathbf{i}\cdot\log\left[\sqrt{2}+1\right]$	$\mathbf{i}$

With given two vectors $\vec{x},\vec{y}\in X$ we consider as before the angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a+\mathbf{i}\cdot b$ with suitable real numbers a,b. Now we express the complex number $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$ in dependence of a and b. For the representation the real valued cosine and hyperbolic cosine are used, and their inverses arccosine and area hyperbolic cosine. For a correct sign we use the signum function. The result is as follows.

THEOREM 1.4. *In a complex normed space $(X,\|\cdot\|)$ we take two elements $\vec{x},\vec{y}\neq\vec{0}$. We assume the angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a+\mathbf{i}\cdot b\in\mathcal{A}$, i.e. $-\frac{\pi}{2}\leq a\leq\frac{\pi}{2}$. (If $a=-\frac{\pi}{2}$ or $a=\frac{\pi}{2}$ since $\angle(\vec{x},\vec{y})\in\mathcal{A}$ it follows $b=0$.) We use the abbreviations H_- and H_+. We get*

$$\angle(\mathbf{i}\cdot\vec{x},\vec{y}) = \frac{\pi}{2}+\frac{1}{2}\cdot\left[-\operatorname{sgn}(b)\cdot\arccos(\mathsf{H}_-)+\mathbf{i}\cdot\operatorname{sgn}(a)\cdot\operatorname{arcosh}(\mathsf{H}_+)\right],$$

$$\mathsf{H}_\pm := \sqrt{\left[\cos^2\left(\frac{\pi}{2}+a\right)+\cosh^2(b)-2\right]^2+4\cdot\cos^2\left(\frac{\pi}{2}+a\right)\cdot\cosh^2(b)} \pm\left[\cos^2\left(\frac{\pi}{2}+a\right)+\cosh^2(b)-1\right].$$

It is worthwhile to look at special cases. We consider a real angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a$ (*i.e.* $b=0$), and an angle on the vertical line of complex numbers with real part $\pi/2$, this means $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+\mathbf{i}\cdot b$ (*i.e.* $a=0$).

COROLLARY 1.5. *For a pure real angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+a$, i.e. $-\frac{\pi}{2}\leq a\leq\frac{\pi}{2}$ and $b=0$, we get a complex angle $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$ with a real part $\pi/2$,*

$$\begin{aligned}\angle(\mathbf{i}\cdot\vec{x},\vec{y}) &= \frac{\pi}{2}+\mathbf{i}\cdot\frac{1}{2}\cdot\operatorname{sgn}(a)\cdot\operatorname{arcosh}\left(2\cdot\cos^2\left(\frac{\pi}{2}+a\right)+1\right)\\ &= \frac{\pi}{2}+\mathbf{i}\cdot\operatorname{sgn}(a)\cdot\log\left[\sqrt{\cos^2\left(\frac{\pi}{2}+a\right)+1}+\left|\cos\left(\frac{\pi}{2}+a\right)\right|\right].\end{aligned}$$

COROLLARY 1.6. *For an angle $\angle(\vec{x},\vec{y})=\frac{\pi}{2}+\mathbf{i}\cdot b$ (i.e. $a=0$) we get a pure real angle $\angle(\mathbf{i}\cdot\vec{x},\vec{y})$, more precisely*

$$\angle(\mathbf{i}\cdot\vec{x},\vec{y}) = \frac{\pi}{2}-\frac{1}{2}\cdot\operatorname{sgn}(b)\cdot\arccos[3-2\cdot\cosh^2(b)]$$

$$= \frac{\pi}{2} - \frac{1}{2} \cdot \operatorname{sgn}(b) \cdot \arccos[2 - \cosh(2 \cdot b)] = \frac{\pi}{2} - \arcsin[\sinh(b)]$$
$$= \arccos[\sinh(b)] \; .$$

Now we are interested in pairs $(\vec{x}, \vec{y})$ with a real valued angle $\angle(\vec{x}, \vec{y})$.

Let us take two vectors $\vec{x}, \vec{y} \neq \vec{0}$ of a complex normed vector space $(X, \|\cdot\|)$. We can prove in Proposition 3.10 that there is a real number $\varphi \in [0, 2\pi]$ such that $(e^{\mathbf{i}\cdot\varphi} \cdot \vec{x}\,, \vec{y})$ has a pure real angle, *i.e.* $\angle(e^{\mathbf{i}\cdot\varphi} \cdot \vec{x}\,, \vec{y}) \in [0, \pi]$.

This fact ensures the existance of many real valued angles in complex normed spaces. The situation improves yet in the special case of complex vector spaces provided with an inner product $< . \,|\, . >$.

The properties of complex inner product spaces $(X, < . \,|\, . >)$ have been studied extensively, and such spaces have many applications in technology and physics.

To do ordinary Euclidean geometry we need real valued angles. The idea is simple. We take an orthogonal basis T of $(X, < . \,|\, . >)$, and we generate the *real span* $\mathcal{L}(\mathbb{R})(\mathsf{T})$, the set of all finite real linear combinations of elements of T. Let

$$\mathcal{L}(\mathbb{R})(\mathsf{T}) := \left\{ \sum_{i=1}^{n} r_i \cdot \vec{x}_i \mid n \in \mathbb{N}, \; r_1, r_2, \ldots, r_n \in \mathbb{R}, \; \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathsf{T} \right\} .$$

If X is a Hilbert space, *i.e.* it is complete, we can even use limits of Cauchy sequences from elements of $\mathcal{L}(\mathbb{R})(\mathsf{T})$, or, in other words, we take the limits of all infinite convergent series of elements of T with real coefficients. It means that we take the closure $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ of $\mathcal{L}(\mathbb{R})(\mathsf{T})$ in X. This creates a real linear subspace $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ of X, where all angles are real:

THEOREM 1.7. *Let $(X, < \cdot \,|\cdot\, >)$ be a complex Hilbert space. Let $\mathsf{T} \subset X$ be an orthonormal system. The set $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ is a real subspace of X, and we get that each pair of non-zero vectors $\vec{y}, \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ has a real angle, i.e. $\angle(\vec{y}, \vec{z}) \in [0, \pi]$.*

Afterwards we consider complex inner product spaces $(X, < . \,|\, . >)$ of finite complex dimension $n \in \mathbb{N}$. Their real dimension is $2 \cdot n$, and we shall state in Corollary 4.4 that the maximal dimension of a real subspace of X with all-real angles is n. The real span $\mathcal{L}(\mathbb{R})(\mathsf{T})$ of an orthogonal basis $\mathsf{T} \subset X$ yields an example.

Finally, we demonstrate two examples of ordinary Euclidean geometry in complex inner product spaces, and to do this we show that real angles are useful.

Note that the concept of this ‘angle’ has been treated first for real normed spaces in [14] and [15].

2. GENERAL DEFINITIONS

Let $X = (X, \tau)$ be an arbitrary complex vector space. This means that the vector space X is provided with a topology τ such that the addition of two vectors and the multiplication with complex numbers are continuous. Further, let $\| \cdot \|$ denote a *norm* on X, *i.e.* $\| \cdot \|$ is a continuous map $X \longrightarrow \mathbb{R}^+ \cup \{0\}$, which fulfils the following axioms $\|z \cdot \vec{x}\| = |z| \cdot \|\vec{x}\|$ ('absolute homogeneity'), $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ ('triangle inequality'), and $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$ ('positive definiteness'), for $\vec{x}, \vec{y} \in X$ and $z \in \mathbb{C}$.

Let $< . | . > : \ X^2 \longrightarrow \mathbb{C}$ be a continuous map from the product space $X \times X$ into the field $\mathbb{C}$. Such a map is called a *product*. We notice the following four properties:

$\overline{(1)}$: For all $z \in \mathbb{C}$ and $\vec{x}, \vec{y} \in X$ it holds $< z \cdot \vec{x} \mid \vec{y} > = \ z \cdot < \vec{x} \mid \vec{y} >$ ('homogeneity').

$\overline{(2)}$: For all $\vec{x}, \vec{y} \in X$ it holds $< \vec{x} \mid \vec{y} > = \overline{< \vec{y} \mid \vec{x} >}$ ('conjugate symmetry').

$\overline{(3)}$: For $\vec{0} \neq \vec{x} \in X$ we get a real number $< \vec{x} | \vec{x} > > \ 0$, ('positive definiteness').

$\overline{(4)}$: For all $\vec{x}, \vec{y}, \vec{z} \in X$ it holds $< \vec{x} | \vec{y} + \vec{z} > = < \vec{x} \mid \vec{y} > + < \vec{x} \mid \vec{z} >$ ('linearity in the second component').

If $< . | . >$ fulfils all properties $\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(4)}$, the map $< . | . >$ is an *inner product* on X. In this case we call the pair $(X, < . | . >)$ a *complex inner product space.*

It is well known that in a complex normed space $(X, \| \cdot \|)$ its norm $\| \cdot \|$ generates an inner product $< . | . >$ by equation (1) if and only if it holds the *parallelogram identity*: For $\vec{x}, \vec{y} \in X$ there is the equation

$$\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2 \cdot \big(\|\vec{x}\|^2 + \|\vec{y}\|^2 \big).$$

In the following section, we need some complex valued functions like the cosine, sine, arccosine, arcsine, etc. We abbreviate them by cos, sin, arccos, arcsin.

In the introduction we have already defined two subset $\mathcal{B}$ and $\mathcal{A}$ of the complex plane $\mathbb{C}$, and we asserted that there exist two homeomorphisms $\cos : \mathcal{A} \stackrel{\cong}{\longrightarrow} \mathcal{B}$ and $\arccos : \mathcal{B} \stackrel{\cong}{\longrightarrow} \mathcal{A}$. Note that $\mathcal{A}$ contains only inner points except two boundary points 0 and π, while $\mathcal{B}$ has 1 and -1.

In the following Definition 2.1 we express the complex functions in detail by known real valued cos, sin, arccos, arcosh, log and exp functions.

Recall the three values of the signum function, $\mathrm{sgn}(0) = 0$, $\mathrm{sgn}(x) = 1$ for real numbers $x > 0$, and $\mathrm{sgn}(x) = -1$ for $x < 0$. We abbreviate the exponential function by $e^x := \exp(x)$. Note that in the next section we prove explicitly that the defined arccosine is truly the inverse function of the cosine. The formulas of the arcsine and arccosine are from [20]. An equivalent definition can be found in [10].

Definition 2.1. For a number $z = a + \mathbf{i} \cdot b \in \mathbb{C}$ its complex cosine and sine can be defined explicitly by

$$\begin{aligned}\cos(a + \mathbf{i} \cdot b) &:= \frac{1}{2} \cdot \left[\cos(a) \cdot \left(e^b + \frac{1}{e^b}\right) - \mathbf{i} \cdot \sin(a) \cdot \left(e^b - \frac{1}{e^b}\right)\right], \\ \sin(a + \mathbf{i} \cdot b) &:= \frac{1}{2} \cdot \left[\sin(a) \cdot \left(e^b + \frac{1}{e^b}\right) + \mathbf{i} \cdot \cos(a) \cdot \left(e^b - \frac{1}{e^b}\right)\right].\end{aligned}$$

For a shorter representation we can use the real hyperbolic cosine and hyperbolic sine. Their abbreviations are the symbols cosh and sinh, their formulas are

$$\cosh(x) := \frac{1}{2} \cdot \left(e^x + \frac{1}{e^x}\right) \quad \text{and} \quad \sinh(x) := \frac{1}{2} \cdot \left(e^x - \frac{1}{e^x}\right), \text{ for } x \in \mathbb{R}.$$

For $r + \mathbf{i} \cdot s \in \mathcal{B}$ we define the functions arcsine and arccosine by

$$\begin{aligned}\arcsin(r + \mathbf{i} \cdot s) &:= \frac{1}{2} \cdot [\, \operatorname{sgn}(r) \cdot \arccos(\mathsf{G}_-) \; + \; \mathbf{i} \cdot \operatorname{sgn}(s) \cdot \operatorname{arcosh}(\mathsf{G}_+) \,], \quad \text{and} \\ \arccos(r + \mathbf{i} \cdot s) &:= \frac{\pi}{2} - \arcsin(r + \mathbf{i} \cdot s).\end{aligned}$$

Here we use the abbreviations G_- and G_+, where

$$\mathsf{G}_- := \sqrt{(r^2+s^2-1)^2+4 \cdot s^2} - (r^2+s^2), \quad \mathsf{G}_+ := \sqrt{(r^2+s^2-1)^2+4 \cdot s^2} + (r^2+s^2).$$

Recall that the symbol arcosh means the real area hyperbolic cosine,

$$\operatorname{arcosh}(x) := \log\left(x + \sqrt{x^2 - 1}\right) \text{ for real numbers } x \geq 1,$$

which is the inverse of the real hyperbolic cosine.

It is useful to mention a few well-known consequences.

We assume the real cosine and sine functions and the complex exp function, all defined by its power series. Since it holds $e^{\mathbf{i} \cdot r} = \cos(r) + \mathbf{i} \cdot \sin(r)$ for real numbers r, with the above Definition 2.1 of the complex sine and cosine we can deduce Euler's formula $e^{\mathbf{i} \cdot z} = \cos(z) + \mathbf{i} \cdot \sin(z)$ for all $z \in \mathbb{C}$. After this, it is easy to prove the identities

$$\cos(z) = \frac{1}{2} \cdot \left[e^{\mathbf{i} \cdot z} + e^{-\mathbf{i} \cdot z}\right] \quad \text{and} \quad \sin(z) = \frac{1}{2 \cdot \mathbf{i}} \cdot \left[e^{\mathbf{i} \cdot z} - e^{-\mathbf{i} \cdot z}\right].$$

We notice the equations $\operatorname{arcosh} \circ \cosh(x) = x$ for real $x \geq 0$, and $\cosh \circ \operatorname{arcosh}(x) = x$ for $x \geq 1$. Further, the complex cosine can be written as

$$\cos(a + \mathbf{i} \cdot b) = \cos(a) \cdot \cosh(b) - \mathbf{i} \cdot \sin(a) \cdot \sinh(b),$$

while the complex sine function is

$$\sin(a + \mathbf{i} \cdot b) = \sin(a) \cdot \cosh(b) + \mathbf{i} \cdot \cos(a) \cdot \sinh(b).$$

3. COMPLEX NORMED SPACES

First, we prove that the cosine and arccosine as they are written in Definition 2.1 are mutually inverse functions.

LEMMA 3.1. *For all $z \in \mathcal{B}$ it holds* $\cos \circ \arccos(z) = z$.

Proof. We take an arbitrary element $z = r + \mathbf{i} \cdot s \in \mathcal{B}$. We use the abbreviations G_- and G_+ of Definition 2.1, and with an easy calculation we get

$$(1 - \mathsf{G}_-) \cdot (\mathsf{G}_+ + 1) = 4 \cdot r^2\,, \quad \text{and} \quad (1 + \mathsf{G}_-) \cdot (\mathsf{G}_+ - 1) = 4 \cdot s^2\,.$$

We assume for $z = r + \mathbf{i} \cdot s \in \mathcal{B}$ that both r and s are positive. We use Definition 2.1 and elementary calculus, and we compute $\cos \circ \arccos(z)$

$$\begin{aligned}
&= \cos\left[\arccos(r + \mathbf{i} \cdot s)\right] = \cos\left[\frac{\pi}{2} - \frac{1}{2} \cdot \arccos(\mathsf{G}_-) - \mathbf{i} \cdot \frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+)\right] \\
&= \cos\left(\frac{\pi}{2} - \frac{1}{2} \cdot \arccos(\mathsf{G}_-)\right) \cdot \cosh\left(-\frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+)\right) \\
&\quad - \mathbf{i} \cdot \sin\left(\frac{\pi}{2} - \frac{1}{2} \cdot \arccos(\mathsf{G}_-)\right) \cdot \sinh\left(-\frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+)\right) \\
&= \sin\left(\frac{1}{2} \cdot \arccos(\mathsf{G}_-)\right) \cdot \cosh\left(\frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+)\right) \\
&\quad + \mathbf{i} \cdot \cos\left(\frac{1}{2} \cdot \arccos(\mathsf{G}_-)\right) \cdot \sinh\left(\frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+)\right) \\
&= \sqrt{\frac{1}{2} \cdot (1 - \mathsf{G}_-)} \cdot \sqrt{\frac{1}{2} \cdot (\mathsf{G}_+ + 1)} + \mathbf{i} \cdot \sqrt{\frac{1}{2} \cdot (1 + \mathsf{G}_-)} \cdot \sqrt{\frac{1}{2} \cdot (\mathsf{G}_+ - 1)} \\
&= \frac{1}{2} \cdot \sqrt{(1 - \mathsf{G}_-) \cdot (\mathsf{G}_+ + 1)} + \mathbf{i} \cdot \frac{1}{2} \cdot \sqrt{(1 + \mathsf{G}_-) \cdot (\mathsf{G}_+ - 1)} = r + \mathbf{i} \cdot s = z.
\end{aligned}$$

The other three cases are $r \cdot s = 0$, $r \cdot s < 0$, and both r and s are negative. By noting the signs, they work in the same manner, and the lemma is established. $\square$

PROPOSITION 3.2. *We have two identities* $\cos \circ \arccos = id_{\mathcal{B}}$ *and* $\arccos \circ \cos = id_{\mathcal{A}}$. *Further, both functions are homeomorphisms, where*

$$\cos : \mathcal{A} \xrightarrow{\cong} \mathcal{B} \quad \textit{and} \quad \arccos : \mathcal{B} \xrightarrow{\cong} \mathcal{A}\,.$$

Proof. By Definition 2.1 we have two continuous maps, $\cos : \mathcal{A} \to \mathcal{B}$ and $\arccos : \mathcal{B} \to \mathcal{A}$. We just proved $\cos \circ \arccos(z) = z$ for all $z \in \mathcal{B}$. If we consider all possible cases for a and b, for $a+\mathbf{i}\cdot b \in \mathcal{A}$, we see that the cosine is injective on its domain $\mathcal{A}$. It follows that the cosine function is a bijective map $\mathcal{A} \to \mathcal{B}$. We have $\cos \circ \arccos = id_{\mathcal{B}}$. Therefore, it holds $\arccos = \cos^{-1} \circ id_{\mathcal{B}} = \cos^{-1}$, and we get that the arccosine is also a bijective map and the inverse of the cosine. Both functions are continuous, hence they are both homeomorphisms. □

We describe now that the cosine and arccosine functions respect the 'center points' $\pi/2$ of $\mathcal{A}$ and 0 of $\mathcal{B}$, respectively. Note that for each complex number z we can write $z = \frac{\pi}{2} + a + \mathbf{i}\cdot b$, with a suitable real number a. It means that the real part of z is $\frac{\pi}{2} + a$.

PROPOSITION 3.3. *For each complex number z in $\mathcal{A}$ we write $z = \frac{\pi}{2} + a + \mathbf{i}\cdot b$, with a suitable real number $-\frac{\pi}{2} \le a \le \frac{\pi}{2}$. If $\cos\left(\frac{\pi}{2} + a + \mathbf{i}\cdot b\right) = r + \mathbf{i}\cdot s$ we have*

$$\cos\left(\frac{\pi}{2} - a - \mathbf{i}\cdot b\right) \;=\; -r - \mathbf{i}\cdot s\ .$$

Correspondingly, for a number $r + \mathbf{i}\cdot s \in \mathcal{B}$ with $\arccos(r + \mathbf{i}\cdot s) = \frac{\pi}{2} + a + \mathbf{i}\cdot b$, it holds $\arccos(-r - \mathbf{i}\cdot s) \;=\; \frac{\pi}{2} - a - \mathbf{i}\cdot b$.

Proof. For the arccosine we see this immediately from Definition 2.1. Note that both the real arccosine and the real area hyperbolic cosine have a non-negative image. Since the cosine is the inverse function of the arccosine, it must act as it is claimed in this proposition. □

Let $(X, \|\cdot\|)$ be a complex normed vector space. In Definition 1.1 we have defined a continuous product $<.\,|\,.>$ on X. In the introduction we already mentioned that this is the inner product in the case that the norm $\|\cdot\|$ generates an inner product by equation (1). We have also noticed some properties of this product. We will discuss them now.

PROPOSITION 3.4. *For all vectors $\vec{x}, \vec{y} \in (X, \|\cdot\|)$ and for real numbers r the product $<.\,|\,.>$ has the following properties.*

(a) $<\vec{x}\,|\,\vec{y}> \;=\; \overline{<\vec{y}\,|\,\vec{x}>}$ *(conjugate symmetry),*

(b) $<\vec{x}\,|\,\vec{x}> \;\ge\; 0$, *and* $<\vec{x}\,|\,\vec{x}> = 0$ *only for* $\vec{x} = \vec{0}$ *(positive definiteness),*

(c) $<r\cdot\vec{x}\,|\,\vec{y}> \;=\; r\cdot <\vec{x}\,|\,\vec{y}> \;=\; <\vec{x}\,|\,r\cdot\vec{y}>$ *(homogeneity for real numbers),*

(d) $<\mathbf{i}\cdot\vec{x}\,|\,\vec{y}> \;=\; \mathbf{i}\cdot <\vec{x}\,|\,\vec{y}> \;=\; <\vec{x}\,|\,-\mathbf{i}\cdot\vec{y}>$ *(homogeneity for the imaginary unit),*

(e) $\|\vec{x}\| = \sqrt{<\vec{x}\,|\,\vec{x}>}$ *(the norm can be expressed by the product).*

Proof. The proof for (a) is easy, and (b) is trivial. For a positive number r, the point (c) is trivial. After that, we can show $<-\vec{x}\,|\,\vec{y}> = -<\vec{x}\,|\,\vec{y}>$, and (c) follows immediately. The point (d) is similar to (c), and (e) is clear. □

In Definition 1.2 we defined the angle $\angle(\vec{x},\vec{y})$, we wrote

$$\angle(\vec{x},\vec{y}) := \arccos\left(\frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}\right) = \arccos\left(\frac{1}{4}\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|}+\frac{\vec{y}}{\|\vec{y}\|}\right\|^2-\left\|\frac{\vec{x}}{\|\vec{x}\|}-\frac{\vec{y}}{\|\vec{y}\|}\right\|^2 + \mathbf{i}\cdot\left(\left\|\frac{\vec{x}}{\|\vec{x}\|}+\mathbf{i}\cdot\frac{\vec{y}}{\|\vec{y}\|}\right\|^2-\left\|\frac{\vec{x}}{\|\vec{x}\|}-\mathbf{i}\cdot\frac{\vec{y}}{\|\vec{y}\|}\right\|^2\right)\right]\right).$$

This complex number $\angle(\vec{x},\vec{y})$ is called the *complex angle* of the pair $(\vec{x},\vec{y})$.

LEMMA 3.5. *For a pair $\vec{x},\vec{y}\neq\vec{0}$ in a complex normed space $(X,\|\cdot\|)$ it holds that both the real part and the imaginary part of* $\cos(\angle(\vec{x},\vec{y}))$ *are in the interval* $[-1,1]$.

Proof. The lemma can be proven easily with the triangle inequality. □

COROLLARY 3.6. *Lemma 3.5 means that* $\left\{\cos(\angle(\vec{x},\vec{y}))\mid \vec{x},\vec{y}\in X\backslash\{\vec{0}\}\right\}$ *is a subset of the 'complex unit square'* $\mathcal{CUS} := \{r+\mathbf{i}\cdot s\in\mathbb{C}\mid -1\leq r,s\leq +1\}\subset\mathcal{B}$. *Hence* $\cos(\angle(\vec{x},\vec{y}))\in\mathcal{B}$ *and* $\angle(\vec{x},\vec{y})\in\mathcal{A}$, *i.e. the angle* $\angle(\vec{x},\vec{y})$ *is defined for each pair* $\vec{x},\vec{y}\neq\vec{0}$. *Further, it holds* $|\cos(\angle(\vec{x},\vec{y}))| \leq \sqrt{2}$.

COROLLARY 3.7. *The values of the 'angles' are from the image* $\arccos(\mathcal{CUS})$, *which forms a symmetric hexagon (with concave sides) in $\mathcal{A}$. Its center is $\pi/2$, two corners are 0 and π. A third corner is* $\arccos(1+\mathbf{i}) =$

$$(\pi/2)-(1/2)\cdot\left[\arccos\left(\sqrt{5}-2\right)+\mathbf{i}\cdot\log\left(\sqrt{5}+2+2\cdot\sqrt{\sqrt{5}+2}\right)\right]\approx 0.90-\mathbf{i}\cdot 1.06.$$

Noting the signs in $\pm1\pm\mathbf{i}\in\mathcal{CUS}$, the other corners can be computed easily with Proposition 3.3 and Definition 2.1, respectively.

In the introduction we stated Theorem 1.3. Here we catch up on its proof:

Proof. The property ($\texttt{An}$ 1) is a consequence of Lemma 3.5. The following two corollaries say that the angle is always defined, and the image of the map $\angle$ is in $\mathcal{A}$. With Proposition 3.3, we get that it is symmetric to $\pi/2$.

The five points ($\texttt{An}$ 2) - ($\texttt{An}$ 6) are rather trivial. We use properties of the product $<\cdot\,|\cdot>$ from Proposition 3.4, and properties of the arccosine.

For the next point ($\texttt{An}$ 7) we have to prove $\angle(\vec{x},\vec{y})+\angle(-\vec{x},\vec{y}) = \pi$, for $\vec{x},\vec{y}\neq\vec{0}$. We use Proposition 3.3 and $<-\vec{x}\,|\,\vec{y}> = -<\vec{x}\,|\,\vec{y}>$ from Proposition 3.4. If $\angle(\vec{x},\vec{y}) = \arccos(r+\mathbf{i}\cdot s) = \frac{\pi}{2}+a+\mathbf{i}\cdot b$, we have $\angle(-\vec{x},\vec{y}) = \arccos(-r-\mathbf{i}\cdot s) = \frac{\pi}{2}-a-\mathbf{i}\cdot b$. It holds ($\texttt{An}$ 7).

To prove the last point ($\texttt{An}$ 8) we use [15]. We take the two linear independent vectors $\vec{x},\vec{y}$, and we generate the set of all its real linear combinations

$$\mathsf{U} := \{r\cdot\vec{x}+s\cdot\vec{y}\mid r,s\in\mathbb{R}\}.$$

This set U is a real subspace of X with the real dimension two.

Instead of $\angle(\vec{x}, \vec{y} + t \cdot \vec{x})$ we consider the real part of the complex number $\cos(\angle(\vec{x}, \vec{y} + t \cdot \vec{x})) \in \mathcal{B}$. We define the real valued map $\widetilde{\Theta}$,

$$\widetilde{\Theta}(t) \ := \ \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y} + t \cdot \vec{x}}{\|\vec{y} + t \cdot \vec{x}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y} + t \cdot \vec{x}}{\|\vec{y} + t \cdot \vec{x}\|} \right\|^2 \right], \quad \text{for } t \in \mathbb{R} \ .$$

By the triangle inequality we can regard this as a map $\widetilde{\Theta} : \mathbb{R} \to [-1, 1]$. Since U is a real normed space we can apply [15]. The main theorem there states that the map $\widetilde{\Theta}$ is an increasing homeomorphism onto the open interval $(-1, 1) \subset \mathcal{B}$. Since the complex arccosine function is a homeomorphism with domain $\mathcal{B}$ and Codomain $\mathcal{A}$, the first claim of (`An` 8) is true.

The limits $\lim_{t\to-\infty} \widetilde{\Theta}(t) = -1$ and $\lim_{t\to\infty} \widetilde{\Theta}(t) = 1$ are mentioned in the proof of the theorem in [15], or we can find one directly in [2], which was the main source of [15]. We use the arccosine, and we get the demanded limits of $\Theta(t)$ in (`An` 8), since the imaginary parts of $\Theta(t)$ vanish if t turns to $\pm\infty$. $\square$

Lemma 3.8. *In a complex normed vector space* $(X, \|\cdot\|)$ *we take two vectors* $\vec{x}, \vec{y} \neq \vec{0}$. *It holds*

$$|\cos(\angle(\vec{x}, \vec{y}))| = |\cos(\angle(-\vec{x}, \vec{y}))| = |\cos(\angle(\mathbf{i} \cdot \vec{x}, \vec{y}))| = |\cos(\angle(\vec{x}, \mathbf{i} \cdot \vec{y}))|.$$

Proof. This fact follows easily with Proposition 3.4. $\square$

In the next proof, the values in Table 1 from the introduction will be discussed.

Again we assume two elements $\vec{x}, \vec{y} \neq \vec{0}$ of a complex normed space $(X, \|\cdot\|)$. Note that in Table 1 the identities $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a + \mathbf{i} \cdot b$ and $\cos(\angle(\vec{x}, \vec{y})) = r + \mathbf{i} \cdot s$ are by definition, as well the expression $\angle(\mathbf{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + \mathbf{i} \cdot w$.

Proof. The last column of Table 1 comes directly from Definition 1.2, e.g. $\cos(\angle(\vec{x}, \vec{x})) = \, < \vec{x} | \vec{x} > / \|\vec{x}\|^2$, etc. We get the values of the fourth column by using the fifth column and applying the arccosine function from Definition 2.1. The other entries of the table have to be calculated.

We show, for instance, that $\angle(\mathbf{i} \cdot \vec{y}, \vec{x}) = \frac{\pi}{2} - v + \mathbf{i} \cdot w$, and $\cos(\angle(\mathbf{i} \cdot \vec{y}, \vec{x})) = s + \mathbf{i} \cdot r$, please see the entries in the bottom row of Table 1. We compute

$$\angle(\mathbf{i} \cdot \vec{y}, \vec{x}) \ = \ \arccos\left(\frac{< \mathbf{i} \cdot \vec{y} \,|\, \vec{x} >}{\|\mathbf{i} \cdot \vec{y}\| \cdot \|\vec{x}\|} \right) \ = \ \arccos\left(\mathbf{i} \cdot \frac{\overline{< \vec{x} \,|\, \vec{y} >}}{\|\vec{y}\| \cdot \|\vec{x}\|} \right),$$

hence it follows

$$\text{(2)} \quad \cos(\angle(\mathbf{i} \cdot \vec{y}, \vec{x})) = \mathbf{i} \cdot \frac{\overline{< \vec{x} \,|\, \vec{y} >}}{\|\vec{y}\| \cdot \|\vec{x}\|} = \mathbf{i} \cdot \overline{\cos(\angle(\vec{x}, \vec{y}))} = \mathbf{i} \cdot \overline{(r + \mathbf{i} \cdot s)} = s + \mathbf{i} \cdot r.$$

With a similar argumentation we get

$$\cos(\angle(\mathbf{i} \cdot \vec{x}, \vec{y})) = -s + \mathbf{i} \cdot r \ .$$

We had defined $\angle(\mathbf{i}\cdot\vec{x},\vec{y}) = \frac{\pi}{2}+v+\mathbf{i}\cdot w$, hence it follows $\frac{\pi}{2}+v+\mathbf{i}\cdot w = \arccos(-s+\mathbf{i}\cdot r)$. We use the arccosine function from Definition 2.1, and by the different signs of s we can deduce $\arccos(s+\mathbf{i}\cdot r) = \frac{\pi}{2}-v+\mathbf{i}\cdot w$. With equation (2) we get $\angle(\mathbf{i}\cdot\vec{y},\vec{x}) = \frac{\pi}{2}-v+\mathbf{i}\cdot w$, and the bottom row of Table 1 is shown. The other rows can be proven with similar considerations. $\square$

With the help of the table, and the properties of the product $<\cdot|\cdot>$ from Proposition 3.4, and Proposition 3.3, we can conclude other values. For instance we get

$$\angle(-\mathbf{i}\cdot\vec{y},\vec{x}) = \frac{\pi}{2}+v-\mathbf{i}\cdot w = \angle(\vec{y},\mathbf{i}\cdot\vec{x}), \text{ and } \angle(\vec{x},-\vec{y}) = \frac{\pi}{2}-a-\mathbf{i}\cdot b = \angle(-\vec{x},\vec{y}).$$

Now we refer to Table 1 again. There we had assumed $\angle(\vec{x},\vec{y}) = \frac{\pi}{2}+a+\mathbf{i}\cdot b$. We want to express $\cos(\angle(\vec{x},\vec{y})) = r+\mathbf{i}\cdot s$ in coordinates of a and b, and we get at once from Definition 2.1 the number $r+\mathbf{i}\cdot s = \cos\left(\frac{\pi}{2}+a+\mathbf{i}\cdot b\right)$

$$= \frac{1}{2}\cdot\left[\cos\left(\frac{\pi}{2}+a\right)\cdot\left(e^b+\frac{1}{e^b}\right)-\mathbf{i}\cdot\sin\left(\frac{\pi}{2}+a\right)\cdot\left(e^b-\frac{1}{e^b}\right)\right].$$

To express the complex number $\angle(\mathbf{i}\cdot\vec{x},\vec{y}) = \frac{\pi}{2}+v+\mathbf{i}\cdot w$ in dependence of a and b requires more effort. In Theorem 1.4 we already have presented the result. Now we need to prove Theorem 1.4.

Proof. Here we also use the real sine and hyperbolic sine functions, abbreviated by sin and sinh, please see Definition 2.1. To shorten the representation of the proof it is useful to introduce more abbreviations.

$$\text{Let } \mathsf{cpi2a} := \cos\left(\frac{\pi}{2}+a\right),\ \mathsf{spi2a} := \sin\left(\frac{\pi}{2}+a\right).$$

From Definition 2.1 we have

$$r+\mathbf{i}\cdot s = \cos(\angle(\vec{x},\vec{y})) = \mathsf{cpi2a}\cdot\cosh(b)-\mathbf{i}\cdot\mathsf{spi2a}\cdot\sinh(b),$$

and we have two real numbers $r = \mathsf{cpi2a}\cdot\cosh(b)$ and $s = -\mathsf{spi2a}\cdot\sinh(b)$. Note $-\frac{\pi}{2}\leq a\leq\frac{\pi}{2}$. Since $\frac{\pi}{2}+a+\mathbf{i}\cdot b\in\mathcal{A}$ it follows from the special cases $a=-\frac{\pi}{2}$ or $a=\frac{\pi}{2}$ that the imaginary part b vanished, *i.e.* $0 = b = s = \mathsf{spi2a} = \sinh(b)$.

We get from Table 1 and from Definition 2.1

$$\begin{aligned}
\angle(\mathbf{i}\cdot\vec{x},\vec{y}) &= \frac{\pi}{2}+v+\mathbf{i}\cdot w = \arccos[-s+\mathbf{i}\cdot r]\\
&= \arccos[\,\mathsf{spi2a}\cdot\sinh(b)+\mathbf{i}\cdot(\mathsf{cpi2a}\cdot\cosh(b))\,]\\
&= \frac{\pi}{2}-\arcsin[\,\mathsf{spi2a}\cdot\sinh(b)+\mathbf{i}\cdot(\mathsf{cpi2a}\cdot\cosh(b))\,]\\
&= \frac{\pi}{2}-\frac{1}{2}\cdot[\,\mathrm{sgn}(\mathsf{spi2a}\cdot\sinh(b))\cdot\arccos(\mathsf{K}_-)+\mathbf{i}\cdot\mathrm{sgn}(\mathsf{cpi2a}\cdot\cosh(b))\cdot\mathrm{arcosh}(\mathsf{K}_+)\,]\\
&= \frac{\pi}{2}-\frac{1}{2}\cdot[\,\mathrm{sgn}(b)\cdot\arccos(\mathsf{K}_-)+\mathbf{i}\cdot\mathrm{sgn}(-a)\cdot\mathrm{arcosh}(\mathsf{K}_+)\,],
\end{aligned}$$

with the abbreviations K_- and K_+,

$$\mathsf{K}_\pm := \sqrt{\left[\mathsf{spi2a}^2 \cdot \sinh^2(b) + \mathsf{cpi2a}^2 \cdot \cosh^2(b) - 1\right]^2 + 4 \cdot \mathsf{cpi2a}^2 \cdot \cosh^2(b)} \\ \pm \left[\mathsf{spi2a}^2 \cdot \sinh^2(b) + \mathsf{cpi2a}^2 \cdot \cosh^2(b)\right] .$$

With the aid of the well-known equations

$$\sin^2(x) + \cos^2(x) \;=\; 1 \;=\; \cosh^2(x) - \sinh^2(x)$$

we finally reach the identities $\mathsf{H}_- = \mathsf{K}_-$ and $\mathsf{H}_+ = \mathsf{K}_+$, which was the last step to prove Theorem 1.4. □

Now we prove Corollary 1.5. We could use the just proven theorem, but we show it more elaborate.

Proof. Since $b = 0$ we have $\sinh(b) = 0$ and $s = 0$, and $\cosh(b) = 1$. It follows

$$\angle(\mathbf{i} \cdot \vec{x}, \vec{y}) \;=\; \frac{\pi}{2} + v + \mathbf{i} \cdot w \;=\; \arccos(\mathbf{i} \cdot r) \;=\; \arccos(\mathbf{i} \cdot \mathsf{cpi2a} \cdot \cosh(b))$$
$$= \frac{\pi}{2} - \frac{1}{2} \cdot [\mathbf{i} \cdot \operatorname{sgn}(-a) \cdot \operatorname{arcosh}(\mathsf{K}_+)] = \frac{\pi}{2} + \frac{1}{2} \cdot \mathbf{i} \cdot \operatorname{sgn}(a) \cdot \operatorname{arcosh}\left[2 \cdot \cos^2\left(\frac{\pi}{2} + a\right) + 1\right] .$$

Please see the definition of the arcosh function in Definition 2.1, and note that the equation

$$\log\left[\sqrt{\cos^2(x) + 1} + |\cos(x)|\right] = \frac{1}{2} \cdot \log\left[2 \cdot \cos^2(x) + 1 + 2 \cdot |\cos(x)| \cdot \sqrt{\cos^2(x) + 1}\right]$$

holds for all real numbers x, which concludes the proof of Corollary 1.5. □

For the proof of the next corollary we need a lemma.

Lemma 3.9. *In the case of* $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + \mathbf{i} \cdot b$, *i.e.* $a = 0$, *the range of* b *is*

$$-\log\left(\sqrt{2} + 1\right) \;\leq\; b \;\leq\; +\log\left(\sqrt{2} + 1\right) \;\approx\; 0.88.$$

Proof. By Lemma 3.5, there is a suitable $-1 \leq s \leq +1$ with $\cos\left(\frac{\pi}{2} + \mathbf{i} \cdot b\right) = \mathbf{i} \cdot s$. Using the arccosine of Definition 2.1, it follows for the modulus of b

$$|b| = \frac{1}{2} \cdot \operatorname{arcosh}(\mathsf{G}_+) = \frac{1}{2} \cdot \operatorname{arcosh}\left(2 \cdot s^2 + 1\right) = \frac{1}{2} \cdot \log\left(2 \cdot s^2 + 1 + 2 \cdot |s| \cdot \sqrt{s^2 + 1}\right),$$

and note $\sqrt{3 + 2 \cdot \sqrt{2}} \;=\; \sqrt{2} + 1$, and the lemma is proven. □

We add the proof of Corollary 1.6.

Proof. We apply Theorem 1.4, and since $a = 0$ we get

$$\angle(\mathbf{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + \frac{1}{2} \cdot [-\operatorname{sgn}(b) \cdot \arccos(\mathsf{H}_-)],$$

$$\text{with } \mathsf{H}_- = \sqrt{\left[\cosh^2(b) - 2\right]^2} - \left[\cosh^2(b) - 1\right] .$$

A consequence of Lemma 3.9 is the fact $\cosh^2(b) \leq 2$, it follows

$$\mathsf{H}_- \;=\; \big[2-\cosh^2(b)\big] - \big[\cosh^2(b)-1\big] \;=\; 3 - 2\cdot\cosh^2(b)\ ,$$

and the first line of Corollary 1.6 is proven. The next line is some calculus. $\square$

Up to now we had defined for each complex normed space X an 'angle' which generally has complex values. The geometrical meaning of a complex angle is unclear. To do the usual known 'Euclidean' geometry we need real valued angles. During the following consideration it turns out that although we deal with complex vector spaces, actually 'a lot' of our angles are purely real. The situation will even improve in inner product spaces, which will be investigated in the next section.

PROPOSITION 3.10. *Let us take two vectors $\vec{x},\vec{y} \neq \vec{0}$ from a complex normed vector space $(X,\|\cdot\|)$. It holds that there is at least one real number $\varphi \in [0,2\pi]$ such that the product $< e^{\mathbf{i}\cdot\varphi}\cdot\vec{x}\,|\,\vec{y} >$ is real, or equivalently $\angle(e^{\mathbf{i}\cdot\varphi}\cdot\vec{x}\,,\,\vec{y}) \in \mathbb{R}$.*

Of course, the parts of $\vec{x}$ and $\vec{y}$ can be exchanged. The proposition means that for $\vec{x},\vec{y} \neq \vec{0}$ we have to 'twist' either $\vec{x}$ or $\vec{y}$ by a suitable complex factor $e^{\mathbf{i}\cdot\varphi}$ to generate a pure real angle in $[0,\pi]$.

Proof. Please see both Proposition 3.3 and Proposition 3.4. Let us assume a complex angle

$$\angle(\vec{x},\vec{y}) \;=\; \arccos\left(\frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}\right) \;=\; \frac{\pi}{2}+a+\mathbf{i}\cdot b\ \in\mathcal{A},\quad \text{with } b\neq 0.$$

From Proposition 3.4 we have $< -\vec{x}\,|\,\vec{y} > \;=\; -<\vec{x}\,|\,\vec{y}>$, *i.e.* with Proposition 3.3 it follows

$$\angle(-\vec{x},\vec{y}) \;=\; \arccos\left(\frac{<-\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}\right) \;=\; \frac{\pi}{2}-a-\mathbf{i}\cdot b.$$

We know $e^{\mathbf{i}\cdot\pi} = -1$. The set $Oval(\vec{x},\vec{y}) := \big\{\angle(e^{\mathbf{i}\cdot\varphi}\cdot\vec{x},\vec{y}) \mid \varphi\in[0,2\pi]\big\} \subset \mathcal{A}$ is the continuous image of the complex unit circle $\big\{e^{\mathbf{i}\cdot\varphi} \mid \varphi\in[0,2\pi]\big\}$, or the interval $[0,2\pi]$, respectively, therefore it has to be connected. This means that $Oval(\vec{x},\vec{y})$ is connected, *i.e.* it must cross the real axis. $\square$

Let's turn to inner product spaces.

4. COMPLEX INNER PRODUCT SPACES

In the introduction we have constructed in Definition 1.1 a continous product for all complex normed spaces $(X,\|\cdot\|)$. There we have already mentioned that in a case of an inner product space $(X,<\cdot\,|\,\cdot>)$ the product from Definition 1.1 coincides with the given inner product $<\cdot\,|\,\cdot>$, which can be written as in equation (1). For all complex normed spaces $(X, \|\cdot\|)$ we have introduced the 'angle' $\angle$ in Definition 1.2. Now we investigate its properties in the special case of an inner product space $(X, <\cdot\,|\,\cdot>)$.

We deal with complex vector spaces X provided with an inner product $<\cdot\,|\,\cdot>$, *i.e.* it has the properties $\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(4)}$. The definition $\|\vec{x}\| := \sqrt{<\vec{x}\,|\vec{x}>}$ generates a norm $\|\cdot\|$, which means that the pair $(X, \|\cdot\|)$ is a normed space, and its angles $\angle$ have at least all properties which have been developed in the previous section. The special conditions of an inner product space open up more possibilities, which we will explore now.

In the previous section we dealt with complex normed spaces, and we ensure by Proposition 3.10 the existence of many pairs of elements with real angles. Now we want to expand on this. Roughly spoken, we seek for subsets $\mathsf{U} \subset X$, such that U is a real subspace of the complex vector space X, and in addition all products in U are real, *i.e.* all angles in U are real.

First we take a second look at Proposition 3.10 and its proof. For two elements $\vec{x}, \vec{y}$ of a complex normed space $(X, \|\cdot\|)$ we know from Proposition 3.10 that there is at least one $\varphi \in [0, 2\pi]$ such that the angle of the pair $(e^{\mathbf{i}\cdot\varphi}\cdot\vec{x}\,,\,\vec{y})$ is real.

LEMMA 4.1. *Let us take two arbitrary vectors $\vec{x}, \vec{y}$ from an inner product space $(X, <\cdot\,|\,\cdot>)$ with $<\vec{x}\,|\vec{y}>\neq 0$. It holds that there exists one number $0 \le \varphi < 2\cdot\pi$ and a suitable number $0 < a \le \frac{\pi}{2}$ such that the set $Oval(\vec{x}, \vec{y})$ has exactly two real angles*

$$\angle\left(e^{\mathbf{i}\cdot\varphi}\cdot\vec{x}\,,\,\vec{y}\right) \;=\; \frac{\pi}{2}+a, \quad \angle\left(e^{\mathbf{i}\cdot(\varphi+\pi)}\cdot\vec{x},\,\vec{y}\right) \;=\; \frac{\pi}{2}-a\,.$$

Proof. For an inner product $<\cdot\,|\,\cdot>$ it holds $<e^{\mathbf{i}\cdot\varphi}\cdot\vec{x}\,|\,\vec{y}> \;=\; e^{\mathbf{i}\cdot\varphi}\cdot<\vec{x}\,|\,\vec{y}>$, hence the cosines $\cos(Oval(\vec{x},\vec{y})) \subset \mathcal{B}$ shape an Euclidean circle with radius $|<\vec{x}\,|\,\vec{y}>|\,/(\|\vec{x}\|\cdot\|\vec{y}\|)$. We map this circle with the arccosine function, and by Proposition 3.3 the image $Oval(\vec{x},\vec{y}) \subset \mathcal{A}$ is symmetrical to $\pi/2$, it crosses the real axis exactly two times. (Note that $Oval(\vec{x},\vec{y})$ is no Euclidean circle.) $\Box$

Albeit we deal with complex inner product spaces we are interested in real subspaces. In a complex normed space $(X, \|\cdot\|)$ let $\mathsf{U} \neq \emptyset$ be any non-empty subset of X. We define $\mathcal{L}(\mathbb{R})(\mathsf{U})$ as the set of all finite real linear combinations of elements from U, while $\mathcal{L}(\mathbb{C})(\mathsf{U})$ is the set of complex linear combinations. The formula is

$$\mathcal{L}(\mathbb{R})(\mathsf{U}) \;:=\; \left\{\sum_{i=1}^{n} r_i\cdot\vec{x}_i \mid n\in\mathbb{N},\; r_1,r_2,\ldots,r_n\in\mathbb{R},\; \vec{x}_1,\vec{x}_2,\ldots,\vec{x}_n\in\mathsf{U}\right\}.$$

The definition of $\mathcal{L}(\mathbb{C})(\mathsf{U})$ is similar, but we use complex numbers instead of real numbers $r_1, r_2, r_3, \ldots$. This definitions mean that $\mathcal{L}(\mathbb{C})(\mathsf{U})$ is a $\mathbb{C}$-linear subspace of the complex vector space X, while $\mathcal{L}(\mathbb{R})(\mathsf{U})$ is a real linear subspace of X, which is also a real vector space.

For both spaces we regard the closure in X. Let $\overline{\mathcal{L}(\mathbb{R})(\mathsf{U})}$ and $\overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$ are the closures of $\mathcal{L}(\mathbb{R})(\mathsf{U})$ and $\mathcal{L}(\mathbb{C})(\mathsf{U})$, respectively. Of course, for a finite set $\mathsf{U} \subset X$ it holds $\mathcal{L}(\mathbb{R})(\mathsf{U}) = \overline{\mathcal{L}(\mathbb{R})(\mathsf{U})}$ and $\mathcal{L}(\mathbb{C})(\mathsf{U}) = \overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$. If we assume an infinite set U, an element $\vec{y} \in X$ belongs to $\overline{\mathcal{L}(\mathbb{R})(\mathsf{U})}$ if and only if there is a countable set $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots\} \subset \mathsf{U}$ and there are real numbers $r_1, r_2, r_3, r_4, \ldots$ such that

$$(3) \qquad \lim_{k\to\infty} \left\| \vec{y} - \sum_{i=1}^{k} r_i \cdot \vec{x}_i \right\| = 0 \; . \quad \text{We can write } \; \vec{y} \; = \; \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i \; .$$

The set $\overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$ is constructed similarly, but we can use complex numbers $z_1, z_2, z_3, \ldots$. Again we get two subspaces of X, $\overline{\mathcal{L}(\mathbb{R})(\mathsf{U})}$ is a real subspace, while $\overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$ is a complex subspace. We have inclusions $\mathcal{L}(\mathbb{R})(\mathsf{U}) \subset \overline{\mathcal{L}(\mathbb{R})(\mathsf{U})}$, and $\mathcal{L}(\mathbb{C})(\mathsf{U}) \subset \overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$, respectively, and generally both inclusions are proper. Note $\overline{\mathcal{L}(\mathbb{R})(\mathsf{U})} \subset \overline{\mathcal{L}(\mathbb{C})(\mathsf{U})}$.

In a complex inner product space $(X, < \cdot | \cdot >)$ we can use the well-known theory of orthogonal systems. Details about this topic can be found in [11] or [19], or in many other books about functional analysis.

Definition 4.2. Let $(X, < \cdot | \cdot >)$ be a complex Hilbert space, *i.e.* it is a complex inner product space which is complete. A subset $\emptyset \neq \mathsf{T} \subset X$ is called an orthonormal system if and only if for each pair of distinct elements $\vec{x}, \vec{y} \in \mathsf{T}$ it holds $< \vec{x} \,|\, \vec{y} >= 0$, and all $\vec{x} \in \mathsf{T}$ are unit vectors, *i.e.* $\|\vec{x}\| = 1$.

An orthonormal system T is called an orthonormal basis if and only if T is maximal. This means that if there is a second orthonormal system V with $\mathsf{T} \subset \mathsf{V}$, it has to be $\mathsf{T} = \mathsf{V}$.

Note that generally an orthonormal basis of X is not a vector space basis of X.

Each unit vector $\vec{x}$ provides an orthonormal system $\{\vec{x}\}$. It is well known that there is an orthonormal basis T with $\{\vec{x}\} \subset \mathsf{T} \subset X$. This shows that there are orthonormal bases in all Hilbert spaces $X \neq \{\vec{0}\}$. Further note that an orthonormal system $\mathsf{T} \subset X$ is an orthonormal basis in $\overline{\mathcal{L}(\mathbb{C})(\mathsf{T})}$.

In Theorem 1.7 we already have described that the real subspace $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ of the complex inner product space $(X, < \cdot | \cdot >)$ has pure real angles. Here is the proof of Theorem 1.7.

Proof. The set $\mathcal{L}(\mathbb{R})(\mathsf{T})$ is a real subspace of X by construction, and it has a vector space basis T. The real vector space $\mathcal{L}(\mathbb{R})(\mathsf{T})$ has the closure $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$

in X. For $\vec{y}, \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ we know that $\vec{y}$ and $\vec{z}$ have representations as it is shown in statement (3). It means that there is a countable set $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots\} \subset \mathsf{T}$ and two sequences $r_1, r_2, r_3, \ldots$ and $s_1, s_2, s_3, \ldots$ of real numbers such that $\vec{y} = \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i$ and $\vec{z} = \sum_{i=1}^{\infty} s_i \cdot \vec{x}_i$. Then it is easy to varify $\vec{y} + \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ and also $r \cdot \vec{y} \in \overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$, for each $r \in \mathbb{R}$. This shows that $\overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$ is a real subspace of X.

Now we want to prove $\angle(\vec{y}, \vec{z}) \in \mathbb{R}$ in the case of two non-zero vectors $\vec{y}, \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(\mathsf{T})}$. We compute the cosine of the angle of the pair $(\vec{y}, \vec{z})$.

$$
\begin{aligned}
\cos(\angle(\vec{y}, \vec{z})) &= \cos\left(\arccos\left(\frac{<\vec{y} \mid \vec{z}>}{\|\vec{y}\| \cdot \|\vec{z}\|}\right)\right) = \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot <\vec{y} \mid \vec{z}> \\
&= \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot \left(< \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i \mid \sum_{j=1}^{\infty} s_j \cdot \vec{x}_j > \right) \\
&= \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot \lim_{k \to \infty} \left(\sum_{i=1}^{k} \sum_{j=1}^{k} r_i \cdot s_j \cdot < \vec{x}_i \mid \vec{x}_j > \right) \\
&= \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot \lim_{k \to \infty} \left(\sum_{i=1}^{k} r_i \cdot s_i \cdot < \vec{x}_i \mid \vec{x}_i > \right) \quad (\mathsf{T} \text{ is orthonormal}) \\
&= \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot \lim_{k \to \infty} \left(\sum_{i=1}^{k} r_i \cdot s_i \right) = \frac{1}{\|\vec{y}\| \cdot \|\vec{z}\|} \cdot \sum_{i=1}^{\infty} r_i \cdot s_i .
\end{aligned}
$$

Since $\cos(\angle(\vec{y}, \vec{z}))$ is an element of $\mathbb{C}$, it is clear that the last infinite series $\sum_{i=1}^{\infty} r_i \cdot s_i$ is convergent. We get $\cos(\angle(\vec{y}, \vec{z})) = (\sum_{i=1}^{\infty} r_i \cdot s_i) \,/\, (\|\vec{y}\| \cdot \|\vec{z}\|)$, and we confirm that indeed the cosine of $\angle(\vec{y}, \vec{z})$ is real, this means that the angle $\angle(\vec{y}, \vec{z})$ is real,too. The proof of Theorem 1.7 is finished. $\square$

The above proof demonstrates the existence of many real angles in each complex inner product space. The following statement is in the opposite direction. It shows that the subsets of $(X, < \cdot \mid \cdot >)$ which have only real angles cannot be 'arbitrarily large'. We formulate this as a precise statement.

Proposition 4.3. *Let $(X, < \cdot \mid \cdot >)$ be a complex inner product space. Let $n \in \mathbb{N}$ be a natural number. The following two properties are equivalent.*

- (1) *There is an orthonormal system $\mathsf{T} := \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_n\} \subset X$ of n vectors.*
- (2) *There exists a set $\mathsf{B} := \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n\} \subset X$ of n elements such that B is a $\mathbb{R}$-linear independent set and all angles in $\mathcal{L}(\mathbb{R})(\mathsf{B})$ are real.*

Before we provide the proof of Proposition 4.3 we formulate a corollary.

Corollary 4.4. *Let $n \in \mathbb{N}$. We assume any n-dimensional complex inner product space $(X, < \cdot | \cdot >)$. (The real dimension of the real vector space X is $2 \cdot n$). It is not possible to find a set $\mathsf{B} := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\} \subset X$ of $n+1$ elements such that both B is $\mathbb{R}$-linear independent, and all angles in $\mathcal{L}(\mathbb{R})(\mathsf{B})$ are real.*

Proof. There are at most n $\mathbb{C}$-linear independent vectors in X. □

Now we prove Proposition 4.3, which is easy.

Proof. From (1) follows (2) trivially, because for the orthonormal system $\mathsf{T} = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ it holds that it is $\mathbb{C}$-linear independent, hence $\mathbb{R}$-linear independent, and the inner product of two elements from T is either 0 or 1.

Also, for complex inner product spaces of infinite dimension the conclusion from (2) to (1) is trivial, since in this case an orthonormal system with the cardinality of $\mathbb{N}$ always exists.

We assume a complex inner product space $(X, < \cdot | \cdot >)$ with finite complex dimension and a set $\mathsf{B} \subset X$ with the properties of (2), *i.e.* all inner products in $\mathcal{L}(\mathbb{R})(\mathsf{B})$ are real. We use the well-known method of Gram-Schmidt to generate a set $\widehat{\mathsf{B}} := \{\widehat{v_1}, \widehat{v_2}, \widehat{v_3}, \ldots, \widehat{v_n}\}$ of n unit vectors. This method yields a set $\widehat{\mathsf{B}}$ which spans the same n-dimensional real subspace as B, *i.e.* $\mathcal{L}(\mathbb{R})(\mathsf{B}) = \mathcal{L}(\mathbb{R})(\widehat{\mathsf{B}})$. By construction, the set $\widehat{\mathsf{B}}$ consists of n $\mathbb{R}$-linear independent vectors. Because their inner products $< \widehat{v_i} | \widehat{v_j} >$ are either 0 or 1, the set $\widehat{\mathsf{B}}$ is even an orthonormal system as defined in Definition 4.2. This was required in (1), and Proposition 4.3 is proven. □

At last we demonstrate that real angles are very useful to do classical Euclidean geometry. We are still investigating complex inner product spaces $(X, < \cdot | \cdot >)$, and we consider the desirable equation

$$\angle(\vec{x}, \vec{y}) = \angle(\vec{x}, \vec{x} + \vec{y}) + \angle(\vec{x} + \vec{y}, \vec{y}) \ . \tag{4}$$

Note that inner product spaces have the property $\overline{(4)}$, the linearity. We know that for real numbers $-1 \leq r, s \leq +1$ there is the identity

$$\arccos(r) + \arccos(s) = \arccos\left(r \cdot s - \sqrt{1 - r^2 - s^2 + r^2 \cdot s^2}\right) .$$

Using this formula, it is a straightforward proof to show that equation (4) is fulfilled for a real angle $\angle(\vec{x}, \vec{y})$, *i.e.* $< \vec{x} | \vec{y} > \in \mathbb{R}$. To demonstrate that a real angle is also necessary for equation (4) it is sufficient to consider the special case of unit vectors $\vec{x}, \vec{y}$. The use of unit vectors simplifies the proof. Before the proof we mention a lemma.

Lemma 4.5. *For all numbers $z \in \mathbb{C}$ it holds the identity*

$$2 \cdot \arccos(z) = \arccos(2 \cdot z^2 - 1) \ .$$

Proof. We use $\cos(z) = \frac{1}{2} \cdot \left[e^{\mathbf{i}\cdot z} + e^{-\mathbf{i}\cdot z}\right]$, and from this we can prove easily $\cos(2 \cdot w) = 2 \cdot \cos^2(w) - 1$, for $w \in \mathbb{C}$. We set $w := \arccos(z)$, and another application of the arccosine function gives the desired equation of Lemma 4.5. $\square$

PROPOSITION 4.6. *In a complex inner product space* $(X, < \cdot | \cdot >)$ *let* $\vec{x}, \vec{y}$ *be two unit vectors. Equation* (4) *holds if and only if their angle* $\angle(\vec{x}, \vec{y})$ *is real.*

Proof. To prove Proposition 4.6 we assume a complex number $r + \mathbf{i} \cdot s := \cos(\angle(\vec{x}, \vec{y}))$ with two unit vectors $\vec{x}, \vec{y}$, *i.e.* $\|\vec{x}\| = 1 = \|\vec{y}\|$. By Definition 1.2 of the angle $\angle(\vec{x}, \vec{y})$ this means $r + \mathbf{i} \cdot s = < \vec{x} \mid \vec{y} >$. We consider the right hand side of equation (4), and we calculate

$$\angle(\vec{x}, \vec{x}+\vec{y}) + \angle(\vec{x}+\vec{y}, \vec{y}) = \arccos\left(\frac{< \vec{x} \mid \vec{x}+\vec{y} >}{\|\vec{x}\| \cdot \|\vec{x}+\vec{y}\|}\right) + \arccos\left(\frac{< \vec{x}+\vec{y} \mid \vec{y} >}{\|\vec{x}+\vec{y}\| \cdot \|\vec{y}\|}\right)$$

$$= \arccos\left(\frac{< \vec{x} \mid \vec{x} > + < \vec{x} \mid \vec{y} >}{\|\vec{x}+\vec{y}\|}\right) + \arccos\left(\frac{< \vec{x} \mid \vec{y} > + < \vec{y} \mid \vec{y} >}{\|\vec{x}+\vec{y}\|}\right)$$

$$= \arccos\left(2 \cdot \left[\frac{1 + < \vec{x} \mid \vec{y} >}{\|\vec{x}+\vec{y}\|}\right]^2 - 1\right) \quad \text{(by Lemma 4.5)}$$

$$= \arccos\left(2 \cdot \left[\frac{1 + 2 \cdot (r + \mathbf{i} \cdot s) + (r + \mathbf{i} \cdot s)^2}{< \vec{x}+\vec{y} \mid \vec{x}+\vec{y} >}\right] - 1\right)$$

$$= \arccos\left(2 \cdot \left[\frac{1 + 2 \cdot r + r^2 - s^2 + \mathbf{i} \cdot (2 \cdot s + 2 \cdot r \cdot s)}{1 + 2 \cdot r + 1}\right] - 1\right) \quad \text{(note } \overline{(2)}\text{)}$$

$$= \arccos\left(\frac{r + r^2 - s^2 + \mathbf{i} \cdot (2 \cdot s + 2 \cdot r \cdot s)}{1 + r}\right) = \arccos\left(r - \frac{s^2}{1+r} + \mathbf{i} \cdot 2 \cdot s\right).$$

Obviously, the last term is equal $\arccos(r + \mathbf{i} \cdot s) = \angle(\vec{x}, \vec{y})$ only in the case of $s = 0$, *i.e.* if and only if the angle $\angle(\vec{x}, \vec{y})$ is real. The proof of Proposition 4.6 is established. $\square$

With the properties of Theorem 1.3 and equation (4) we can do ordinary Euclidean geometry even in complex Hilbert spaces, but only in those parts where real angles occur. We will give two examples. As a first example we prove that the sum of inner angles in a triangle is π.

PROPOSITION 4.7. *Let us assume a real angle* $\angle(\vec{x}, \vec{y})$ *in a complex Hilbert space, i.e.* $< \vec{x} \mid \vec{y} > \in \mathbb{R}$. *We get*

$$\angle(\vec{x}, \vec{y}) + \angle(-\vec{x}, \vec{y} - \vec{x}) + \angle(-\vec{y}, \vec{x} - \vec{y}) \;=\; \pi.$$

Proof. We use equation (4), and Theorem 1.3 (`An` 4),(`An` 6),(`An` 7). If we regard $\overline{(4)}$, the linearity (from the section 'General Definitions'), we see that all angles in the following equation are real. We compute

$$
\begin{aligned}
& \angle(\vec{x},\vec{y}) + [\angle(-\vec{x},\vec{y}-\vec{x})] + [\angle(-\vec{y},\vec{x}-\vec{y})] \\
= \; & \angle(\vec{x},\vec{y}) + [\angle(-\vec{x},\vec{y}) - \angle(-\vec{x}+\vec{y},\vec{y})] + [\angle(-\vec{y},\vec{x}) - \angle(-\vec{y}+\vec{x},\vec{x})] = \pi. \quad \square
\end{aligned}
$$

As a second example we consider the 'Law of Cosines'.

PROPOSITION 4.8. *Let $\vec{x},\vec{y} \neq \vec{0}$ be two vectors in a complex Hilbert space. It holds the 'Law of Cosines'*

$$
\|\vec{x}-\vec{y}\|^2 \; = \; \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2 \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos(\angle(\vec{x},\vec{y})) \tag{5}
$$

if and only if the angle $\angle(\vec{x},\vec{y})$ is real, or in other words $<\vec{x} \mid \vec{y}> \in \mathbb{R}$.

Proof. If the angle $\angle(\vec{x},\vec{y})$ is real, the proof of the Law of Cosines is straightforward. In the case of a proper complex angle $\angle(\vec{x},\vec{y})$ the right hand side of equation (5) is complex, while the left hand side is real. $\square$

Note that the above theorems can be adapted to the complex situation, e.g. the Law of Cosines can be expressed by $\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2+\|\vec{y}\|^2-\|\vec{x}\|\cdot\|\vec{y}\|\cdot [\cos(\angle(\vec{x},\vec{y})) + \cos(\angle(\vec{y},\vec{x}))]$, or alternatively $<\vec{x}|\vec{y}> = \cos(\angle(\vec{x},\vec{y}))\cdot\|\vec{x}\|\cdot\|\vec{y}\|$.

At the end we should mention that in the last decades other concepts of generalized angles in complex inner product spaces have been considered. Note that the following concepts of angles (except $\angle_3$, since the value of ϱ may be larger than 1, see Corollary 3.6) can be used in all complex normed spaces, provided with the product of Definition 1.1.

There were attempts to enforce pure real angles by the definitions $\angle_1$, $\angle_2$, and $\angle_3$, where

$$
\angle_1(\vec{x},\vec{y}) \; := \; \text{the real part of } (\angle(\vec{x},\vec{y})) \; = \; \text{real part of } \left(\arccos\left(\frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}\right)\right),
$$

$$
\angle_2(\vec{x},\vec{y}) \; := \; \text{the arccosine of the real part of } \left(\frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}\right),
$$

$$
\angle_3(\vec{x},\vec{y}) \; := \; \arccos(\varrho) \;, \quad \text{for} \quad \frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|} \; = \; \varrho \cdot e^{\mathbf{i}\cdot\varphi} \; \in \mathbb{C} \;.
$$

For more information and references see a paper [12] by Scharnhorst.

An interesting position is held by Froda in [4]. For the complex number $\frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|} = r + \mathbf{i} \cdot s$ he defined the complex angle $\angle_4(\vec{x},\vec{y})$,

$$
\angle_4(\vec{x},\vec{y}) \; := \; \arccos(r) + \mathbf{i} \cdot \arcsin(s).
$$

Note that in the special case of a pure real non-negative product $<\vec{x}|\vec{y}>$ all four angles coincide with our angle $\angle(\vec{x},\vec{y})$.

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