EXISTENCE OF A RENORMALISED SOLUTION FOR A CLASS OF NONLINEAR DEGENERATED PARABOLIC PROBLEM WITH UNBOUNDED NONLINEARITIES

Y. AKDIM, J. BENNOUNA, M. MEKKOUR and H. REDWANE

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In this work, we study the existence of renormalized solutions for a class of nonlinear degenerated parabolic problem in the form

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + \operatorname{div}(\phi(u)) = f \quad \text{in } Q, \tag{0.1}$$

where $b(x,u)$ is unbounded function on u, the Carathéodory function a satisfying the coercivity condition, the general growth condition and only the large monotonicity, the function ϕ is assumed to be continuous on $\mathbb{R}$ and not belong to $(L^1_{loc}(Q))^N$. The data belongs to $L^1(Q)$.

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1. INTRODUCTION

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$, $Q = \Omega\times]0,T[$ and $w = \{w_i(x),\ 0 \le i \le N\}$ be a vector of weight functions (*i.e.*, every component $w_i(x)$ is a measurable function which is positive *a.e.* in Ω) satisfying some integrability conditions. The objective of this paper is to study the following problem in the weighted Sobolev space:

$$\begin{aligned} &\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + \operatorname{div}(\phi(u)) = f \quad \text{in } Q,\\ &b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega \\ &u = 0 \quad \text{in } \partial\Omega\times]0,T[. \end{aligned} \tag{1.1}$$

The data f and $b(x,u_0)$ lie in $L^1(Q)$ and $L^1(\Omega)$, respectively. The functions ϕ is just assumed to be continnous of $\mathbb{R}$ with values in $\mathbb{R}^N$. The operator $\operatorname{div}(a(x,t,u,Du))$ is a Leray-Lions operator which is coercive, and which grows like $|Du|^{p-1}$ with respect to $|Du|$, but which is not restricted by any growth condition with respect to u and only the large monotonicity (see assumption (H_2)) and $b(x,u)$ is unbounded function on u.

Let us point out, the difficulties that arise in problem (1.1) are due to the following facts: the data f and u_0 only belong to L^1, a satisfies the large monotonicity that is

$$[a(x,t,s,\xi)-a(x,t,s,\eta)](\xi-\eta)\geq 0 \quad \text{for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N.$$

and the function $\phi(u)$ does not belong to $(L^1_{loc}(Q))^N$ (because the function ϕ is just assumed to be continuous on $\mathbb{R}$). To overcome this difficulty, we will apply Landes's technical (see [14, 24]) and the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [20] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by L. Boccardo *et al.* [8] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [27] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [19] for the case of right hand side is general measure data.

For the parabolic equation (1.1) the existence of weak solution has been proved by J.-M. Rakotoson [26] with the strict monotonicity and a measure data, the existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [10] in the case where $a(x,t,u,Du)$ is independent of u, $\phi=0$, $b(x,u)=u$, and by D.Blanchard, F. Murat and H. Redwane [11] with the large monotonicity on a.

For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch *et al.* [3] in the case where a is strictly monotone, $\phi=0$, $b(x,u)=u$ and $f\in L^{p'}(0,T,W^{-1,\,p'}(\Omega,w^*))$. See also the existence of renormalized solution by Y. Akdim *et al.* [7] in the case where $a(x,t,u,Du)$ is independent of u and $\phi=0$, $b(x,u)=u$.

Note that, this paper can be seen as a generalization of [3, 29] in weighted case and as a continuation of [7].

The plan of the paper is as follows. In Section 2 we give some preliminaries and the definition of weighted Sobolev spaces. In Section 3 we make precise all the assumptions on a, ϕ, f and u_0. In Section 4 we give some technical results. In Section 5 we give the definition of a renormalized solution of (1.1) and we establish the existence of such a solution (Theorem 5.3). Section 6 is devoted to an example which illustrates our abstract result.

2. PRELIMINARIES

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2<p<\infty$ and $w=\{w_i(x),\,0\leq i\leq N\}$ be a vector of weight functions, *i.e.*, every component $w_i(x)$ is a measurable function which is strictly positive *a.e.* in Ω. Further, we suppose in all our considerations that, there exits

$$(2.1)\qquad r_0 > \max(N,p) \ \text{ such that } \ w_i^{\frac{-r_0}{r_0-p}} \in L^1_{\text{loc}}(\Omega),$$

$$(2.2)\qquad w_i \in L^1_{\text{loc}}(\Omega),$$

$$(2.3)\qquad w_i^{\frac{-1}{p-1}} \in L^1(\Omega),$$

for any $0 \le i \le N$. We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \quad \text{for } i = 1,\dots,N.$$

Which is a Banach space under the norm

$$(2.4)\qquad \|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,\mathrm{d}x\Big]^{1/p}.$$

The condition (2.2) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $V = W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4). Moreover, condition (2.3) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0,\dots,N\}$ and where p' is the conjugate of p *i.e.* $p' = \frac{p}{p-1}$ (see [23]).

3. BASIC ASSUMPTIONS

Assumption (H1). For $2 \le p < \infty$, we assume that the expression

$$(3.1)\qquad |||u|||_V = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,\mathrm{d}x\Big)^{1/p}$$

is a norm defined on V which equivalent to the norm (2.4), and there exist a weight function σ on Ω such that,

$$\sigma \in L^1(\Omega) \ \text{ and } \ \sigma^{-1} \in L^1(\Omega).$$

We assume also the Hardy inequality,

$$(3.2)\qquad \Big(\int_\Omega |u(x)|^q \sigma\, dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,\mathrm{d}x\Big)^{1/p},$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$W^{1,\,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{3.3}$$

expressed by the inequality (3.2) is compact. Note that $(V, |||.|||_V)$ is a uniformly convex (and thus, reflexive) Banach space.

Remark 3.1. If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, +\infty[\cap[\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \;\text{ and }\; w_i^{\frac{N}{N-1}} \in L^1_{loc}(\Omega) \quad \text{for all } i = 1, \dots, N. \tag{3.4}$$

Note that the assumptions (2.2) and (3.4) imply that,

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,\mathrm{d}x\Big)^{1/p}, \tag{3.5}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (2.4) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega), \tag{3.6}$$

is compact for all $1 \le q \le p_1^*$ if $p.\nu < N(\nu+1)$ and for all $q \ge 1$ if $p.\nu \ge N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of p_1 (see [22], pp. 30–31).

Assumption (H2).

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \tag{3.7}$$

is a Carathéodory function such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1 – function with $b(x, 0) = 0$.

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \le \frac{\partial b(x,s)}{\partial s} \le A_k(x) \;\text{ and }\; \left|D_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \le B_k(x) \tag{3.8}$$

for almost every $x \in \Omega$, for every s such that $|s| \le k$, we denote by $D_x\left(\frac{\partial b(x,s)}{\partial s}\right)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions.

For $i = 1, ..., N$ and for any $k > 0$ there exist $\beta_k > 0$ and a function $C_k(x,t) \in L^{p'}(Q)$ such that,

$$|a_i(x,t,s,\xi)| \le \beta_k w_i^{\frac{1}{p}}(x)[C_k(x,t) + \sum_{j=1}^N w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{3.9}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq k$ and $\xi \in \mathbb{R}^N$.

(3.10) $$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) \geq 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N,$$

(3.11) $$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p,$$

(3.12) $$\phi : \ \mathbb{R} \to \ \mathbb{R}^N \quad \text{is a continuous function,}$$

(3.13) $$f \quad \text{is an element of } \ L^1(Q),$$

(3.14) u_0 is measurable function defined on Ω such that $b(x,u_0) \in L^1(\Omega)$.

Where α is strictly positive constant. We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as,

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

4. SOME TECHNICAL RESULTS

Characterization of the time mollification of a function u. In order to deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. Thus, we define for all $\mu \geq 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_\infty^t \tilde{u}(x,s) \exp(\mu(s-t)) \mathrm{d}s, \ \text{where} \ \ \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

PROPOSITION 4.1 ([3]).
1) *If $u \in L^p(Q,w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and,*

$$\|u_\mu\|_{L^p(Q,w_i)} \leq \|u\|_{L^p(Q,w_i)}.$$

2) *If $u \in W_0^{1,p}(Q,w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q,w)$ as $\mu \to \infty$.*
3) *If $u_n \to u$ in $W_0^{1,p}(Q,w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q,w)$.*

Some weighted embedding and compactness results. In this section, we establish some embedding and compactness results in weighted Sobolev spaces, some trace results, Aubin's and Simon's results [30].

Let $V = W_0^{1,\,p}(\Omega,w)$, $H = L^2(\Omega,\sigma)$ and let $V^* = W^{-1,p'}$, with $(2 \leq p < \infty)$.

Let $X = L^p(0,T;W_0^{1,\,p}(\Omega,w))$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : \ v' \in X^*\}$ endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u, *i.e.*,

$$\int_0^T u'(t)\varphi(t)\mathrm{d}t = -\int_0^T u(t)\varphi'(t)\mathrm{d}t \quad \text{for all} \quad \varphi \in C_0^\infty(0,T).$$

Lemma 4.2 ([31]).
1) *The evolution triple $V \subseteq H \subseteq V^*$ is verified.*
2) *The imbedding $W_p^1(0,T,V,H) \subseteq C(0,T,H)$ is continuous.*
3) *The imbedding $W_p^1(0,T,V,H) \subseteq L^p(Q,\sigma)$ is compact.*

Lemma 4.3 ([3]). *Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \leq C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Q, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$.*

Lemma 4.4 ([3]). *Assume that,*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad in \quad D'(Q)$$

where α_n and β_n are bounded respectively in X^* and in $L^1(Q)$. If v_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, then $v_n \to v$ in $L^p_{loc}(Q,\sigma)$.
Further $v_n \to v$ strongly in $L^1(Q)$.

Definition 4.5. A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u,T(u)) \in X \times X^* \quad \text{for all} \quad u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$.

Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset $D(L) = \{v \in X : v' \in X^*, v(0) = 0\}$ of X into X^* by

$$\langle Lu, v\rangle_X = \int_0^T \langle u'(t), v(t)_V \mathrm{d}t\rangle \quad u \in D(L), \quad v \in X.$$

Lemma 4.6 ([31]). *L is a closed linear maximal monotone map.*

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subset X$ to X^* and S is a bounded demicontinuous map of monotone type from X to X^*.

Definition 4.7. A mapping S is called pseudo-monotone with $u_n \rightharpoonup u$, $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0$, that we have

$$\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle = 0 \quad \text{and} \quad S(u_n) \rightharpoonup S(u) \text{ as } n \to \infty.$$

5. MAIN RESULTS

Consider the problem

$$b(x,u_0)\in L^1(\Omega),\quad f\in L^1(Q)$$

(5.1) $$\frac{\partial b(x,u)}{\partial t}-\operatorname{div}(a(x,t,u,Du))+\operatorname{div}(\phi(u))=f\quad\text{in } Q$$

$$u=0\quad\text{on } \partial\Omega\times]0,T[,$$

$$b(x,u(x,0))=b(x,u_0)\quad\text{on } \Omega$$

Definition 5.1. Let $f\in L^1(Q)$ and $b(x,u_0)\in L^1(\Omega)$. A real-valued function u defined on $\Omega\times]0,T[$ is a renormalized solution of problem (5.1) if

(5.2)
$$T_k(u)\in L^p(0,T;W_0^{1,\,p}(\Omega,w))\ \text{ for all }\ (k\geq 0)\ \text{ and }\ b(x,u)\in L^\infty(0,T;L^1(\Omega));$$

(5.3) $$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Du\,\mathrm{d}x\mathrm{d}t\to 0\ \text{ as }\ m\to\ +\infty;$$

$$\frac{\partial B_S(x,u)}{\partial t}-div\left(S'(u)a(u,Du)\right)+S''(u)a(u,Du)Du$$

(5.4) $$+\operatorname{div}(S'(u)\phi(u))-S''(u)\phi(u)Du=fS'(u)\ \text{ in }\ D'(Q);$$

for all functions $S\in W^{2,\,\infty}(\mathbb{R})$ which compact support in $\mathbb{R}$, where $B_S(x,z)=\int_0^z\frac{\partial b(x,r)}{\partial r}S'(r)\mathrm{d}r$ and

(5.5) $$B_S(x,u)(t=0)=B_S(x,u_0)\ \text{ in }\ \Omega.$$

Remark 5.2. Equation (5.4) is formally obtained through pointwise multiplication of equation (5.1) by $S'(u)$. However, while $a(u,Du)$ and $\phi(u)$ does not in general make sense in (5.1), all the terms in (5.4) have a meaning in $D'(Q)$.

Indeed, if M is such that $supp(S')\subset[-M,M]$, the following identifications are made in (5.4):

- $S(u)\in L^\infty(Q)$ since S is a bounded function.
- $S'(u)a(u,Du)$ identifies with $S'(u)a(T_M(u),DT_M(u))$ *a.e.* in Q. Since $|T_M(u)|\leq M$ *a.e.* in Q, assumptions (3.9) imply that

$$|a_i(x,t,T_M(u),DT_M(u))|\leq\beta_M w_i^{\frac{1}{p}}(x)\left(C_M(x,t)+\sum_{i=1}^N w_j^{\frac{1}{p'}}(x)\left|\frac{\partial T_M(u)}{\partial x_j}\right|^{p-1}\right).$$

We obtain that $S'(u)a(T_M(u),DT_M(u))\in\prod_{i=1}^N L^{p'}(Q,w_i^*)$.

- $S''(u)a(u,Du)Du$ identifies with $S''(u)a(T_M(u),DT_M(u))DT_M(u)$ we have $S''(u)a(T_M(u),DT_M(u))DT_M(u)\in L^1(Q)$.

• $S''(u)\phi(u)Du$ *and* $S'(u)\phi(u)$ respectively identify with $S''(u)\phi(T_M(u))$ $DT_M(u)$ and $S'(u)\phi(T_M(u))$. Due to the properties of S' and to (3.12), the functions S', S'' *and* $\phi o T_M$ are bounded on $\mathbb{R}$ so that (5.2) implies that $S'(u)\phi(T_M(u)) \in (L^\infty(Q))^N$, and $S''(u)\phi(T_M(u))DT_M(u) \in L^p(Q,w)$

• $S'(u)f$ belongs to $L^1(Q)$.

The above considerations show that equation (5.4) holds in $D'(Q)$ and that

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w_i^*)) + L^1(Q).$$

Due to the properties of S and (5.4), $\frac{\partial S(u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w_i^*)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (5.5) makes sense, since, due to the properties of S (increasing) and (3.8), we have

$$\left|B_S(x,r) - B_S(x,r')\right| \le A_k(x)\left|S(r) - S(r')\right| \quad \text{for all} \quad r,r' \in \mathbb{R}. \tag{5.6}$$

THEOREM 5.3. *Let $f \in L^1(Q)$ and $b(x,u_0) \in L^1(\Omega)$. Assume that (H1) and (H2) hold true. then, there exists at least a renormalized solution u of the problem (5.1) (in the sense of Definition 5.1).*

Remark 5.4. The statement of Theorem 5.3 generalized in weighted case the analogous in [29] and [7] (with $b(x,u) = u$).

Remark 5.5. Since, the function $\phi(u)$ does note belong to $\left(L^1_{loc}(Q)\right)^N$. Then the problem (5.1) can have a renormalized solution, but not a weak solution.

Proof. Step 1: The approximate problem.

For $n > 0$, let us define the following approximation of b, a, ϕ, f and u_0;

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r \quad \text{for} \quad n > 0, \tag{5.7}$$

In view of (5.7), b_n is a Carathéodory function and satisfies (3.8), there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \quad \text{and} \quad \left|D_x\left(\frac{\partial b_n(x,s)}{\partial s}\right)\right| \le B_n(x) \quad \text{a.e. in} \quad \Omega, \quad s \in \mathbb{R}.$$

$$a_n(x,t,s,d) = a(x,t,T_n(s),d) \quad \text{a.e. in} \quad Q, \quad \forall s \in \mathbb{R}, \ \forall d \in \mathbb{R}^N, \tag{5.8}$$

In view of (5.8), a_n satisfy (3.11) and (3.9), there exists $C_n \in L^{p'}(Q)$ and $\beta_n > 0$ such that

(5.9)

$$|a_i^n(x,t,s,\xi)| \le \beta_n w_i^{\frac{1}{p}}(x)[C_n(x,t) + \sum_{j=1}^N w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \quad \text{for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N,$$

(5.10) ϕ_n is a Lipschitz continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$,

such that ϕ_n uniformly converges to ϕ on any compact subset of $\mathbb{R}$ as n tends to $+\infty$,

(5.11)

$$f_n \in L^{p'}(Q) \ \text{ and } \ f_n \to f \ \ a.e. \ \text{ in } \ Q \ \text{ and strongly in } \ L^1(Q) \ \ as \ \ n \to +\infty,$$

$$u_{0n} \in D(\Omega): \quad \|b_n(x,u_{0n})\|_{L^1} \leq \|b(x,u_0)\|_{L^1},$$

(5.12) $$b_n(x,u_{0n}) \to b(x,u_0) \ \ a.e. \ \text{ in } \ \Omega \ \text{ and strongly in } \ L^1(\Omega).$$

Let us now consider the approximate problem:

(5.13)
$$\begin{gathered}\frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a_n(x,t,u_n,Du_n)) + \operatorname{div}(\phi_n(u_n)) = f_n \ \text{ in } \ D'(Q),\\ u_n = 0 \ \text{ in } \ (0,T)\times\partial\Omega,\\ b_n(x,u_n(t=0)) = b_n(x,u_{0n}) \ \text{ in } \ \Omega.\end{gathered}$$

As a consequence, proving existence of a weak solution $u_n \in L^p(0,T; W_0^{1,\,p}(\Omega,w))$ of (5.13) is an easy task (see *e.g.* [25, 28]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (5.13).

Using in (5.13) the test function $T_k(u_n)\chi_{(0,\tau)}$, we get, for every $\tau \in [0,T]$.

(5.14)
$$\begin{gathered}\langle\frac{\partial b_n(x,u_n)}{\partial t}, T_k(u_n)\chi_{(0,\tau)}\rangle + \int_{Q_\tau} a(x,t,T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}t\\ + \int_{Q_\tau}\phi_n(u_n)DT_k(u_n)\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} f_nT_k(u_n)\mathrm{d}x\mathrm{d}t,\end{gathered}$$

which implies that,

(5.15)
$$\begin{gathered}\int_\Omega B_k^n(x,u_n(\tau))\mathrm{d}x + \int_0^\tau\int_\Omega a(x,t,T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}t\\ + \int_{Q_\tau}\phi_n(u_n)DT_k(u_n)\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} f_nT_k(u_n)\mathrm{d}x\mathrm{d}t + \int_\Omega B_k^n(x,u_{0n})\mathrm{d}x,\end{gathered}$$

where $B_k^n(x,r) = \int_0^r T_k(s)\frac{\partial b_n(x,s)}{\partial s}\mathrm{d}s$. The Lipschitz character of ϕ_n and Stokes' formula together with the boundary condition 2 of problem (5.13) give

(5.16) $$\int_0^\tau\int_\Omega \phi_n(u_n)DT_k(u_n)\mathrm{d}x\mathrm{d}t = 0.$$

Due to the definition of B_k^n we have

(5.17) $$0 \leq \int_\Omega B_k^n(x,u_{0n})\mathrm{d}x \leq k\int_\Omega |b_n(x,u_{0n})|\,\mathrm{d}x \leq k\,\|b(x,u_0)\|_{L^1(\Omega)}.$$

Using (5.16), (5.17) and $B_k^n(x,u_n) \geq 0$, it follows from (5.15) that

$$(5.18)\qquad \int_0^T\int_\Omega a(x,t,T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}t \le k(\|f_n\|_{L^1(Q)}+\|b_n(x,u_{0n})\|_{L^1(\Omega)})\le Ck,$$

Thanks to (3.11) we have

$$(5.19)\qquad \alpha\int_Q\sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p \mathrm{d}x\mathrm{d}t\le Ck,\quad \forall k\ge 1.$$

We deduce from that above inequality (5.15) and (5.17) that

$$(5.20)\qquad \int_\Omega B_k^n(x,u_n)\mathrm{d}x\le k(\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)})\equiv Ck.$$

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$, $T_k(u_n)\rightharpoonup v_k$ in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$, and by the compact imbedding (3.6) gives,

$$T_k(u_n)\to v_k \ \text{ strongly in } \ L^p(Q,\sigma) \ \text{ and } \ a.e. \ \text{ in } \ Q.$$

Let $k>0$ large enough and B_R be a ball of Ω, we have,

$$\begin{aligned} k\ \ meas(\{|u_n|>k\}\cap B_R\times[0,T]) &= \int_0^T\int_{\{|u_n|>k\}\cap B_R}|T_k(u_n)|\,\mathrm{d}x\mathrm{d}t \\ &\le \int_0^T\int_{B_R}|T_k(u_n)|\,\mathrm{d}x\mathrm{d}t \\ &\le \left(\int_Q|T_k(u_n)|^p\,\sigma\mathrm{d}x\mathrm{d}t\right)^{\frac1p}\left(\int_0^T\int_{B_R}\sigma^{1-p'}\mathrm{d}x\mathrm{d}t\right)^{\frac1{p'}} \\ &\le Tc_R\left(\int_Q\sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p\mathrm{d}x\mathrm{d}t\right)^{\frac1p}\le ck^{\frac1p}, \end{aligned}$$

which implies that,

$$meas(\{|u_n|>k\}\cap B_R\times[0,T])\le\frac{c_1}{k^{1-\frac1p}},\quad \forall k\ge 1.$$

So, we have

$$\lim_{k\to+\infty}(meas(\{|u_n|>k\}\cap B_R\times[0,T]))=0.$$

Now, we turn to prove the almost every convergence of u_n and $b_n(x,u_n)$. Consider now a function non decreasing $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le\frac{k}{2}$ and $g_k(s)=k$ for $|s|\ge k$. Multiplying the approximate equation by $g_k'(b_n(x,u_n))$, we get

$$(5.21)\qquad \frac{\partial g_k(b_n(x,u_n))}{\partial t}-div(a(x,t,u_n,Du_n)g_k'(b_n(x,u_n)))$$

$$+ a(x,t,u_n,Du_n)g_k''(b_n(x,u_n))D_x\left(\frac{\partial b_n(x,u_n)}{\partial s}\right)Du_n$$

$$- div(g_k'(b_n(x,u_n))\phi_n(u_n)) + g_k''(b_n(x,u_n))D_x\left(\frac{\partial b_n(x,u_n)}{\partial s}\right)\phi_n(u_n)Du_n$$

$$= f_n g_k'(b_n(x,u_n))$$

in the sense of distributions, which implies that

$$g_k(b_n(x,u_n)) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.22}$$

and

$$\frac{\partial g_k(b_n(x,u_n))}{\partial t} \text{ is bounded in } X^* + L^1(Q), \tag{5.23}$$

independently of n as soon as $k < n$. Due to Definition (3.7) and (5.7) of b_n, it is clear that

$$\{|b_n(x,u_n)| \le k\} \subset \{|u_n| \le k^*\}$$

as soon as $k < n$ and k^* is a constant independent of n. As a first consequence we have

(5.24)

$$Dg_k(b_n(x,u_n)) = g_k'(x,b_n(u_n))D_x\left(\frac{\partial b_n(x,T_{k^*}(u_n))}{\partial s}\right)DT_{k^*}(u_n) \quad a.e. \text{ in } Q$$

as soon as $k < n$. Secondly, the following estimate holds true

$$\left\|g_k'(b_n(x,u_n))D_x\left(\frac{\partial b_n(x,T_{k^*}(u_n))}{\partial s}\right)\right\|_{L^\infty(Q)} \le \|g_k'\|_{L^\infty(Q)}\left(\max_{|r|\le k^*}\left(D_x\left(\frac{\partial b_n(x,s)}{\partial s}\right)\right)+1\right).$$

As a consequence of (5.19), (5.24) we then obtain (5.22). To show that (5.23) holds true, due to (5.21) we obtain

$$\frac{\partial g_k(b_n(x,u_n))}{\partial t} = div(a(x,t,u_n,Du_n)g_k'(b_n(x,u_n))) \tag{5.25}$$

$$- a(x,t,u_n,Du_n)g_k''(b_n(u_n))D_x\left(\frac{\partial b_n(x,u_n)}{\partial s}\right) + div(g_k'(b_n(x,u_n))\phi_n(u_n)$$

$$- g_k''(b_n(u_n))D_x\left(\frac{\partial b_n(x,u_n)}{\partial s}\right)\phi_n(u_n)Du_n + f_n g_k'(b_n(x,u_n)).$$

Since $\text{supp} g_k'$ and $\text{supp} g_k''$ are both included in $[-k,k]$, u_n may be replaced by $T_{k^*}(u_n)$ in each of these terms. As a consequence, each term on the right-hand side of (5.25) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ or in $L^1(Q)$. Hence, lemma 4.4 allows us to conclude that $g_k(b_n(x,u_n))$ is compact in $L^p_{loc}(Q,\sigma)$.

Thus, for a subsequence, it also converges in measure and almost every where in Q, due to the choice of g_k, we conclude that for each k, the sequence $T_k(b_n(x,u_n))$ converges almost everywhere in Q (since we have, for every $\lambda > 0$,)

$$meas(\{|b_n(x,u_n)-b_m(x,u_m)|>\lambda\}\cap B_R\times[0,T])$$
$$\le meas(\{|b_n(x,u_n)|>k\}\cap B_R\times[0,T])+meas(\{|b_m(x,u_m)|>k\}\cap B_R\times[0,T])$$
$$+\,meas(\{|g_k(b_n(x,u_n))-g_k(b_m(x,u_m))|>\lambda\}).$$

Let $\varepsilon>0$, then, there exist $k(\varepsilon)>0$ such that,

$$meas(\{|b_n(x,u_n)-b_m(x,u_m)|>\lambda\}\cap B_R\times[0,T])\le\varepsilon$$
$$\text{for all } \ n,m\ge n_0(k(\varepsilon),\lambda,R).$$

This proves that $(b_n(x,u_n))$ is a Cauchy sequence in measure in $B_R\times[0,T]$, thus converges almost everywhere to some measurable function v. Then for a subsequence denoted again u_n,

$$u_n\to u \ \ a.e. \ \text{in} \ \ Q, \tag{5.26}$$

and

$$b_n(x,u_n)\to b(x,u) \ \ a.e. \ \text{in} \ \ Q, \tag{5.27}$$

we can deduce from (5.19) that,

$$T_k(u_n)\rightharpoonup T_k(u) \ \ \text{weakly in} \ \ L^p(0,T;W_0^{1,\,p}(\Omega,w)) \tag{5.28}$$

and then, the compact imbedding (3.3) gives,

$$T_k(u_n)\to T_k(u) \ \ \text{strongly in} \ \ L^q(Q,\sigma) \ \ \text{and} \ \ a.e. \ \ \text{in} \ \ Q.$$

Which implies, by using (3.9), for all $k>0$ that there exists a function $h_k\in\prod_{i=1}^{N}L^{p'}(Q,w_i^*)$, such that

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup h_k \ \ \text{weakly in} \ \ \prod_{i=1}^{N}L^{p'}(Q,w_i^*). \tag{5.29}$$

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (5.26) and passing to the limit-inf in (5.20) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_k(x,u)(\tau)\mathrm{d}x\le[\|f\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}]\equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(x,s)$ and the fact that $\frac{1}{k}B_k(x,u)$ converges pointwise to $b(x,u)$, as k tends to $+\infty$, shows that $b(x,u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Step 3: This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ and to establish the following limits:
(5.30)

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup a(x,t,T_k(u),DT_k(u)) \ \ \text{weakly in} \ \ \prod_{i=1}^{N}L^{p'}(Q,w_i^*),$$

as n tends to $+\infty$. This proof is devoted to introduce for $k \geq 0$ fixed, a time regularization of the function $T_k(u)$ in order to perform the monotonicity method.

Firstly, we prove the following lemma:

LEMMA 5.6.

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t=0, \tag{5.31}$$

for any integer $m \geq 1$,

Proof. Taking $T_1(u_n - T_m(u_n))$ as a test function in (5.13), we obtain

$$\begin{aligned}&\left\langle \frac{\partial b_n(x,u_n)}{\partial t}, T_1(u_n - T_m(u_n))\right\rangle + \int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&+\int_Q \operatorname{div}\left[\int_0^{u_n}\phi(r)T_1'(r-T_m(r))\right]\mathrm{d}x\mathrm{d}t=\int_Q f_nT_1(u_n-T_m(u_n)).\end{aligned} \tag{5.32}$$

Using the fact that $\int_0^{u_n}\phi(r)T_1'(r-T_m(r))\mathrm{d}x\mathrm{d}t \in L^p(0,T;W_0^{1,\,p}(\Omega,w))$ and Stokes' formula, we get

$$\begin{aligned}&\int_\Omega B_n^m(x,u_n)(T)\mathrm{d}x+\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&\leq \int_Q |f_nT_1(u_n-T_m(u_n))|\,\mathrm{d}x\mathrm{d}t+\int_\Omega B_n^m(x,u_{0n})\mathrm{d}x,\end{aligned} \tag{5.33}$$

where $B_n^m(r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}T_1(s - T_m(s))\mathrm{d}s$. In order to pass to the limit as n tends to $+\infty$ in (5.33), we use $B_n^m(x,u_n)(T) \geq 0$ and (5.11),(5.12), we obtain that

$$\begin{aligned}&\lim_{m\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t\\&\leq \int_{\{|u(x)|>m\}}|f|\,\mathrm{d}x\mathrm{d}t+\int_{\{|u_0(x)|>m\}}|b(x,u_0(x))|\,\mathrm{d}x.\end{aligned} \tag{5.34}$$

Finally, by (3.14), (3.13) and (5.34) we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,Du_n)Du_n\mathrm{d}x\mathrm{d}t=0. \tag{5.35}$$

The very definition of the sequence $(T_k(u))_\mu$ for $\mu > 0$ (and fixed k) we establish the following lemma.

LEMMA 5.7. *Let $k \geq 0$ be fixed. Let $(T_k(u))_\mu$ the mollification of $T_k(u)$. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \leq k$ and supp S' is compact. Then,*

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\left\langle\frac{\partial b_n(x,u_n)}{\partial t}, S'(u_n)(T_k(u_n)-(T_k(u))_\mu)\right\rangle\mathrm{d}x\mathrm{d}t\geq 0, \tag{5.36}$$

where $\langle .,. \rangle$ *denotes the duality pairing between* $L^1(\Omega) + W^{-1,p'}(\Omega, w^*)$ *and* $L^\infty(\Omega) \cap W_0^{1,\,p}(\Omega, w)$.

Proof. See H. Redwane [29].

We prove the following lemma, which is the key point in the monotonicity arguments.

LEMMA 5.8. *The subsequence of* u_n *satisfies for any* $k \geq 0$

(5.37)
$$\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(T_k(u_n), DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t \leq \int_0^T\int_0^t\int_\Omega h_k DT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t,$$
where h_k *is defined in (5.29).*

Proof. In the following we adapt the above-mentionned method to problem (5.1) and we first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that

$$S_m(r) = r \quad if \quad |r| \leq m,$$
$$suppS'_m \subset [-(m+1), m+1],$$
$$\left\|S''_m\right\|_{L^\infty} \leq 1, \quad \text{for any} \quad m \geq 1.$$

We use the sequence $T_k(u)_\mu$ of approximations of $T_k(u)$, and plug the test function $S'_m(u_n)(T_k(u_n) - (T_k(u))_\mu)$ (for $n > 0$ and $\mu > 0$) in (5.13). Through setting, for fixed $k \leq 0$,

$$W_\mu^n = T_k(u_n) - (T_k(u))_\mu,$$

we obtain upon integration over $(0,\, t)$ and then over $(0,\, T)$:

(5.38)
$$\int_0^T\int_0^t\left\langle\frac{\partial b_n(x,u_n)}{\partial t}, S'_m(u_n)W_\mu^n\right\rangle\mathrm{d}t\mathrm{d}s+\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n,Du_n)DW_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$+\int_0^T\int_0^t\int_\Omega S''_m(u_n)a_n(u_n,Du_n)Du_nW_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t-\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi_n(u_n)DW_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$-\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

In the following we pass the limit in (5.38) as n tends to $+\infty$, then μ tends to $+\infty$ and then m tends to $+\infty$, the real number $k \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $k \geq 0$:

$$\text{(5.39)}\quad \liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\left\langle\frac{\partial B_m^n(x,u_n)}{\partial t}, W_\mu^n\right\rangle\mathrm{d}t\mathrm{d}s \geq 0, \quad \text{for any} \quad m \geq k,$$

$$\text{(5.40)}\quad \lim_{n\to+\infty}\lim_{\mu\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi_n(u_n)DW_\mu^n\mathrm{d}x\mathrm{d}s\mathrm{d}t = 0, \quad \text{for any} \quad m \geq 1,$$

(5.41)
$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t=0, \quad \text{for any} \quad m\geq 1,$$
(5.42)
$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\left|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(u_n,Du_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t\right|=0, \quad m\geq 1$$

(5.43) $$\lim_{\mu\to+\infty}\lim_{m\to+\infty}\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t=0.$$

Proof of (5.39). The function S_m belongs to $C^\infty(\mathbb{R})$ and is increasing. We have for $m\geq k$, $S_m(r)=r$ *for* $|r|\leq k$ while $suppS'_m$ is compact. In view of the definition of W^n_μ, lemma 5.7 applies with $S=S_m$ for fixed $m\geq k$. As a consequence (5.39) holds true.

Proof of (5.40). In order to avoid repetitions in the proofs of (5.43), let us summarize the properties of W^n_μ. For fixed $\mu>0$

$$W^n_\mu \rightharpoonup T_k(u)-(T_k(u))_\mu \quad \text{weakly in} \quad L^p(0,T;W_0^{1,\,p}(\Omega,w)), \quad \text{as} \quad n\to+\infty$$
$$\|W^n_\mu\|_{L^\infty(Q)}\leq 2k, \quad \text{for any} \quad n>0 \quad \text{and for any } \mu>0$$

we deduce that for fixed $\mu>0$

$$W^n_\mu\to T_k(u)-(T_k(u))_\mu \quad a.e. \text{ in } Q \text{ and in } L^\infty(Q) weak-*, \ as \quad n\to+\infty$$

one has $suppS''_m\subset[-(m+1),-m]\cup[m,m+1]$ for any fixed $m\geq 1$, we have

(5.44) $$S'_m(u_n)\phi_n(u_n)DW^n_\mu=S'_m(u_n)\phi_n(T_{m+1}(u_n))DW^n_\mu \quad a.e. \text{ in } Q,$$

since $\mathrm{supp}S'_m\subset[-m-1,m+1]$. Since S'_m is smooth and bounded, (3.12), (5.10), and $u_n\to u$ *a.e.* in Q lead to
(5.45)
$$S'_m(u_n)\phi_n(T_{m+1}(u_n))\to S'_m(u)\phi(T_{m+1}(u)) \ a.e. \text{ in } Q \text{ and in } L^\infty(Q)\text{weak}-*,$$
as n tends to $+\infty$. As a consequence of (5.47) and (5.45), we deduce that

(5.46) $$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi_n(u_n)DW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t=$$
$$=\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)\phi\left(T_{m+1}(u)\right)\left(DT_k(u)-D(T_k(u))_\mu\right)\mathrm{d}x\mathrm{d}s\mathrm{d}t,$$
for any $\mu>0$. Passing to the limit as $\mu\to+\infty$ in (5.46) we conclude that (5.40) holds true.

Proof of (5.41). For fixed $m\geq 1$, and by the same arguments that those that lead to (5.47), we have
(5.47)
$$S''_m(u_n)\phi_n(u_n)Du_nW^n_\mu=S''_m(u_n)\phi_n(T_{m+1}(u_n))DT_{m+1}(u_n)W^n_\mu \quad a.e. \text{ in } Q.$$

From (3.12), $u_n \to u$ *a.e.* in Q and (5.28), it follows that for any $\mu > 0$

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(u_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t$$

$$=\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi\left(T_{m+1}(u)\right)\left(DT_k(u)-D(T_k(u))_\mu\right)\mathrm{d}x\mathrm{d}s\mathrm{d}t,$$

for any $\mu > 0$. Passing to the limit as $\mu \to +\infty$ in (5.46) we conclude that (5.41) holds true.

Proof of (5.42). One has $suppS''_m \subset [-(m+1), -m] \cup [m, m+1]$ for any $m \geq 1$. As a consequence

$$\left|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(u_n, Du_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t\right|$$

$$\leq T\left\|S''_m(u_n)\right\|_{L^\infty}\left\|W^n_\mu\right\|_{L^\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(u_n, Du_n)Du_n\mathrm{d}x\mathrm{d}t,$$

for any $m \geq 1$, any $\mu > 0$ and any $n \geq 1$. It is possible to obtain

$$\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\left|\int_0^T\int_0^t\int_\Omega S''_m(u_n)a(u_n, Du_n)Du_nW^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t\right|$$

$$\leq C\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(u_n, Du_n)Du_n\mathrm{d}x\mathrm{d}t,$$

for any $m \geq 1$, where C is a constant independent of m.

Appealing now to (5.31) it possible to pass the limit as m tends to $+\infty$ to establish (5.42).

Proof of (5.43). Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $m \geq 1$

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega f_nS'_m(u_n)W^n_\mu \mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_0^T\int_0^t\int_\Omega fS'_m(u)(T_k(u)-(T_k(u)_\mu))\mathrm{d}x\mathrm{d}s\mathrm{d}t$$

Now, for fixed $m \geq 1$, using lemma 4.1 and passing to the limit as $\mu \to +\infty$ in the above equality to obtain (5.43).

We now turn back to the proof of lemma 5.8. Due to (5.39)–(5.42) and (5.43), we are in a position to pass the limit-sup when n tends to $+\infty$, then to the limit-sup when μ tends $+\infty$ and then to the limit as m tends to $+\infty$ in (5.38). We obtain by using the definition of W^n_μ that for any $k \geq 0$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n, Du_n)(DT_k(u_n)-D(T_k(u))_\mu)\mathrm{d}x\mathrm{d}s\mathrm{d}t \leq 0.$$

Since $S'_m(u_n)a_n(u_n, Du_n)DT_k(u_n) = a(u_n, Du_n)DT_k(u_n)$ for $k \le n$ and $k \le m$, the above inequality implies that for $k \le m$

$$\begin{aligned}&\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a_n(u_n, Du_n)DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t\\ &\le \lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n, Du_n)D(T_k(u))_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t.\end{aligned} \tag{5.48}$$

The right-hand side of (5.48) is computed as follows. We have for $n \ge m+1$:

$$S'_m(u_n)a_n(u_n, Du_n) = S'_m(u_n)a(T_{m+1}(u_n), DT_{m+1}(u_n)) \;\; a.e. \;\text{ in } Q.$$

Due to the weak convergence of $a(DT_{m+1}(u_n))$ it follows that for fixed $m \ge 1$

$$S'_m(u_n)a_n(u_n, Du_n) \rightharpoonup S'_m(u_n)h_{m+1} \;\text{ weakly in }\; \prod_{i=1}^N L^{p'}(Q, w_i^*),$$

when n tends to $+\infty$. The strong convergence of $(T_k(u))_\mu$ to $T_k(u)$ in $L^p(0,T;W_0^{1,\,p}(\Omega, w))$ as μ tends to $+\infty$, then we conclude that

$$\begin{aligned}&\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(u_n, Du_n)D(T_k(u))_\mu\mathrm{d}x\mathrm{d}s\mathrm{d}t\\ &= \int_0^T\int_0^t\int_\Omega S'_m(u_n)h_{m+1}DT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t,\end{aligned} \tag{5.49}$$

as soon as $k \le m$, $S'_m(r) = 1$ for $|r| \le m$. Now for $k \le m$ we have,

$$a(T_{m+1}(u_n), DT_{m+1}(u_n))\chi_{\{|u_n|<k\}} = a(T_k(u_n), DT_k(u_n))\chi_{\{|u_n|<k\}} \;\; a.e. \;\text{ in }\; Q,$$

which implies that, passing to the limit as $n \to +\infty$,

$$h_{m+1}\chi_{\{|u_n|<k\}} = h_k\chi_{\{|u_n|<k\}} \;\; a.e. \;\text{ in }\; Q - \{|u| = k\} \;\text{ for }\; k \le m. \tag{5.50}$$

As a consequence of (5.50) we have for $k \le m$,

$$h_{m+1}DT_k(u) = h_kDT_k(u) \;\; a.e. \;\text{ in }\; Q. \tag{5.51}$$

Recalling (5.48), (5.49), (5.51) we conclude that (5.37) holds true and the proof of Lemma 5.8 is complete.

In this Lemma we prove the following monotonicity estimate:

Lemma 5.9. *The subsequence of u_n satisfies for any $k \ge 0$*

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega[a(T_k(u_n), DT_k(u_n)) - a(T_k(u_n), DT_k(u))][DT_k(u_n) - DT_k(u)]\mathrm{d}x\mathrm{d}s\mathrm{d}t = 0. \tag{5.52}$$

Proof. Let $k \geq 0$ be fixed. The character (3.10) of $a(x,t,s,d)$ with respect to d implies that

(5.53)
$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega [a(T_k(u_n),DT_k(u_n))-a(T_k(u_n),DT_k(u))][DT_k(u_n)-DT_k(u)]\mathrm{d}x\mathrm{d}s\mathrm{d}t \geq 0.$$

To pass to the limit-sup as n tends to $+\infty$ in (5.53) imply that

$$a(T_k(u_n),DT_k(u)) \to a(T_k(u),DT_k(u)) \;\; a.e. \;\text{ in }\; Q,$$

and that,

$$|a_i(T_k(u_n),DT_k(u))| \leq \beta w_i^{\frac{1}{p}}(x)\left(C_k(x,t)+\sum_{j=1}^N w_j^{\frac{1}{p'}}(x)\left|\frac{\partial T_k(u)}{\partial x_j}\right|^{p-1}\right) \;\; a.e. \text{ in } Q,$$

uniformly with respect to n. It follows that when n tends to $+\infty$

$$(5.54) \quad a(T_k(u_n),DT_k(u)) \to a(T_k(u),DT_k(u)) \;\text{ strongly in }\; \prod_{i=1}^N L^{p'}(Q,w_i^*).$$

Lemma 5.8, weak convergence of $DT_k(u_n)$, $a(T_k(u_n),DT_k(u_n))$ and (5.54) make it possible to pass to the limit-sup as $n \to +\infty$ in (5.53) and to obtain the result .

In this lemma we identify the weak limit h_k and we prove the weak-L^1 convergence of the “truncated” energy $a(T(u_n),\; DT_k(u_n))DT(u_n)$ as n tends to $+\infty$.

LEMMA 5.10. *For fixed $k \geq 0$, we have*

$$(5.55) \qquad h_k = a(T(u),DT_k(u)) \;\; a.e. \;\; in \;\; Q,$$

(5.56)
$$a(T(u_n),DT_k(u_n))DT(u_n) \rightharpoonup a(T(u),DT_k(u))DT_k(u) \;\; weakly \;\; in \;\; L^1(Q).$$

Proof. The proof is standard once we remark that for any $k \geq 0$, any $n > k$ and any $d \in \mathbb{R}^N$

$$a_n(T_k(u_n),d) = a(T_k(u_n),d) \;\; a.e. \;\text{ in }\; Q$$

which together with weak convergence of $(T_k(u_n))$ and $a(DT_k(u_n))$ and (5.54) we obtain from (5.52)

(5.57)
$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(T_k(u_n),DT_k(u_n))DT_k(u_n)\mathrm{d}x\mathrm{d}s\mathrm{d}t = \int_0^T\int_0^t\int_\Omega h_k DT_k(u)\mathrm{d}x\mathrm{d}s\mathrm{d}t.$$

The usual Minty's argument applies in view of weak convergence of $(T_k(u_n))$ and $a(DT_k(u_n))$ and (5.57). It follows that (5.55) hold true.

In order to prove (5.56), we observe that monotone character of a and (5.52) give that for any $k \geq 0$ and any $T' < T$

$$[a(T_k(u_n), DT_k(u_n)) - a(T_k(u), DT_k(u))][DT_k(u_n) - DT_k(u)] \to 0 \tag{5.58}$$

strongly in $L^1((0,T') \times \Omega)$ as $n \to +\infty$.

Moreover, weak convergence of $(T_k(u_n))$ and $a(DT_k(u_n))$, (5.58), (5.54) and (5.55) imply that

$$a(T_k(u_n), DT_k(u_n))DT_k(u) \rightharpoonup a(T_k(u), DT_k(u))DT_k(u) \text{ weakly in } L^1(Q),$$

and

$$a(T_k(u_n), DT_k(u))DT_k(u) \to a(T_k(u_n), DT_k(u))DT_k(u) \text{ strongly in } L^1(Q)$$

as $n \to +\infty$.

Using the above convergence results in (5.58) shows that for any $k \geq 0$ and any $T' < T$

$$a(T_k(u_n), DT_k(u_n))DT_k(u_n) \rightharpoonup a(T_k(u), DT_k(u))DT_k(u) \text{ weakly in } L^1((0,T') \times \Omega), \tag{5.59}$$

as $n \to +\infty$.

At the possible expense of extending the functions $a(x,t,s,d)$, f on a time interval $(0,\bar{T})$ with $\bar{T} > T$ in such a way that assumptions with a and f hold true with $\bar{T}$ in place of T, we can show that the convergence result (5.59) is still valid in $L^1(Q)$-weak, namely that (5.56) holds true.

Step 4: In this step we prove that u satisfies (5.3).

LEMMA 5.11. *The limit u of the approximate solution u_n of (5.13) satisfies*

$$\lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(u, Du)Du \, \mathrm{d}x\mathrm{d}t = 0.$$

Proof. To this end, observe that for any fixed $m \geq 0$ one has

$$\int_{\{m \leq |u_n| \leq m+1\}} a(u_n, Du_n)Du_n \mathrm{d}x\mathrm{d}t = \int_Q a(u_n, Du_n)(DT_{m+1}(u_n) - DT_m(u_n))\mathrm{d}x\mathrm{d}t$$

$$= \int_Q a(T_{m+1}(u_n), DT_{m+1}(u_n))DT_{m+1}(u_n)\mathrm{d}x\mathrm{d}t - \int_Q a(T_m(u_n), DT_m(u_n))DT_m(u_n)\mathrm{d}x\mathrm{d}t.$$

According to (5.56), one is at liberty to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$ and to obtain

(5.60)

$$\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n, Du_n)Du_n \mathrm{d}x\mathrm{d}t$$

$$= \int_Q a(T_{m+1}(u), DT_{m+1}(u))DT_{m+1}(u)\mathrm{d}x\mathrm{d}t - \int_Q a(T_m(u), DT_m(u))DT_m(u)\mathrm{d}x\mathrm{d}t$$

$$= \int_{\{m\leq|u_n|\leq m+1\}} a(u, Du)Du\mathrm{d}x\mathrm{d}t.$$

Taking the limit as $m \to +\infty$ in (5.60) and using the estimate (5.31) show that u satisfies (5.3) and the proof of Lemma is complete.

Step 5: In this step, u is shown to satisfy (5.4) and (5.5). Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let M be a positive real number such that $\operatorname{supp}(S') \subset [-M, M]$. Pointwise multiplication of the approximate equation (5.13) by $S'(u_n)$ leads to

$$\begin{aligned}\frac{\partial B_S^n(x,u_n)}{\partial t} &- div[S'(u_n)a(u_n, Du_n)] + S''(u_n)a(u_n, Du_n)Du_n \\ &+ div(S'(u_n)\phi_n(u_n)) - S''(u_n)\phi_n(u_n)Du_n = fS'(u_n) \text{ in } D'(Q).\end{aligned} \tag{5.61}$$

It was follows we pass to the limit as in (5.61) n tends to $+\infty$.

• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$. Since S is bounded and continuous, $u_n \to u$ *a.e.* in Q implies that $B_S^n(x, u_n)$ converges to $B_S(x, u)$ it a.e. in Q and L^∞ weak–*. Then $\frac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

• Limit of $-div[S'(u_n)a_n(u_n, Du_n)]$. Since $\operatorname{supp}(S') \subset [-M, M]$, we have for $n \geq M$

$$S'(u_n)a_n(u_n, Du_n) = S'(u_n)a(T_M(u_n), DT_M(u_n)) \ \ a.e. \text{ in } Q.$$

The pointwise convergence of u_n to u and (5.55) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(u_n, Du_n) \rightharpoonup S'(u)a(T_M(u), DT_M(u)) \text{ in } \prod_{i=1}^{N} L^{p'}(Q, w_i^*), \tag{5.62}$$

as n tends to $+\infty$. $S'(u)a(T_M(u), DT_M(u))$ has been denoted by $S'(u)a(u, Du)$ in equation (5.4).

• Limit of $S''(u_n)a(u_n, Du_n)Du_n$. As far as the 'energy' term

$$S''(u_n)a(u_n, Du_n)Du_n = S''(u_n)a(T_M(u_n), DT_M(u_n))DT_M(u_n) \ \ a.e. \text{ in } Q.$$

The pointwise convergence of $S'(u_n)$ *to* $S'(u)$ and (5.56) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

(5.63)
$$S''(u_n)a_n(u_n, Du_n)Du_n \rightharpoonup S''(u)a(T_M(u), DT_M(u))DT_M(u) \text{ weakly in } L^1(Q).$$

Recall that

$$S''(u)a(T_M(u), DT_M(u))DT_M(u) = S''(u)a(u, Du)Du \quad a.e. \text{ in } Q.$$

• Limit of $S'(u_n)\phi_n(u_n)$. Since $\operatorname{supp}(S') \subset [-M, M]$, we have

$$S'(u_n)\phi_n(u_n) = S'(u)\phi_n(T_M(u)) \quad a.e. \text{ in } Q.$$

As a consequence of (5.10) and $u_n \to u$ *a.e.* in Q, it follows that

$$S'(u_n)\phi_n(u_n) \to S'(u)\phi(T_M(u)) \text{ strongly in } \prod_{i=1}^{N} L^{p'}(Q, w_i^*),$$

as n tends to $+\infty$. The term $S'(u)\phi(T_M(u))$ is denoted by $S'(u)\phi(u)$.

• Limit of $S''(u_n)\phi_n(u_n)Du_n$. Since $S' \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp}(S') \subset [-M, M]$, we have

$$S''(u_n)\phi_n(u_n)Du_n = \phi_n(T_M(u_n))DS'(u_n) \quad a.e. \text{ in } Q.$$

Moreover, $DS'(u_n)$ converges to $DS'(u)$ weakly in $L^p(Q, w)$ as n tends to $+\infty$, while $\phi_n(T_M(u_n))$ is uniformly bounded with respect to n and converges *a.e.* in Q to $\phi(T_M(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\phi_n(u_n)Du_n \rightharpoonup \phi(T_M(u))DS'(u) \text{ weakly in } L^p(Q, w).$$

The term $\phi(T_M(u))DS'(u) = S''(u_n)\phi(u)Du$.

• Limit of $S'(u_n)f_n$. Due to (5.11) and $u_n \to u$ *a.e.* in Q, we have

$$S'(u_n)f_n \to S'(u)f \text{ strongly in } L^1(Q) \text{ as } n \to +\infty.$$

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.61) and to conclude that u satisfies (5.4).

It remains to show that $B_S(x, u)$ satisfies the initial condition (5.5). To this end, firstly remark that, S being bounded, $B_S^n(x, u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.61) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0, T; W^{-1,p'}(\Omega, w^*))$. As a consequence, an Aubin's type lemma (see, *e.g.* [30]) implies that $B_S^n(x, u_n)$ lies in a compact set of $C^0([0, T], L^1(\Omega))$. It follows that on the one hand, $B_S^n(x, u_n)(t = 0) = B_S^n(x, u_0^n)$ converges to $B_S(x, u)(t = 0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that

$$B_S(x, u)(t = 0) = B_S(x, u_0) \text{ in } \Omega.$$

As a conclusion of step 1 to step 5, the proof of theorem 5.3 is complete. □

Remark 5.12. We obtain the same result if the data is the forme $f-div(F)$, whith $f\in L^1(\Omega)$ and $F\in\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.

Remark 5.13. Under the assumption of theorem 5.3, if we suppose that the seconde member are nonnegative, then we obtain a nonnegative solution. Indeed, if we take $T_k(u-T_h(u^+))\chi_{(0,\tau)}$ a test function in (5.1), we have

(5.64)

$$\int_\Omega B_k^h(x,u(\tau))\mathrm{d}x+\int_0^\tau\int_\Omega a(x,t,u,Du)DT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t$$
$$+\int_{Q_\tau}\phi(u)DT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t=\int_{Q_\tau} fT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t+\int_\Omega B_k^h(x,u_0)\mathrm{d}x,$$

where $B_k^h(x,r)=\int_0^r T_k(s-T_h(s^+))\frac{\partial b(x,s)}{\partial s}\mathrm{d}s$. The Lipschitz character of ϕ and stokes' formula together with the boundary condition 2 of problem (5.1) give

$$\int_0^\tau\int_\Omega \phi(u)DT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t=0. \tag{5.65}$$

Using (5.65), and $B_k^h(x,u)\geq 0$, it follows that

(5.66)

$$\int_Q a(x,t,u,Du)DT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t\leq\int_Q fT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t+\int_\Omega B_k^h(x,u_0)\mathrm{d}x,$$

we remark also, by using $f\geq 0$

$$\int_Q fT_k(u-T_h(u^+))\mathrm{d}x\mathrm{d}t\leq\int_{\{u\geq h\}} fT_k(u-T_h(u))\mathrm{d}x\mathrm{d}t.$$

On the other hand, thanks to (3.11), we conclude

(5.67)

$$\alpha\int_Q\sum_{i=1}^N w_i(x)\left|\frac{\partial T_k(u^-)}{\partial x_i}\right|^p\mathrm{d}x\mathrm{d}t\leq\int_{\{u\geq h\}} fT_k(u-T_h(u))\mathrm{d}x\mathrm{d}t+\int_\Omega B_k^h(x,u_0)\mathrm{d}x.$$

Letting h tend to infinity, we can easily deduce

$$T_k(u^-)=0,\quad\forall\, k>0,$$

which implies that

$$u\geq 0.$$

6. EXAMPLE

Let us consider the following special case: $b(x,r) = Z(x)C(s)$ where $Z \in W^{1,\,p}(\Omega, w)$, $Z(x) \geq \alpha > 0$ and $C \in C^1(\mathbb{R})$ such that $\forall \;\; k > 0 \;\; : \;\; 0 < \lambda_k \equiv \inf\limits_{|s|\leq k} C'(s)$ and $C(0) = 0$.

$$(6.1) \qquad 0 < \lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left|\nabla_x \left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_k(x)$$

$$\phi : \; r \in \mathbb{R} \to (\phi_i)_{i=1,\dots,N} \in \mathbb{R}^N,$$

where

$$\phi_i(r) = \exp(\alpha_i r) \;\; i = 1, ..., N, \;\;\; \alpha_i \in \mathbb{R}$$

ϕ is a continuous function.

And

$$a_i(x,t,s,d) = w_i(x)\, |d_i|^{p-1}\, sgn(d_i), \;\;\; i = 1, ..., N,$$

with $w_i(x)$ a weight function ($i = 1, ..., N$).

For simplicity, we suppose that

$$w_i(x) = w(x) \;\;\; for \;\;\; i = 1, ..., N-1, \;\;\; w_N(x) \equiv 0.$$

It is easy to show that the $a_i(x,t,s,d)$ are Carathéodory functions satisfying the growth condition (3.9) and the coercivity (3.11). On the order hand the monotonicity condition is verified. In fact,

$$\sum_{i=1}^{N} \left(a_i(x,t,s,d) - a(x,t,s,d')\right)(d_i - d_i')$$

$$= w(x) \sum_{i=1}^{N-1} \left(|d_i|^{p-1}\, sgn(d_i) - \left|d_i'\right|^{p-1} sgn(d_i')\right)(d_i - d_i') \geq 0,$$

for almost all $x \in \Omega$ and for all $d, d' \in \mathbb{R}^N$. This last inequality can not be strict, since for $d \neq d'$ with $d_N \neq d_N'$ and $d_i = d_i'$, $\;i = 1, ..., N-1$, the corresponding expression is zero.

In particular, let us use special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $w(x) = d^{\lambda}(x)$, such that

$$(6.2) \qquad \lambda < \min\left(\frac{p}{N}, p-1\right).$$

Remark 6.1. The condition (6.2) is sufficient to show the integrability condition (3.4). Finally, the hypotheses of Theorem 5.3 are satisfied. Therefore, for all $f \in L^1(Q)$, the following problem:

$$(6.3)\quad \begin{cases} b(x,u) \in L^\infty([0,T]; L^1(\Omega)); \\ T_k(u) \in L^p(0,T; W_0^{1,\ p}(\Omega, w)), \\ \lim\limits_{m\to+\infty} \int_{\{m\le|u|\le m+1\}} \sum\limits_{i=1}^{N} w_i \left|\frac{\partial u}{\partial x_i}\right|^{p-1} \frac{\partial u}{\partial x_i} sgn\left(\frac{\partial u}{\partial x_i}\right) \mathrm{d}x\mathrm{d}t = 0; \\ B_S(x,r) = \int_0^r \frac{\partial b(x,\sigma)}{\partial \sigma} S'(\sigma)\mathrm{d}\sigma, \\ -\int_Q B_S(x,u)\frac{\partial \varphi}{\partial t}\mathrm{d}x\mathrm{d}t + \int_Q S(u) \sum\limits_{i=1}^{N} w_i \left|\frac{\partial u}{\partial x_i}\right|^{p-1} sgn\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial \varphi}{\partial x_i}\mathrm{d}x\mathrm{d}t \\ +\int_Q S'(u) \sum\limits_{i=1}^{N} w_i \left|\frac{\partial u}{\partial x_i}\right|^{p-1} sgn\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i}\varphi\mathrm{d}x\mathrm{d}t \\ +\int_Q \sum\limits_{i=1}^{N} S(u) exp(\alpha_i u)\frac{\partial \varphi}{\partial x_i}\mathrm{d}x\mathrm{d}t - \int_Q \sum\limits_{i=1}^{N} S'(u) exp(\alpha_i u)\frac{\partial u}{\partial x_i}\varphi\mathrm{d}x\mathrm{d}t \\ = \int_Q f S'(u)\varphi\mathrm{d}x\mathrm{d}t, \\ B_S(u)(t=0) = B_S(u_0) \ \text{ in } \ \Omega, \\ \forall\ \ \varphi \in C_0^\infty(Q) \ \text{ and } \ S \in W^{1,\infty}(\mathbb{R}) \ \text{ with } \ S' \in C_0^\infty(\mathbb{R}), \end{cases}$$

has at least one renormalised solution.

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"Sidi Mohamed Ben Abdellah University",
Poly-disciplinary Faculty of Taza,
Laboratory "Laboratoire des Sciences de l'Ingènieur (LSI)",
Department MPI,
P.O. ox 1223 Taza Gare,
Morocco
akdimyoussef@yahoo.fr
mekkour.mounir@yahoo.fr

"Sidi Mohamed Ben Abdellah University",
Faculty of Sciences Dhar El Mahraz,
Laboratory LAMA,
Department of Mathematics,
B.P. 1796, Atlas Fez,
Morocco
jbennouna@hotmail.com

Faculté des Sciences Juridiques,
Economiques et Sociales
Université Hassan 1,
B.P 784, Settat
Morocco
redwane_hicham@yahoo.fr