QUASI-STATIONARY DISTRIBUTIONS: AN APPLICATION OF THE REVIVAL TECHNIQUE

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We use the framework and results developed by Meyer in [7] to establish for general irreducible Markov processes some properties of quasi-stationary distributions well known in the continuous time denumerable space case. We also set out a class of processes that do not satisfy the (usually assumed) condition of asymptotic remoteness of absorbtion, nevertheless still having quasi-stationary distributions.

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1. INTRODUCTION

Quasi-stationary-distributions were introduced to describe the long-term behaviour of Markov processes expected to die out, but maintaining a sort of equilibrium up to their death.

To make things precise let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, X_t, P^x)$ be a right Markov processes with state space $(E, \mathcal{E})$, semigroup $(P_t)_{t\geq 0}$, resolvent $(U^\alpha)_{\alpha\geq 0}$ and lifetime ζ. Denote as usual $P^\nu(\cdot) = \int \nu(\mathrm{d}x) P^x(\cdot)$.

Definition 1. A *quasi-stationary distribution* (in short a QSD) for X is a probability measure ν on $(E, \mathcal{E})$ satisfying

$$P^\nu(X_t \in A, t < \zeta) = \nu(A) P^\nu(t < \zeta)$$

for any $t > 0,\ A \in \mathcal{E}$.

In order to eventually associate a QSD with the Markov process X, the latter will be subject to certain conditions that we now introduce:

(i) *m-irreducibility* with respect to a measure m satisfying $mU^1 \ll m$, *i.e.*

$$m(A) > 0 \Rightarrow U^1(x, A) > 0 \text{ for every } x \in E.$$

(ii) $P_t 1(x) > 0$ for every $t > 0,\ x \in E$.

(iii) $P^x(\zeta) < \infty$ for every $x \in E$.

Comments on the hypotheses. The condition of m-irreducibility is traditionally imposed on the process when looking for QSD's. According to it whenever A is an absorbing set we have either $A = \emptyset$ or $m(A^c) = 0$. Applying this property to $A_t = \{x : P_t 1(x) = 0\}$ for $t > 0$ (which is absorbing since $x \to P_t 1(x)$ is excessive) we get that either $P_t 1(x) > 0$ for each x, or $P_t 1 = 0$ a.e. m, the latter being excluded by the long term analysis we have in mind. Similarly, $x \to P^x(\zeta)$ being excessive we either have $P^x(\zeta) = \infty$ for every $x \in E$, or $P^x(\zeta) < \infty$ a.e.m which leads to the assumption (iii).

We end up this introduction by recalling some results from [6]. First, we have the following

Definition 2. A probability measure ν on $(E, \mathcal{E})$ is *a quasi-limiting distribution* for X (in short a QLD) if there exists a probability measure α on $(E, \mathcal{E})$ such that for every $A \in \mathcal{E}$

$$\lim_{t\to\infty} P^\alpha (X_t \in A \mid t < \zeta) = \nu(A).$$

From Proposition 1 in [6] we get

PROPOSITION 1. *A probability measure ν on $(E, \mathcal{E})$ is a QLD for X if and only if it is a QSD for X.*

Also, we shall repeatedly make use of the following result which is Proposition 3 in [6].

PROPOSITION 2. *For any QSD ν there exists $\lambda_\nu \in]0, \infty[$ such that*

$$P^\nu (\zeta > t) = \exp[-t\lambda_\nu].$$

In particular, for any $\gamma \in]0, \lambda_\nu[$ we have $P^\nu(\exp(\gamma\zeta)) < \infty$.

This proposition entails the equivalence between QSDs and γ-invariant probabilities and as a consequence for any QSD ν we have $\lambda_\nu \le \lambda$, where λ stands for the decay parameter of the irreducible process X.

In Section 2 we recall the framework of reviving a process having $\zeta \in]0, \infty[$, by means of a given probability measure μ on the state space and use this technique to give the precise form of the invariant probability measure $\widetilde{\mu}$ of the revived process as well as to establish the fact that within the class of probabilities μ satisfying $P^\mu(\zeta) < \infty$, μ is a QSD for X if and only if $\mu = \widetilde{\mu}$.

In Section 3, we discuss a *class* of processes that have QSD's although these processes do not satisfy the condition of asymptotic remoteness of extinction. Particular examples of such processe are given in [9].

2. CLASSICAL REVIVAL TECHNIQUE

In the context for QSD's for continuous time denumerable Markov chains a very useful tool is the "returned" or "resurrected" process: a process that behaves like the initial one up to the moment of extinction when it returns to the state space according to a probability μ (see [1, 2, 4, 8]). This technique is essentially based on the connection between the Q-matrix of the resurrected chain and the Q-matrix of the initial one. Since for general Markov processes perturbing a generator with a measure is a very delicate matter we propose instead the existing technique of revival of a process (see [5] and [7]). Actually a very particular case of this theory will be involved.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, X_t, P^x)$ be the initial right Markov process subject to the conditions set out in section 1 and let μ be a probability on $(E, \mathcal{E})$. Recall that this means that we may asumme $\zeta \in]0, \infty[$.

Obtaining the revived process $(\widetilde{X}, \mu) = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{X}_t, \widetilde{\theta}_t, \widetilde{X}_t, \widetilde{P}^x)$ requires the following elements: consider

$$\begin{aligned} \widetilde{\Omega} &:= \prod_{i=1}^{\infty} \Omega_i, \ \ \Omega_i = \Omega \text{ for every } i = 1, 2... \\ s_{-1}(\widetilde{\omega}) &:= 0, \ s_0(\widetilde{\omega}) := \zeta(\omega), \ s_i(\widetilde{\omega}) := \sum_{j=0}^{i} \zeta(\omega_j), \quad \widetilde{\omega} \in \widetilde{\Omega} \end{aligned}$$

and for $s_{i-1}(\widetilde{\omega}) \le t < s_i(\widetilde{\omega})$ define

$$\begin{aligned} \widetilde{\theta}_t(\widetilde{\omega}) &:= \left(\theta_{t-s_{i-1}(\widetilde{\omega})}(\omega_i, \omega_{i+1}, ...)\right) \\ \widetilde{X}_t(\widetilde{\omega}) &:= X_{t-s_{i-1}(\widetilde{\omega})}(\omega_i) \end{aligned}$$

Finally, the probabilities associated with $(\widetilde{X}, \mu)$ are $\widetilde{P}^x := P^x \bigotimes \bigotimes_{i=1}^{\infty} P^{\mu}$.

The particular form of the σ-algebras associated with $(\widetilde{X}, \mu)$ will be irrelevant in the sequel and we shall omit their definitions (which however are crucial in [7] in the proof of the fact that $(\widetilde{X}, \mu)$ is a right process).

Note that the measure μ appears only in the definition of the probabilities (and consequently in the definition of the associated semigroup and resolvent as well); the other elements are defined in a standard way independent of it.

Due to the strong law of the large numbers applied to the sequences $(s_n - s_{n-1})_{n \ge 1}$ and $\left(\int_{s_{n-1}}^{s_n} 1_B\left(\widetilde{X}_s\right) \mathrm{d}s\right)_{n \ge 1}$ we have $\widetilde{\zeta} = \infty$ $\widetilde{P}^x$-a.s. and $(\widetilde{X}, \mu)$ is Harris recurrent with respect to the measure m. Denote by $\widetilde{\mu}$ be the unique (up to multiplicative constants) invariant measure of $(\widetilde{X}, \mu)$.

For reasons that will be subsequently clear we shall restrict our attention to measures in the class

$$\mathcal{M} := \{\mu : \text{probability on } (E, \mathcal{E}) \text{ such that } P^{\mu}(\zeta) < \infty\}.$$

We shall denote by $F_\mu(v) := P^\mu(\zeta \le v)$.

Proposition 3. *For any $\mu \in \mathcal{M}$ the measure $\widetilde{\mu}$ is given by*

$$\widetilde{\mu}(B) := \left[P^\mu(\zeta)\right]^{-1} P^\mu\left(\int_0^\zeta 1_B(X_s)\,\mathrm{d}s\right).$$

Proof. It is enough to check that $\widetilde{\mu}\widetilde{P}_t(f) \le \widetilde{\mu}(f)$ for any bounded, non-negative f and any t, since the process $\widetilde{X}$ is Harris recurrent the only excessive probability measure is the invariant one.

$$\begin{aligned}\int \mu U(\mathrm{d}x)\,\widetilde{P}_t f(x) &= \sum_{\kappa=0}^{\infty} \int \mu U(\mathrm{d}x)\,\widetilde{P}^x\left[f\left(\widetilde{X}_t\right); s_{k-1} \le t < s_k\right] = \\ &= \sum_{\kappa=0}^{\infty} \int_0^{\infty} P^\mu\left(\widetilde{P}^{X_u}\left[f\left(\widetilde{X}_t\right); s_{k-1} \le t < s_k\right]; u < \zeta\right)\mathrm{d}u = \\ &= \sum_{\kappa=0}^{\infty} \int_0^{\infty} \widetilde{P}^\mu\left(\widetilde{P}^{\widetilde{X}_u}\left[f\left(\widetilde{X}_t\right); s_{k-1} \le t < s_k\right]; u < s_0\right)\mathrm{d}u = \\ &= \sum_{\kappa=0}^{\infty} \int_0^{\infty} \widetilde{P}^\mu\left[f\left(\widetilde{X}_{t+u}\right); s_{k-1} \le t+u < s_k; u < s_0\right]\mathrm{d}u.\end{aligned}$$

In the last equality we have applied the simple Markov property of $(\widetilde{X}, \mu)$ and the fact that $s_j \circ \widetilde{\theta}_u = s_j - u$ whenever $u < s_0$. We now separate the first term of the sum and get

$$\int_0^{\infty} \widetilde{P}^\mu\left(f\left(\widetilde{X}_{t+u}\right); 0 \le t+u < s_0\right) = P^\mu \int_t^\infty f(X_v)\,1_{v<\zeta}\mathrm{d}v = \int_t^\infty \mu P_v(f)\,\mathrm{d}v.$$

As for the remaining part we have

$$\begin{aligned}&\sum_{\kappa=1}^{\infty} \int_0^{\infty} \widetilde{P}^\mu\left(f\left(\widetilde{X}_{t+u}\right); s_{k-1} \le t+u < s_k; u < s_0\right)\mathrm{d}u = \\ &= \sum_{\kappa=1}^{\infty} \widetilde{P}^\mu \int_{s_{k-1}}^{s_k}\left(f\left(\widetilde{X}_v\right); t \le v < t+s_0\right)\mathrm{d}v = \\ &= \sum_{\kappa=1}^{\infty} \widetilde{P}^\mu \int_{s_{k-1}(\widetilde{\omega})}^{s_k(\widetilde{\omega})}\left(f\left(X_{v-s_{k-1}(\widetilde{\omega})}(\omega_k)\right); t \le v < t+s_0(\widetilde{\omega})\right)\mathrm{d}v \le \\ &\le \sum_{\kappa=1}^{\infty} \int_0^t P^\mu\left(f\left(X_s(\omega_k)\right), s < \zeta(\omega_k)\right)\widetilde{P}^\mu\left(s_{k-1} - s_0 \le t-s < s_{k-1}\right)\mathrm{d}s.\end{aligned}$$

Since $F_\mu^{(n)}(v) = \widetilde{P}^\mu(s_{n-1} \le v)$ and the last sum equals

$$\int_0^t \mu P_s(f) \sum_{\kappa=1}^{\infty} \left[F_\mu^{(k-1)}(v) - F_\mu^{(k)}(v)\right] = \int_0^t \mu P_s(f). \quad \square$$

We now turn to the second previously announced result, namely

PROPOSITION 4. *Let $\mu \in \mathcal{M}$. Then μ is a QSD for X if and only if $\mu = \widetilde{\mu}$.*

Proof. Suppose first that μ is a QSD for X. To get $\mu = \widetilde{\mu}$ it is enough to check that in this case μ is an invariant measure for $(\widetilde{X}, \mu)$. To this end let

$$\begin{aligned}
\mu\widetilde{P}_t(f) &= \widetilde{P}^\mu\left(f\left(\widetilde{X}_t\right)\right) = \sum_{k=0}^{\infty} \widetilde{P}^\mu\left(f\left(\widetilde{X}_t\right); s_{k-1} \le t < s_k\right) = \\
&= \sum_{k=0}^{\infty} \widetilde{P}^\mu\left(f\left(X_{t-s_{k-1}(\widetilde{\omega})}(\omega_k)\right); s_{k-1}(\widetilde{\omega}) \le t < s_k(\widetilde{\omega})\right) = \\
&= \sum_{k=0}^{\infty} \int_0^t P^\mu\left(f\left(X_{t-s}(\omega_k)\right); 0 \le t-s < \zeta(\omega_k)\right) \mathrm{d}\widetilde{P}^\mu(s_{k-1} \le s) = \\
&= \sum_{k=0}^{\infty} \int_0^t \mu P_{t-s}(f)\, \mathrm{d}F_\mu^{(k)}(s) = \sum_{k=0}^{\infty} \int_0^t \mu(f)\, \mu P_{t-s}(1)\, \mathrm{d}F_\mu^{(k)}(s) = \\
&= \mu(f) \sum_{k=0}^{\infty} \left(F_\mu^{(k)}(t) - F_\mu^{(k+1)}(t)\right) = \mu(f).
\end{aligned}$$

To prove the converse, note that $\mu = \widetilde{\mu}$ implies, due to the particular form of $\widetilde{\mu}$, that $\mu(U1_A) = \mu(A)\mu(U1)$. By the resolvent equation we get for every $\lambda \ge 0$ that

$$\mu\left(U^\lambda 1_A\right) = \mu(A)\mu(U1)\left[1 + \lambda\mu(U1)\right]^{-1}$$

whence $\mu\left(U^\lambda 1_A\right) = \mu(A)\mu\left(U^\lambda 1\right)$ for any $\lambda \ge 0$, $A \in \mathcal{E}$. Also, the function $t \to \mu(P_t 1_A)$ is right continuous since from the equality $\mu = \widetilde{\mu}$ we get that the measure μ is excessive for (P_t) : $\mu(P_t 1_A) = \widetilde{\mu}(P_t 1_A) \le \widetilde{\mu}\left(\widetilde{P}_t 1_A\right) = \widetilde{\mu}(A) = \mu(A)$. The uniqueness of the Laplace transform now implies $\mu(P_t 1_A) = \mu(A)\mu(P_t 1)$ for every $A \in \mathcal{E}$, $t \ge 0$ and thus, μ is a QSD. $\square$

3. ASYMPTOTICAL REMOTENESS OF EXTINCTION VERSUS FAST EXPLOSION

In this section the state space $(E, \mathcal{E})$ of the process is assumed locally compact with a countable base. As usual a point Δ is adjoined to E as a point to infinity if E is not compact and as an isolated point if E is compact.The space E_Δ thus, obtained is endowed with the σ-algebra of Borel sets $\mathcal{E}_\Delta$.

In discussing the existence of QSD's in case of a continuous time denumerable Markov chain the following condition was imposed in [4]:

$$(\mathrm{AR}) : \lim_{x \to \Delta} P_t 1(x) = 1 \text{ for every } t > 0$$

(this was called asymptotic remoteness of the absorbing state in [9]).

It is also proved in [4] that under (AR) a necessary and sufficient condition for the existence of a QSD is :

$$(*)\ \exists\, \gamma > 0 \text{ such that } \{x : P^x (\exp(\gamma\zeta)) < \infty\} \neq \emptyset.$$

But Pakes in [9] gives examples of processes for which (AR) does not hold without preventing the existence of QSD's.

On the other hand in the context of Feller processes irreducible by open sets Sato [10] shows that imposing the condition

$$\text{(FE)} : \lim_{x\to\Delta} P_t 1(x) = 0 \text{ for every } t > 0$$

there exists a probability measure μ which is λ_0-invariant, where

$$\lambda_0 := \lim_{t\to\infty} \left[-t^{-1} \log \sup_{x\in E} P_t 1(x) \right].$$

As $\lambda_0 > 0$ under (FE) μ turns out to be a QSD for X. In a way condition (FE) (coming from fast explosion) is more compatible with $(*)$ since it actually implies

$$(**) \text{ for every } \gamma<\lambda_0 \text{ we have } \sup_{x\in E} P^x (\exp(\gamma\zeta)) < \infty.$$

Further, adding the condition that the process is strong Feller (which is always true in the denumerable case) it is proved in [3] that the exponential decay λ of the process is equal to λ_0 and that (by the results in [12])

$$\lim_{t\to\infty} \frac{P_t(x, A)}{P_t 1(x)} = \mu(A) \text{ for every } x \in E.$$

Actually in [12] this holds up to an m-null set, which due to the absolute continuity underlying the strong Feller property vanishes here.

Thus, in this particular case, we have the best situation, *i.e.* what is called in [6] the existence of Yaglom limits.

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