GLOBAL EXISTENCE AND STABILITY FOR NEUTRAL FUNCTIONAL EVOLUTION EQUATIONS

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Communicated by Gabriela Marinoschi

In this paper, we prove the global existence and attractivity of mild solutions for neutral semilinear evolution equations with delay in a Banach space.

AMS 2010 Subject Classification: 34G20, 34G25, 34K20, 34K30.

Key words: semilinear functional differential equations. mild solution, attractivity, evolution system, fixed-point, infinite delay, infinite interval.

1. INTRODUCTION

Differential equations with delay are used in many fields of science specifically equations with delay. The latter are called neutral differential equations which the delayed argument occurs in the derivative of the state variable as well as in the independent variable. We refer the reader to the books Hale and Lunel [17], Lakshmikantham *et al.* [24], Kolmanovskii and Myshkis [23]. On the other hand, the neutral differential equations have become more important in some mathematic models of real phenomena, especially in control, biological, and medical domains. For more details, we refer the reader to [9, 14, 16, 17].

In the literature there are many papers study the problems of neutral differential equations using different methods. Among them, the fixed point method combined by semigroup theory in Fréchet space, see for exemple Baghli and Benchohra [4, 5, 6] and Hernandez *et al.* [18, 19, 20].

In [26, 27], Travis and Webb are the first who considered the existence and stability of the differential equations with delay, recently the study of equations of this type attracted the attention of many authors for their consideration in the study of different real phenomena see [2, 7, 8, 13, 21].

Inspired by the above-mentioned works, we consider in this paper some sufficient conditions for the existence and attractivity of mild solutions of the following neutral evolution equation

$$(1) \qquad \begin{cases} \frac{d}{dt}[y(t) - g(t, y_t)] - A(t)y(t) = f(t, y_t), & t \in J := [0, \infty) \\ y(t) = \phi(t) & t \in (-\infty, 0], \end{cases}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, f and g is a given function from $J \times \mathcal{B}$ into E, and $\phi \in \mathcal{B}$ is a given functions and $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $0 \leq s \leq t < +\infty$.

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$: Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

As far as we know, there are few papers dealing with global existence results for the problem (1). Most of these results are stated in the Fréchet space setting. The present paper provides sufficient conditions for the existence and attractivity mild solutions to problem (1) in the Banach space setting.

2. PRELIMINARY

Let E a Banach space with the norm $|\cdot|$ and $BC(J,E)$ the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t\in J} |y(t)|.$$

Let $\mathcal{X}$ be the space defined by

$$\mathcal{X} = \{y : \mathbb{R} \to E \text{ such that } y|_J \in BC(J,E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [15] and follow the terminology used in [22]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0,b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0,b)$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0,b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1. In the sequel we assume that K and M are bounded on J and

$$\gamma := \max\left\{ \sup_{t\in\mathbb{R}_+}\{K(t)\}, \sup_{t\in\mathbb{R}_+}\{M(t)\} \right\}.$$

For other details we refer, for instance to the book by Hino *et al* [22].

In what follows, we assume that $\{A(t),\ t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.1. A family of bounded linear operators

$$\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : E \to E\ (t,s) \in \Delta := \{(t,s) \in J{\times}J : 0 \leq s \leq t < +\infty\}$$

is called en evolution system if the following properties are satisfied:

1. $U(t,t) = I$ where I is the identity operator in E,
2. $U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,
3. $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(s,t) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\ y$ is continuous.

More details on evolution systems and their properties could be found on the books of Ahmed [1], Engel and Nagel [12] and Pazy [25].

LEMMA 2.1 ([10]). *Let $C \subset BC(J,E)$ be a set satisfying the following conditions:*

(i) *C is bounded in $BC(J,E)$;*

(ii) *the functions belonging to C are equicontinuous on any compact interval of J;*

(iii) *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of J;*

(iv) *the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.*

Then C is relatively compact in $BC(J,E)$.

THEOREM 2.1 ([3] Buton-Kirk's fixed point theorem). *Let X Banach space, and $A, B : X \to X$ two operators. Suppose that B is a contraction and A a compact operator. Then either*

(i) *$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$ has a solution for $\lambda = 1$, or*

(ii) *the set $\{x \in X : x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax,\ \ \lambda \in (0,1)\}$ is unbounded.*

3. MAIN RESULT

Definition 3.1. A function $y \in \mathcal{X}$ is said to be a mild solution of the problem (1), if

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ U(t,0)(\phi(0) - g(0,\phi)) + g(t, y_t) \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_t)\mathrm{d}s + \int_0^t U(t,s)f(s,y_t)\mathrm{d}s, & \text{if } t \in J. \end{cases} \tag{2}$$

To prove our results we introduce the following conditions:

(H_1) There exists a constant $\widehat{M} \geq 1$ and $\omega > 0$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M}e^{-\omega(t-s)} \quad \text{for every } (s,t) \in \Delta.$$

(H_2) There exists a function $p \in L^1(J, \mathbb{R}_+)$ such that:

$$|f(t,u)| \leq p(t)(\|u\|_{\mathcal{B}} + 1) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H_3) For each $(t,s) \in \Delta$ we have: $\displaystyle\lim_{t \to +\infty} \int_0^t e^{-w(t-s)}p(s)\mathrm{d}s = 0.$

(H_4) There exists a constant $\widetilde{M} > 0$ such that:

$$\|A^{-1}(t)\|_{B(E)} \leq \widetilde{M} \quad \text{for all} \quad t \in J.$$

(H_5) There exists a constant $\ell > 0$ such that

$$|A(t)g(t,\phi) - A(s)g(s,\varphi)| \leq \ell(|t-s| + \|\phi - \varphi\|_{\mathcal{B}})$$

for all $t, s \in J$ and $\phi, \varphi \in \mathcal{B}$.

(H_6) There exists a bounded continuous function $\zeta : J \to \mathbb{R}_+$ such that:

$$|A(t)g(t,\phi)| \leq \zeta(t)\|\phi\|_{\mathcal{B}} \quad \text{for all} \quad t \in J, \ \phi \in \mathcal{B}.$$

THEOREM 3.1. *Assume $(H_1) - (H_6)$ are satisfied, and if*

$$\gamma\left(\ell\widetilde{M} + \frac{\widehat{M}}{\omega}\right) < 1,$$

and

$$\widetilde{M}\zeta^*\gamma + \frac{\widehat{M}\zeta^*}{\omega} + \widehat{M}\gamma < 1.$$

Then the problem (1) admits at least one mild solution.

Proof. It is clear that we will obtain the results if we show that the operator $T : \mathcal{X} \to \mathcal{X}$ defined by:

$$Ty(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ U(t,0)(\phi(0) - g(0,\phi)) + g(t,y_t) \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_t)\mathrm{d}s \\ + \displaystyle\int_0^t U(t,s)f(s,y_t)\mathrm{d}s, & \text{if } t \in J. \end{cases} \tag{3}$$

has a fixed point.

For $\phi \in \mathcal{B}$, we can introduce the following function $x : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0] \\ U(t,0)\phi(0) & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \mathcal{X}$, set

$$y(t) = x(t) + z(t).$$

It is obvious that y satisfies (2) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$\begin{aligned} z(t) &= U(t,0)g(0,\phi) + g(t, x_t + z_t) + \int_0^t U(t,s)A(s)g(s, x_s + z_s)\mathrm{d}s \\ &+ \int_0^t U(t,s)f(s, x_s + z_s)\mathrm{d}s. \end{aligned}$$

Let

$$\mathcal{X}_0 = \{z \in \mathcal{X} : \ z_0 = 0\}.$$

The $\mathcal{X}_0$ is a Banach space with norm

$$\|z\|_{\mathcal{X}_0} = \sup_{t \in J} |z(t)| + \|z_0\|_{\mathcal{B}} = \sup_{t \in J} |z(t)|$$

Now, define the operators $F, L : \mathcal{X}_0 \to \mathcal{X}_0$ by

$$Fz(t) = \int_0^t U(t,s)f(s, z_s + x_s)\mathrm{d}s, \text{ for } t \in J,$$

and

$$Lz(t) = U(t,0)g(0,\phi) + g(t, x_t + z_t) + \int_0^t U(t,s)A(s)g(s, x_s + z_s)\mathrm{d}s.$$

Obviously, the problem (1) has a solution is equivalent to $F + L$ has a fixed point. To prove this end, we start with the following estimation.

For each $z \in \mathcal{X}_0$ and $t \in J$, we have

$$\begin{aligned}
\|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\
&\leq K(t)|z(t)| + K(t)\|U(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} + M(t)\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma\widehat{M}e^{-\omega t}\|\phi\|_{\mathcal{B}} + \gamma\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma(\widehat{M}+1)\|\phi\|_{\mathcal{B}}. \qquad (4)
\end{aligned}$$

Now, we prove that the operators F, L satisfied conditions of Theorem 2.1.

Step 1. F is continuous and compact.

• F **is continuous**. Let $(z^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{X}_0$ such that $z^k \to z$ in $\mathcal{X}_0$. We get for every $t \in J$

$$\begin{aligned}
|F(z^k)(t) - F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_s^k + x_s) - f(s, z_s + x_s)|\,\mathrm{d}s \\
&\leq \widehat{M}\int_0^t e^{-\omega(t-s)}|f(s, z_s^k + x_s) - f(s, z_s + x_s)|\,\mathrm{d}s.
\end{aligned}$$

Since f is continuous, we obtain by the Lebesgue dominated convergence theorem that

$$\|Fz_k - Fz\|_{\mathcal{X}_0} \to 0 \quad \text{as} \quad k \to +\infty.$$

Thus, F is continuous.

• $F(D)$ **relatively compact**. Let D is a bounded sub set of $\mathcal{X}_0$. Then, we will Lemma 2.1.

Let $\eta \geq 0$ such that $D = \{z \in \mathcal{X}_0 : \|x\|_{\mathcal{X}_0} \leq \eta\}$, then for $z \in D$ we have

$$\begin{aligned}
|F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_s + x_s)|\,\mathrm{d}s \\
&\leq \widehat{M}\int_0^t e^{-\omega(t-s)}p(s)(\|z_s + x_s\|_{\mathcal{B}} + 1)\mathrm{d}s \\
&\leq \widehat{M}(\gamma\|z\|_{\mathcal{X}_0} + \gamma(\widehat{M}+1)\|\phi\|_{\mathcal{B}} + 1)\int_0^t e^{-\omega(t-s)}p(s)\mathrm{d}s \\
&\leq \widehat{M}\xi\|p\|_{L^1},
\end{aligned}$$

with $\xi := \gamma\eta + \gamma(\widehat{M}+1)\|\phi\|_{\mathcal{B}} + 1$.

Thus, $F(D)$ is bounded.

• $F(D)$ is **equicontinuous**. Let $s, t \in [0, b]$ with $t > s$ and $z \in D$. Then

$$\begin{aligned}
|(Fz)(t) - (Fz)(s)| &= \left|\int_0^s (U(t,\tau) - U(s,\tau))f(\tau, z_\tau + x_\tau)\mathrm{d}\tau \right. \\
&\quad + \left.\int_s^t U(t,\tau)f(\tau, z_\tau + x_\tau)\mathrm{d}\tau\right|
\end{aligned}$$

$$\leq \int_0^s \|U(t,\tau)-U(s,\tau)\|_{B(E)}\, p(\tau)(\|z_\tau+x_\tau\|_{\mathcal{B}}+1)\mathrm{d}\tau$$
$$+ \quad \widehat{M}\int_s^t e^{-\omega(t-\tau)}\, p(\tau)(\|z_\tau+x_\tau\|_{\mathcal{B}}+1)\mathrm{d}\tau.$$

Using (4) we get

$$|(Fz)(t)-(Fz)(s)| \;\leq\; \xi\int_0^s \|U(t,\tau)-U(s,\tau)\|_{B(E)}\, p(\tau)\,\mathrm{d}\tau+\widehat{M}\xi\int_s^t p(\tau)\mathrm{d}\tau.$$

The right-hand side of the above inequality tends to zero as $t-s\to 0$, then $F(D)$ is equicontinuous.

Now, we will prove that $\Lambda := \{(Fz)(t):\; z\in D\}$ is relatively compact in E. Let $t\in J$ be a fixed and let $0<\varepsilon<t\leq b$. For $z\in D$ we define

$$F_\varepsilon(z)(t)=U(t,t-\varepsilon)\int_0^{t-\varepsilon} U(t-\varepsilon,s)f(s,z_s+x_s)\mathrm{d}s.$$

Since $U(t,s)$ is a compact operator, and the set $\Lambda_\varepsilon := \{(F_\varepsilon z)(t):\; z\in D\}$ is the image of bounded set of E by $U(t,s)$ then Λ_ε is precompact in E for every $0<\varepsilon<t$. Furthermore, for $z\in D$, we have

$$\begin{aligned}|F(z)(t)-F_\varepsilon(z)(t)| &\leq \int_{t-\varepsilon}^t \|U(t,s)\|_{B(E)}|f(s,z_s+x_s)|\mathrm{d}s\\ &\leq \int_{t-\varepsilon}^t \|U(t,s)\|_{B(E)}p(s)(\|z_s+x_s\|_{\mathcal{B}}+1)\mathrm{d}s\\ &\leq \xi\widehat{M}\int_{t-\varepsilon}^t e^{-\omega(t-s)}p(s)\mathrm{d}s.\end{aligned}$$

The right-hand side tends to zero as $\varepsilon\to 0$, then $F_\varepsilon(z)$ converge uniformly to $F(z)$ which implies that $D(t)$ is precompact in E.

Finally, F is **equiconvergent**.

Let $z\in D$, then from (H_1), (H_2) and (4) we have

$$|(Fz)(t)|\leq \widehat{M}\xi\int_0^t e^{-\omega(t-s)}p(s)\mathrm{d}s,$$

it follows immediately by (4) that $|(Fz)(t)|\longrightarrow 0$ as $t\to+\infty$.

Then

$$\lim_{t\to+\infty}|(Fz)(t)-(Fz)(+\infty)|=0$$

which implies that F is equiconvergent.

Step 2. L is a contraction. Take $z,\bar{z}\in\mathcal{X}_0$, then for each $t\in J$ and by (H_1), (H_4), (H_5) and (4)

$$\begin{aligned}
|(Lz)(t)-(L\bar{z})(t)| &\leq |g(t,x_t+z_t)-g(t,x_t+\bar{z}_t)| \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s,x_s+z_s)-A(s)g(s,x_s+\bar{z}_s)|\mathrm{d}s \\
&\leq \|A^{-1}(t)\|_{B(E)}|A(t)g(t,x_t+z_t)-A(t)g(t,x_t+\bar{z}_t)| \\
&+ \widehat{M}\int_0^t e^{-\omega(t-s)}|A(s)g(s,x_s+z_s)-A(s)g(s,x_s+\bar{z}_s)|\mathrm{d}s \\
&\leq \ell\widetilde{M}\|z_t-\bar{z}_t\|_{\mathcal{B}}+\widehat{M}\int_0^t e^{-\omega(t-s)}\|z_s-\bar{z}_s\|_{\mathcal{B}}\mathrm{d}s \\
&\leq \gamma\left(\ell\widetilde{M}+\widehat{M}\int_0^t e^{-\omega(t-s)}ds\right)\|z-\bar{z}\|_{\mathcal{X}_0} \\
&\leq \gamma\left(\ell\widetilde{M}+\frac{\widehat{M}}{\omega}\right)\|z-\bar{z}\|_{\mathcal{X}_0}.
\end{aligned}$$

Therefore,

$$\|Lz-L\bar{z}\|_{\mathcal{X}_0}\leq\gamma\left(\ell\widetilde{M}+\frac{\widehat{M}}{\omega}\right)\|z-\bar{z}\|_{\mathcal{X}_0}.$$

Thus, the operator L is a contraction.

For applying the Theorem 2.1, we must check hypothesis (ii) is not hold, *i.e.* prove that the set

$$D_\lambda=\left\{z\in\mathcal{X}_0 : z=\lambda L\left(\frac{z}{\lambda}\right)+\lambda F(z) \text{ for } \lambda\in(0,1)\right\},$$

is bounded. Let $z\in D_\lambda$ then for each $t\in J$, $z(t)=\lambda L\left(\frac{z}{\lambda}\right)(t)+\lambda F(z)(t)$ then we obtain

$$\begin{aligned}
|z(t)| &\leq \lambda\|U(t,0)\|_{B(E)}\|A^{-1}(t)\|_{B(E)}|A(t)g(0,\phi)| \\
&+ \|A^{-1}(t)\|_{B(E)}|A(t)g(t,x_t+\frac{z_t}{\lambda})| \\
&+ \lambda\int_0^t\|U(t,s)\|_{B(E)}|A(s)g(s,x_s+\frac{z_s}{\lambda})|\mathrm{d}s \\
&+ \lambda\int_0^t\|U(t,s)\|_{B(E)}|f(s,x_s+z_s)|\mathrm{d}s \\
&\leq \lambda\widehat{M}\widetilde{M}\zeta(t)\|\phi\|_{\mathcal{B}}+\lambda\widetilde{M}\zeta(t)\|x_t+\frac{z_t}{\lambda}\|_{\mathcal{B}} \\
&+ \lambda\widehat{M}\int_0^t e^{-\omega(t-s)}\zeta(s)\|x_s+\frac{z_s}{\lambda}\|_{\mathcal{B}}\mathrm{d}s \\
&+ \lambda\widehat{M}\int_0^t e^{-\omega(t-s)}p(s)(\|x_s+z_s\|_{\mathcal{B}}+1)\mathrm{d}s
\end{aligned}$$

$$\begin{aligned} &\leq \widehat{M}\widetilde{M}\zeta^*\|\phi\|_{\mathcal{B}} + \widetilde{M}\zeta^*\gamma(\|z\|_{\mathcal{X}_0} + \mu) \\ &+ \frac{\widehat{M}\zeta^*}{\omega}(\|z\|_{\mathcal{X}_0} + \mu) + \widehat{M}\gamma(\|z\|_{\mathcal{X}_0} + \mu + 1)\|p\|_{L^1}, \end{aligned}$$

with $\mu := (\widehat{M}+1)\|\phi\|_{\mathcal{B}}$ and $\zeta^* := \sup_{t\in J}|\zeta(t)|$.

Therefore,

$$\|z\|_{\mathcal{X}_0} \leq \frac{\widehat{M}\widetilde{M}\zeta^*\|\phi\|_{\mathcal{B}} + \widetilde{M}\zeta^*\gamma\mu + \frac{\widehat{M}\zeta^*}{\omega}\mu + \widehat{M}\gamma(\mu+1)\|p\|_{L^1}}{(1 - \widetilde{M}\zeta^*\gamma - \frac{\widehat{M}\zeta^*}{\omega} - \widehat{M}\gamma)} := c,$$

which implies that D_λ is bounded.

Thus, by Theorem 3.1 the operator T has at least one fixed point which is a mild solution of problem (1). □

4. **ATTRACTIVITY OF SOLUTIONS**

In this section, we study the attractivity of solutions the problem (1)

Definition 4.1 ([11]). We say that solutions of (1) are locally attractive if there exists a closed ball $\overline{B}(z^*,\rho)$ in the space $\mathcal{X}_0$ for some $z^* \in \mathcal{X}$ such that for arbitrary solutions z and $\widetilde{z}$ of (1) belonging to $\overline{B}(z^*,\rho)$ we have that

$$\lim_{t\to+\infty}(z(t) - \widetilde{z}(t)) = 0.$$

Under the assumption of Section 3, let z^* a solution of (1) and $\overline{B}(z^*,\rho)$ the closed ball in $\mathcal{X}_0$ witch ρ satisfies the following inequality

$$\rho \geq \frac{2\widehat{M}\|p\|_{L^1}}{1 - \widetilde{M}\ell\gamma - \frac{\widehat{M}\ell\gamma}{\omega} - 2\widehat{M}\gamma\|p\|_{L^1}}.$$

Moreover, we assume that

$$(5) \qquad \lim_{t\to\infty}\zeta(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty}\int_0^t e^{-\omega(t-s)}\zeta(s)\mathrm{d}s = 0,$$

which ζ is the function in (H_6). Then, for $z \in \overline{B}(z^*,\rho)$ by (H_1)–(H_2) and (4) we have

$$\begin{aligned} |(Tz)(t) - z^*(t)| &= |(Tz)(t) - (Tz^*)(t)| \\ &\leq \|A^{-1}(t)\|_{B(E)}|A(t)g(t,z_t+x_t) - A(t)g(t,z_t^*+x_t)| \\ &+ \int_0^t \|U(t,s)\|_{B(E)}|f(s,z_s+x_s) - f(s,z_s^*+x_s)|\mathrm{d}s \\ &+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s,z_s+x_s) - A(s)g(s,z_s^*+x_s)|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leq \widetilde{M}\ell\|z_t - z_t^*\|_{\mathcal{B}} + \widehat{M}\ell\int_0^t e^{-\omega(t-s)}\|z_s - z_s^*\|_{\mathcal{B}}\mathrm{d}s \\
&+ \widehat{M}\int_0^t e^{-\omega(t-s)}p(t)(\|z_s + x_s\|_{\mathcal{B}} + \|z_s^* + x_s\|_{\mathcal{B}} + 2)\mathrm{d}s \\
&\leq \widetilde{M}\ell\gamma\rho + \frac{\widehat{M}\ell\gamma\rho}{\omega} + 2\widehat{M}(\gamma\rho + 1)\|p\|_{L^1} \\
&\leq \widetilde{M}\ell\gamma\rho + \frac{\widehat{M}\ell\gamma\rho}{\omega} + 2\widehat{M}(\gamma\rho + 1)\|p\|_{L^1} \leq \rho.
\end{aligned}
$$

Therefore, we get $T(\overline{B}(z^*,\rho)) \subset \overline{B}(z^*,\rho)$. So, for each $z \in \overline{B}(z^*,\rho)$ solution of problem (1) and $t \in J$, we have

$$
\begin{aligned}
|z(t) - z^*(t)| &= |(Tz)(t) - (Tz^*)(t)| \\
&\leq \|A^{-1}(t)\|_{B(E)}|A(t)g(t, z_t + x_t) - A(t)g(t, z_t^* + x_t)| \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_s + x_s) - f(s, z_s^* + x_s)|\mathrm{d}s \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s, z_s + x_s) - A(s)g(s, z_s^* + x_s)|\mathrm{d}s \\
&\leq \widetilde{M}\zeta(t)(\|z_t + x_t\|_{\mathcal{B}} + \|z_t^* + x_t\|_{\mathcal{B}}) \\
&+ \widehat{M}\int_0^t e^{-\omega(t-s)}p(s)(\psi(\|z_s + x_s\|_{\mathcal{B}}) + \psi(\|z_s^* + x_s\|_{\mathcal{B}}))\mathrm{d}s \\
&+ \widehat{M}\int_0^t e^{-\omega(t-s)}\zeta(s)(\|z_s + x_s\|_{\mathcal{B}} + \|z_s^* + x_s\|_{\mathcal{B}})\mathrm{d}s \\
&\leq 2\gamma\widetilde{M}(\rho + \mu)\zeta(t) + 2\widehat{M}\psi(\gamma\rho + \gamma\mu)\int_0^t e^{-\omega(t-s)}p(s)\mathrm{d}s \\
&+ 2\gamma\widehat{M}(\rho + \mu)\int_0^t e^{-\omega(t-s)}\zeta(s)\mathrm{d}s.
\end{aligned}
$$

Hence, from (H_3) and (5), we conclude that

$$\lim_{t\to\infty} |z(t) - \widetilde{z}(t)| = 0.$$

Consequently, the solutions of equation (1) are locally attractive.

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Received 14 November 2013

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