ON AN ERGODIC DECOMPOSITION DEFINED IN TERMS OF CERTAIN GENERATORS: THE CASE WHEN THE GENERATOR IS DEFINED ON THE ENTIRE SPACE $C_b(X)$

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In this note, we consider Feller transition functions $(P_t)_{t\in[0,+\infty)}$ defined on a Polish space (X,d), the associated families $((S_t,T_t))_{t\in[0,+\infty)}$ of Markov-Feller pairs, we think of $(S_t)_{t\in[0,+\infty)}$ as a semigroup of positive contractions of $C_b(X) =$ the Banach space of all real-valued bounded continuous functions defined on X, and we assume that $(S_t)_{t\in[0,+\infty)}$ has a generator A defined on the entire $C_b(X)$. For this type of transition function, we characterize completely the sets Γ_{cpi} and Γ_{cpie} that appear in the KBBY (Krylov-Bogolioubov-Beboutoff-Yosida) ergodic decomposition defined by $(P_t)_{t\in[0,+\infty)}$ in terms of A, only. The above-mentioned characterizations of Γ_{cpi} and Γ_{cpie} are a first step in a research program that consists of articulating the KBBY decomposition in terms of the generator and then using the results in order to study the decomposition for various continuous-time time-homogeneous Markov processes (for a description of this research direction, see the subsection 2. *Transition functions of Markov processes* in section 4. *Future research* of the second author's paper *Transition probabilities, transition functions, and an ergodic decomposition,* Bull. of the Transilvania Univ. of Braşov, Vol. **15**(**50**), Series B-2008; see also *Introduction* of the second author's monograph *Invariant Probabilities for Transition Functions,* Springer, 2014.

We then use the characterizations of the sets Γ_{cpi} and Γ_{cpie} in terms of A in order to study certain exponential one-parameter convolution semigroups of probability measures, and to extend and strengthen a result of Hunt discussed in Heyer's 1977 monograph *Probability Measures on Locally Compact Groups.*

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1. INTRODUCTION

It has been pointed out in the monograph [11] in *Introduction* that, even though all the general results obtained in the above-mentioned monograph are valid for various transition functions defined by continuous-time

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time-homogeneous Markov processes, most of the examples in [11] are transition functions defined by flows or by one-parameter convolution semigroups of probability measures, and that transition functions defined by Markov processes are not studied in the monograph at all. The reason that transition functions defined by Markov processes do not appear among the examples discussed in [11] stems from the fact that these transition functions cannot be known explicitly (for details, see Subsection 2, *Transition functions of Markov processes,* of Section 4, *Future research,* of the paper [10]). We will now explain briefly the situation.

Let (X, d) be a locally compact separable metric space, let $(P_t)_{t\in[0,+\infty)}$ be a Feller transition function, and let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$.

In general, S_t is a $B_b(X)$-valued linear operator defined on $B_b(X)$ for every $t \in [0, +\infty)$, where $B_b(X)$ is the Banach space of all real-valued Borel measurable bounded functions on X (the norm on $B_b(X)$ is the usual sup (uniform) norm). Since $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function, we also have that $S_t f \in C_b(X)$ for every $f \in C_b(X)$ and every $t \in [0, +\infty)$, where $C_b(X)$ is the Banach subspace of $B_b(X)$ of all continuous functions in $B_b(X)$. Thus, we may and, most of the time in this paper, we do think of the restriction of S_t to $C_b(X)$ as a positive contraction of $C_b(X)$ for every $t \in [0, +\infty)$.

Clearly, if we think of S_t, $t \in [0, +\infty)$, as positive contractions of $C_b(X)$, then $(S_t)_{t\in[0,+\infty)}$ is a semigroup of positive contractions of $C_b(X)$.

As usual, we say that $(S_t)_{t\in[0,+\infty)}$, as a semigroup of contractions of $C_b(X)$, is *strongly continuous* (see Engel and Nagel's monograph [1], p. 36) if $\lim\limits_{t\to t_0} S_t f$ exists in the norm topology of $C_b(X)$ for every $f \in C_b(X)$ and $t_0 \in [0, +\infty)$.

We will assume that $(S_t)_{t\in[0,+\infty)}$ is strongly continuous.

We will also assume that $P_0(x, A) = \mathbf{1}_A(x)$ for every $x \in X$ and $A \in \mathcal{B}(X) =$ the σ-algebra of all the Borel subsets of X; this means that, as an operator on $C_b(X)$, S_0 is the identity operator $\mathrm{Id}_{C_b(X)}$ on $C_b(X)$.

$$\text{Set } D = \left\{ f \in C_b(X) \;\middle|\; \begin{array}{l} \lim\limits_{\substack{t\to 0\\ t>0}} \frac{S_t f - f}{t} \text{ exists in the} \\ \text{norm topology of } C_b(X) \end{array} \right\}.$$

Now define the $C_b(X)$-valued operator A on D as follows: $Af = \lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{S_t f - f}{t}$ for every $f \in D$. A is called the *generator of the semigroup* $(S_t)_{t\in[0,+\infty)}$, and it is the custom (to which we adhere, as well) to use the notation $D(A)$ for D.

Since we assume that $(S_t)_{t\in[0,+\infty)}$ is strongly continuous and that $S_0 = \mathrm{Id}_{C_b(X)}$, using Theorem 1.4, pp. 51–52 of Engel and Nagel [1], we obtain that $D(A)$ is a dense subspace of $C_b(X)$ and that the generator A determines the semigroup uniquely.

Now, assume that $(P_t)_{t\in[0,+\infty)}$ is the transition function of some continuous-time time-homogeneous Markov process. As pointed out on p. 160 in Ethier and Kurtz's monograph [2], in general, in most cases of interest, the transition functions are not known explicitly. What is usually known is that $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function, that the semigroup $(S_t)_{t\in[0,+\infty)}$, thought of as a semigroup on $C_b(X)$, is strongly continuous, and that $S_0 = \mathrm{Id}_{C_b(X)}$; moreover, even though the transition function $(P_t)_{t\in[0,+\infty)}$ cannot be studied directly, the generator A of $(S_t)_{t\in[0,+\infty)}$ can be obtained for the transition function $(P_t)_{t\in[0,+\infty)}$; also, since $(P_t)_{t\in[0,+\infty)}$ is defined by a Markov process, it follows that $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a. and is pointwise continuous, so $(P_t)_{t\in[0,+\infty)}$ defines an ergodic (KBBY) decomposition of X (that is, the results of Chapter 5 and Section 6.2 of [11] hold true for $(P_t)_{t\in[0,+\infty)}$).

Finally, since the generator A defines the semigroup $(S_t)_{t\in[0,+\infty)}$ on $C_b(X)$ uniquely, extending a bit the discussion that appears in Subsection 2, *Transition Functions of Markov Processes* of Section 4, *Future Research* of [10] and using Proposition 2.1.2 of [11], we obtain that, actually, the generator A of $(S_t)_{t\in[0,+\infty)}$ defines uniquely the transition function $(P_t)_{t\in[0,+\infty)}$.

From the above discussion, it follows that, in theory, we should be able to obtain the sets that appear in the KBBY ergodic decomposition defined by $(P_t)_{t\in[0,+\infty)}$ using the generator A, only. This research direction is the topic of Subsection 2 of Section 4 of [10].

In the present paper, we take a first step in using the generator to study the sets that appear in the decomposition.

Thus, we will characterize the elements of the sets Γ_{cpi} and Γ_{cpie} that appear in the KBBY decomposition in terms of the generator A in the case when $D(A) = C_b(X)$; that is, in the case when A satisfies the conditions of Corollary 2.1.5, p. 52, of Engel and Nagel [1]. By taking advantage of various results discussed in the Ph.D. thesis [5], we will obtain the characterizations in the more general situation when X is a Polish space (that is, when X is a separable complete metrizable topological space; in order to simplify the exposition, we will always assume given a metric d compatible with the topology on X).

Then, we will exhibit a family of Feller transition functions $(P_t)_{t\in[0,+\infty)}$ that have the property that the generator A of the semigroup $(S_t)_{t\in[0,+\infty)}$ (where $(S_t)_{t\in[0,+\infty)}$ is thought of as a semigroup of positive contractions of $C_b(X)$) satisfies the conditions of the above-mentioned Corollary 2.1.5 of Engel and Nagel [1]. The transition functions $(P_t)_{t\in[0,+\infty)}$ under consideration here are defined by exponential one-parameter convolution semigroups $(\mu_t)_{t\in[0,+\infty)}$

of probability measures defined by an element $\mu \in \mathcal{M}(H)$, $\mu \geq 0$, $\|\mu\| = 1$, where H is a locally compact separable metric semigroup that has a neutral element (for the definition and various properties of $(P_t)_{t\in[0,+\infty)}$, see Example 2.2.14 and Proposition 2.2.15, both in [11]). By proving that the generator A of the semigroup $(S_t)_{t\in[0,+\infty)}$ of such a transition function $(P_t)_{t\in[0,+\infty)}$ satisfies the conditions of Corollary 2.1.5, p. 52, of Engel and Nagel [1], we extend and strengthen (i) of Part A of Theorem 4.2.8 (the Hunt representation theorem) on pp. 268–269 of Heyer's monograph [3] for this type of transition functions (see also Theorem 4.1.14, Theorem 4.1.16, and Theorem 4.2.1, all of them in [3]).

Finally, we discuss an application of all the above-mentioned results in the case when $(P_t)_{t\in[0,+\infty)}$ is the transition function defined by an exponential one-parameter convolution semigroup $(\mu_t)_{t\in[0,+\infty)}$ of probability measures generated by $\mu \in \mathcal{M}(H)$, $\mu \geq 0$, $\|\mu\| = 1$, in the case when H is a compact metric group and $H = \bigcup_{n=1}^{\infty}(\text{supp }(\mu^n))$.

The paper is organized as follows: in the next section (Section 2) we make an attempt to unify the terminology used by each of the authors of this paper earlier and we obtain the general results (the characterizations of the elements of Γ_{cpi} and of Γ_{cpie}), and in the last section (Section 3) we discuss the transition functions defined by exponential one-parameter convolution semigroups $(\mu_t)_{t\in[0,+\infty)}$ of probability measures generated by a probability measure $\mu \in \mathcal{M}(H)$, where H is a locally compact separable metric semigroup that has a neutral element.

2. THE SETS Γ_{cpi} AND Γ_{cpie}

We start with a lemma.

LEMMA 2.1. *Let $t \in \mathbb{R}$, $t > 0$, and let $(g_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of real-valued bounded measurable functions defined on $[0, t]$ such that the series $\sum_{n=0}^{\infty} |g_n(s)|$ converges for every $s \in [0, t]$ and such that the limit function $h : [0, t] \to \mathbb{R}$, $h(s) = \sum_{n=0}^{\infty} |g_n(s)|$ for every $s \in [0, t]$, is a bounded function. Then the function $g : [0, t] \to \mathbb{R}$, $g(s) = \sum_{k=0}^{\infty} g_k(s)$ for every $s \in [0, t]$, is well-defined (in the sense that the series $\sum_{k=0}^{\infty} g_k(s)$ converges (conditionally) for every $s \in [0, t]$)*

and is Lebesgue integrable on $[0,t]$*. Moreover,* $\int\limits_0^t g(s)\,\mathrm{d}s = \sum\limits_{n=0}^{\infty}\int\limits_0^t g_n(s)\,\mathrm{d}s$ *(that is* $\int\limits_0^t \sum\limits_{n=0}^{\infty} g_n(s)\,\mathrm{d}s = \sum\limits_{n=0}^{\infty}\int\limits_0^t g_n(s)\,\mathrm{d}s$*, so we can switch the integration and summation signs).*

Proof. Clearly, the series $\sum\limits_{k=0}^{\infty} g_k(s)$ converges conditionally because the series converges absolutely for every $s \in [0,t]$. Therefore, the function $g : [0,t] \to \mathbb{R}$, $g(s) = \sum\limits_{k=0}^{\infty} g_k(s)$ for every $s \in [0,t]$, is well-defined.

Let $f_n : [0,t] \to \mathbb{R}$ be defined by $f_n(s) = \sum\limits_{k=0}^{n} g_k(s)$ for every $s \in [0,t]$ and $n \in \mathbb{N}\cup\{0\}$.

Then $|f_n(s)| \le \sum\limits_{k=0}^{n} |g_k(s)| \le h(s)$ for all $n \in \mathbb{N}$ and $s \in [0,t]$; moreover, the function h is integrable with respect to the Lebesgue measure on $[0,t]$ because h is bounded. Therefore, we can apply the Lebesgue dominated convergence theorem to the functions f_n, $n \in \mathbb{N}\cup\{0\}$, g, and h in order to obtain that the functions g_n and f_n, $n \in \mathbb{N}\cup\{0\}$, and g are integrable, and that the sequence $\left(\int\limits_0^t f_n(s)\,\mathrm{d}s\right)_{n\in\mathbb{N}\cup\{0\}}$ converges to $\int\limits_0^t g(s)\,\mathrm{d}s$ (that is, that the sequence $\left(\int\limits_0^t \left(\sum\limits_{k=0}^{n} g_k(s)\right)\,\mathrm{d}s\right)_{n\in\mathbb{N}\cup\{0\}}$ converges to $\int\limits_0^t g(s)\,\mathrm{d}s$).

Since $\int\limits_0^t \left(\sum\limits_{k=0}^{n} g_k(s)\right)\,\mathrm{d}s = \sum\limits_{k=0}^{n}\int\limits_0^t g_k(s)\,\mathrm{d}s$ for every $n \in \mathbb{N}\cup\{0\}$, we obtain that the sequence of partial sums $\left(\sum\limits_{k=0}^{n}\int\limits_0^t g_k(s)\,\mathrm{d}s\right)_{n\in\mathbb{N}}$ converges to $\int\limits_0^t g(s)\,\mathrm{d}s$; that is, $\sum\limits_{k=0}^{\infty}\int\limits_0^t g_k(s)\,\mathrm{d}s = \int\limits_0^t \left(\sum\limits_{k=0}^{\infty} g_k(s)\right)\,\mathrm{d}s$. $\square$

As mentioned in Introduction, in this paper, a Polish space is a metric space (X, d) such that the topology defined by d is complete (every Cauchy sequence of elements of X is convergent (in X)) and separable.

Given a Polish space (X, d), we use the notation $C_b(X)$ for the Banach space of all real-valued continuous bounded functions on X, where the norm on $C_b(X)$ is the usual uniform (sup) norm $\| \, \|_\infty$: $\|f\|_\infty = \sup_{x \in X} |f(x)|$ for every $f \in C_b(X)$.

LEMMA 2.2. *Let (X, d) be a Polish space, and let $Q : C_b(X) \to C_b(X)$ be a linear bounded operator. Then, for every $f \in C_b(X)$, $x \in X$, and $t \in (0, +\infty)$, the function $g : [0, t] \to \mathbb{R}$ defined by $g(s) = \sum_{k=0}^{\infty} \frac{s^k Q^k f(x)}{k!}$ for every $s \in [0, t]$ (where $0^0 = 0$ and $Q^0 f = f$) is well-defined (in the sense that the series $\sum_{k=0}^{\infty} \frac{s^k Q^k f(x)}{k!}$ converges whenever $s \in [0, t]$), is measurable, and is Lebesgue integrable on $[0, t]$; also, the functions $g_k : [0, t] \to \mathbb{R}$ defined by $g_k(s) = \frac{s^k Q^k f(x)}{k!}$ for every $s \in [0, t]$ and $k \in \mathbb{N} \cup \{0\}$ are integrable with respect to the Lebesgue measure on $[0, t]$, and $\int_0^t g(s)\, \mathrm{d}s = \sum_{k=0}^{\infty} \int_0^t g_k(s)\, \mathrm{d}s$ (that is,*

$$\int_0^t \left(\sum_{k=0}^{\infty} \frac{s^k Q^k f(x)}{k!} \right) \mathrm{d}s = \sum_{k=0}^{\infty} \int_0^t \frac{s^k Q^k f(x)}{k!}\, \mathrm{d}s).$$

Proof. Let $f \in C_b(X)$, $x \in X$, and $t \in (0, +\infty)$. Also, set $M = \|Q\|$, and consider the functions g and g_k, $k \in \mathbb{N} \cup \{0\}$, defined in the lemma.

Since

$$\sum_{k=0}^{\infty} |g_k(s)| = \sum_{k=0}^{\infty} \frac{s^k |Q^k f(x)|}{k!} \leq \sum_{k=0}^{\infty} \frac{s^k M^k \|f\|_\infty}{k!} = e^{sM} \|f\|_\infty,$$

it follows that the series $\sum_{k=0}^{\infty} g_k(s) = \sum_{k=0}^{\infty} \frac{s^k Q^k f(x)}{k!}$ converges conditionally (because it converges absolutely) for every $s \in [0, t]$. Therefore, the function $g : [0, t] \to \mathbb{R}$, $g(s) = \sum_{k=0}^{\infty} g_k(s)$ for every $s \in [0, t]$, is well-defined.

Clearly, g is measurable because each g_k is measurable, $k \in \mathbb{N} \cup \{0\}$, and g is the pointwise limit of the sequence of functions $\left(\sum_{k=0}^{n} g_k \right)_{n \in \mathbb{N} \cup \{0\}}$.

Using the fact that

$$(2.1) \qquad |g(s)| \le \sum_{k=0}^{\infty} |g_k(s)| \le e^{sM}\|f\|_\infty \le e^{tM}\|f\|_\infty$$

for every $s \in [0,t]$, we obtain that g is a bounded function. Since the Lebesgue measure on $[0,t]$ is finite, it follows that g is Lebesgue integrable on $[0,t]$.

The functions g_k, $k \in \mathbb{N} \cup \{0\}$, are bounded measurable functions because these functions are continuous and defined on the closed bounded interval $[0,t]$.

Using (2.1) we obtain that the function $h : [0,t] \to \mathbb{R}$ defined by $h(s) = \sum_{k=0}^{\infty} |g_k(s)|$ for every $s \in [0,t]$, is well-defined (in the sense that $\sum_{k=0}^{\infty} |g_k(s)|$ converges for every $s \in [0,t]$) and is a bounded function. Since h is also measurable, it follows that we can apply Lemma 2.1 to h, g, and g_k, $k \in \mathbb{N} \cup \{0\}$. Thus, we obtain that $\int_0^t g(s)\,\mathrm{d}s = \sum_{k=0}^{\infty} \int_0^t g_k(s)\,\mathrm{d}s$. $\square$

Let (X,d) be a Polish space.

As usual, we denote by $\mathcal{M}(X)$ the Banach space of all real-valued signed Borel measures on X, where the norm on $\mathcal{M}(X)$ is the total variation norm.

Also as usual (see, for instance, p. 30 of [8]), a linear operator $T : \mathcal{M}(X) \to \mathcal{M}(X)$ is called a *Markov operator* if the following two conditions are satisfied:

(MO1) T is a positive operator; that is, $T\mu \ge 0$ for every $\mu \in \mathcal{M}(X)$, $\mu \ge 0$.

(MO2) $\|T\mu\| = \|\mu\|$ whenever $\mu \in \mathcal{M}(X)$, $\mu \ge 0$.

Recall that $B_b(X)$ denotes the Banach space of all real-valued bounded Borel measurable functions on X, where the norm on $B_b(X)$ is the usual sup (uniform) norm: $\|f\| = \sup_{x \in X} |f(x)|$ for every $f \in B_b(X)$.

We will use the notation $\langle f, \mu \rangle$ for $\int_X f(x)\,\mathrm{d}\mu(x)$, where $f \in B_b(X)$ and $\mu \in \mathcal{M}(X)$.

A Markov operator $T : \mathcal{M}(X) \to \mathcal{M}(X)$ is said to be *regular* if there exists a map $S : B_b(X) \to B_b(X)$ such that $\langle Sf, \mu \rangle = \langle f, T\mu \rangle$ for every $f \in B_b(X)$ and $\mu \in \mathcal{M}(X)$.

It can be shown that if T is a regular Markov operator, the map S that appears in the definition of the regularity of T is unique, is a linear positive contraction, and has the property that $S\mathbf{1}_X = \mathbf{1}_X$.

Given a regular Markov operator $T : \mathcal{M}(X) \to \mathcal{M}(X)$ and the operator

$S : B_b(X) \to B_b(X)$ that appears in the definition of the regularity of T, the ordered pair (S, T) is called a *Markov pair.*

We will denote by $\mathcal{B}(X)$ the σ-algebra of all Borel subsets of X.

As in [5], p. 50 (see also [8], p. 4, and Section 1.1 of [11]), a map $P : X \times \mathcal{B}(X) \to \mathbb{R}$ is called a *transition probability* if the following two conditions are satisfied:

(TP1) For every $x \in X$, the map $\mu_x : \mathcal{B}(X) \to \mathbb{R}$ defined by $\mu_x(\mathbf{A}) = P(x, \mathbf{A})$ for every $\mathbf{A} \in \mathcal{B}(X)$ is a probability measure.

(TP2) For every $\mathbf{A} \in \mathcal{B}(X)$, the function $g_{\mathbf{A}} : X \to \mathbb{R}$ defined by $g_{\mathbf{A}}(x) = P(x, \mathbf{A})$ for every $x \in X$ is Borel measurable.

For every $x \in X$, we denote by δ_x the Dirac probability measure concentrated at x.

Given a transition probability $P : X \times \mathcal{B}(X) \to \mathbb{R}$, for every $\mu \in \mathcal{M}(X)$ and $f \in B_b(X)$, we define the maps $\nu_\mu : \mathcal{B}(X) \to \mathbb{R}$ and $g_f : X \to \mathbb{R}$ as follows:

$$\nu_\mu(\mathbf{A}) = \int_X P(x, \mathbf{A})\, \mathrm{d}\mu(x)$$

for every $\mathbf{A} \in \mathcal{B}(X)$, and

$$g_f(x) = \int_X f(y) P(x) \mathrm{d}y$$

for every $x \in X$ (where $P(x, \mathrm{d}y)$ stands for $\mathrm{d}\mu_x(y)$ and μ_x is the probability measure that appears in condition (TP1) in the definition of a transition probability); it is easy to see that ν_μ is a real-valued signed Borel measure on X (so, $\nu_\mu \in \mathcal{M}(X)$) and that $g_f \in B_b(X)$ for every $\mu \in \mathcal{M}(X)$; therefore, it makes sense to define the maps $T : \mathcal{M}(X) \to \mathcal{M}(X)$, $T\mu = \nu_\mu$ for every $\mu \in \mathcal{M}(X)$, and $S : B_b(X) \to B_b(X)$, $Sf = g_f$ for every $f \in B_b(X)$. It is easy to see that T is a Markov operator, and that (S, T) is a Markov pair; therefore, T is actually a regular Markov operator; we say that T and (S, T) are the (*regular*) *Markov operator* and the *Markov pair defined* (or *generated*) *by* P, respectively.

Conversely, given a regular Markov operator $T : \mathcal{M}(X) \to \mathcal{M}(X)$, let (S, T) be the Markov pair generated by T, and consider the map $P : X \times \mathcal{B}(X) \to \mathbb{R}$ defined by $P(x, \mathbf{A}) = T\delta_x(\mathbf{A})$ $(= S\mathbf{1}_{\mathbf{A}}(x))$ for every $x \in X$ and $\mathbf{A} \in \mathcal{B}(X)$; using Proposition 3.3.1, pp. 49–50 of [5], we obtain that P is a transition probability; we refer to P as the *transition probability defined* (or *generated*) *by* T (or *by* (S, T)).

Given a transition probability P and the Markov pair (S, T) defined by P, we say that P is a *Feller transition probability* if the operator S has the property that $Sf \in C_b(X)$ whenever $f \in C_b(X)$. If the transition probability P is Feller, the pair (S, T) is said to be a *Markov-Feller pair.*

A family $(P_t)_{t\in[0,+\infty)}$ of transition probabilities on (X,d) is called a *transition function* if the *Chapman-Kolmogorov equations* hold true; that is, if

$$P_{s+t}(x,\mathbf{A}) = \int\limits_X P_s(y,\mathbf{A})P_t(x,\,\mathrm{d}y)$$

for every $s\in[0,+\infty)$, $t\in[0,+\infty)$, $x\in X$, and $A\in\mathcal{B}(X)$.

Let $(P_t)_{t\in[0,+\infty)}$ be a transition function on (X,d). As in Section 2.1 of [11], we say that $(P_t)_{t\in[0,+\infty)}$ satisfies the *standard measurability assumption* (*s.m.a.*) if for every $A\in\mathcal{B}(X)$, the real-valued map $(t,x)\mapsto P_t(x,A)$, $(t,x)\in[0,+\infty)\times X$, is jointly measurable with respect to t and x; that is, the map is measurable with respect to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ and the product σ-algebra $\mathcal{L}([0,+\infty))\otimes\mathcal{B}(X)$, where $\mathcal{L}([0,+\infty))$ is the σ-algebra of all Lebesgue measurable subsets of $[0,+\infty)$.

A transition function that satisfies the s.m.a. is said to be *standard.*

We say that a transition function $(P_t)_{t\in[0,+\infty)}$ is a *Markov transition function* if the following two conditions are satisfied:

(MTP1) $P_0(x,\mathbf{A}) = \mathbf{1}_{\mathbf{A}}(x)$ for every $x\in X$ and $\mathbf{A}\in\mathcal{B}(X)$.

(MTP2) $(P_t)_{t\in[0,+\infty)}$ is a standard transition function.

Let $(P_t)_{t\in[0,+\infty)}$ be a transition function, and, for every $t\in[0,+\infty)$, let (S_t,T_t) be the Markov pair defined by P_t. It is easy to see that condition (MTP1) in the definition of a Markov transition function means that $S_0: B_b(X)\to B_b(X)$ and $T_0:\mathcal{M}(X)\to\mathcal{M}(X)$ are the identity operators of $B_b(X)$ and $\mathcal{M}(X)$, respectively.

We note that a standard transition function may or may not satisfy condition (MTP1). Most transition functions that we deal with do satisfy (MTP1); an example of a standard transition function that does not appears in Example 2.2.3 of [11].

If a transition function $(P_t)_{t\in[0,+\infty)}$ has the property that P_t is a Feller transition probability for every $t\in[0.+\infty)$, then $(P_t)_{t\in[0,+\infty)}$ is called a *Feller transition function.*

Let $(T_t)_{t\in[0,+\infty)}$ be a semigroup of operators on $\mathcal{M}(X)$, $T_t:\mathcal{M}(X)\to\mathcal{M}(X)$ for every $t\in[0,+\infty)$.

We say that $(T_t)_{t\in[0,+\infty)}$ is a *Markov semigroup* if T_t is a Markov operator for every $t\in[0,+\infty)$ and if T_0 is the identity operator on $\mathcal{M}(X)$ (that is, if $T_0\mu=\mu$ for every $\mu\in\mathcal{M}(X)$).

We say that $(T_t)_{t\in[0,+\infty)}$ is a *regular Markov semigroup* if $(T_t)_{t\in[0,+\infty)}$ is a Markov semigroup, and T_t is a regular operator for every $t\geq 0$.

The semigroup $(T_t)_{t\in[0,+\infty)}$ is said to be *jointly measurable* if, for every $A\in\mathcal{B}(X)$, the map $(t,x)\mapsto T_t\delta_x(A)$, $(t,x)\in[0,+\infty)\times X$, is jointly measurable with respect to t and x; that is, if the map is measurable with respect to the product σ-algebra $\mathcal{L}([0,+\infty)) \otimes \mathcal{B}(X)$.

Assume that $(T_t)_{t\in[0,+\infty)}$ is a jointly measurable regular Markov semigroup and, for every $t \in [0,+\infty)$, let $P_t : X \times \mathcal{B}(X) \to \mathbb{R}$ be defined by $P_t(x,A) = T_t\delta_x(A)$ for every $(x,A) \in X \times \mathcal{B}(X)$. Using several arguments that appear in the proof of Proposition 2.1.2 of [11], we obtain that $(P_t)_{t\in[0,+\infty)}$ is a transition function (note that even though in Proposition 2.1.2 of [11] we consider X to be only a locally compact separable metric space, the arguments in the proof of the proposition are valid even in the more general situation when X is a Polish space). Since $(T_t)_{t\in[0,+\infty)}$ is jointly measurable, it follows that $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a., so $(P_t)_{t\in[0,+\infty)}$ is a standard transition function. Since $(T_t)_{t\in[0,+\infty)}$ is a Markov semigroup, it follows that $(P_t)_{t\in[0,+\infty)}$ is a Markov transition function. Thus, given a jointly measurable regular Markov semigroup, we can associate to it a Markov transition function.

Conversely, given a Markov transition function $(P_t)_{t\in[0,+\infty)}$, let $((S_t,T_t))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$. Using Proposition 2.1.1 of [11] (the proposition and its proof in [11] are valid even if X is Polish rather than a locally compact separable metric space), we obtain that $(T_t)_{t\in[0,+\infty)}$ is a one-parameter semigroup of operators. As pointed out in Section 1.1 of [11], the operators T_t, $t \in [0,+\infty)$, are Markov, and using the equality (1.1.3) of [11], we obtain that T_t, $t \in [0,+\infty)$, are regular, as well; taking into consideration that $(P_t)_{t\in[0,+\infty)}$ is a Markov transition function, we obtain that T_0 is the identity operator, so $(T_t)_{t\in[0,+\infty)}$ is a (regular) Markov semigroup; finally, since we assume that $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a., since $T_t\delta_x(A) = S_t\mathbf{1}_A(x)$ for every $t \in [0,+\infty)$, $x \in X$, and $A \in \mathcal{B}(X)$, and using Proposition 2.1.5 of [11], we obtain that $(T_t)_{t\in[0,+\infty)}$ is jointly measurable. Thus, $(T_t)_{t\in[0,+\infty)}$ is a jointly measurable regular Markov semigroup.

Given a Markov transition function $(P_t)_{t\in[0,+\infty)}$ and the jointly measurable regular Markov semigroup $(T_t)_{t\in[0,+\infty)}$ obtained by the procedure outlined above, we call $(T_t)_{t\in[0,+\infty)}$ the *Markov semigroup defined* (or *generated*) *by* $(P_t)_{t\in[0,+\infty)}$; since the procedure is reversible, we might also say that $(P_t)_{t\in[0,+\infty)}$ *is the* (*Markov*) *transition function defined* (or *generated*) *by* $(T_t)_{t\in[0,+\infty)}$.

Let $(P_t)_{t\in[0,+\infty)}$ be a Markov transition function and let $((S_t,T_t))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$.

If $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function, then we say that $(T_t)_{t\in[0,+\infty)}$ *is a Feller* (*jointly measurable regular*) *Markov semigroup* and, of course, we say that $((S_t,T_t))_{t\in[0,+\infty)}$ *is the family of Markov-Feller pairs defined by* $(P_t)_{t\in[0,+\infty)}$; in this case (as pointed out in Introduction for the situation when X is a locally compact separable metric space), we may, and in this paper we often do think of $(S_t)_{t\in[0,+\infty)}$ as a semigroup of positive contractions of $C_b(X)$.

Using the terminology discussed in Section 1.1 of [11] (see also Section 2.1 of [5]), we consider the dual system $(\mathcal{M}(X), C_b(X))$ (with respect to the function $(\mu, f) \mapsto \langle f, \mu\rangle$ for every $\mu \in \mathcal{M}(X)$ and $f \in C_b(X)$), and we call $\sigma(\mathcal{M}(X), C_b(X))$ the $C_b(X)$-*weak topology of* (or *on*) $\mathcal{M}(X)$.

Set

$$\Gamma_{\mathrm{cp}} = \left\{ x \in X \;\middle|\; \begin{array}{l} \left(\frac{1}{t}\left(\text{P-}\int\limits_0^t T_u\delta_x \,\mathrm{d}u\right)\right)_{t\in[0,+\infty)} \text{ converges in the} \\ C_b(X)\text{-weak topology of } \mathcal{M}(X) \text{ as } t \to +\infty \end{array} \right\},$$

where $\text{P-}\int\limits_0^t T_u\delta_x \,\mathrm{d}u$, $t \in (0,+\infty)$, are pointwise integrals (for the definition and basic properties of the pointwise integrals, see Subsection 3.3.1 of [11]; for the definition of Γ_{cp} stated here, see Subsection 6.3.2 of [5].

For every $x \in \Gamma_{\mathrm{cp}}$, set $\varepsilon_x = \lim\limits_{\substack{t\to+\infty\\ t>0}} {}_{C_b(X)\text{-}w} \left(\frac{1}{t}\left(\text{P-}\int\limits_0^t T_u\delta_x \,\mathrm{d}u\right)\right)$, where $\lim_{C_b(X)\text{-}w}$ denotes the limit in the $C_b(X)$-weak topology of $\mathcal{M}(X)$. As pointed out at the beginning of Subsection 6.3.2 of [5], ε_x is a probability measure for every $x \in \Gamma_{\mathrm{cp}}$. Taking into consideration that $\frac{1}{t}\left(\text{P-}\int\limits_0^t T_u\delta_x \,\mathrm{d}u\right)$, $t \in (0,+\infty)$, $x \in X$, are probability measures, and using Proposition 3.3.7 of [11] and a well-known result (discussed, for instance, on p. 71 of Högnäs and Mukherjea's monograph [4]), we obtain that Γ_{cp} and the probability measures ε_x, $x \in \Gamma_{\mathrm{cp}}$, defined above are precisely the corresponding subset of X and the corresponding probability measures discussed in Section 5.1 of [11], respectively, whenever X is a locally compact separable metric space.

Set

$$\Gamma_{\mathrm{cpi}} = \left\{ x \in \Gamma_{\mathrm{cp}} \,\middle|\, \varepsilon_x \text{ is an invariant probability measure for } (P_t)_{t\in[0,+\infty)} \right\}.$$

Note that if $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function, then $\Gamma_{\mathrm{cp}} = \Gamma_{\mathrm{cpi}}$. Indeed, assume that $(P_t)_{t\in[0,+\infty)}$ is Feller, and let R be the resolvent operator of $(P_t)_{t\in[0,+\infty)}$ (for the definition of R, see Section 6.2 of [5]). Using Proposition 6.2.2 of [5], we obtain that R is a regular Markov operator. Further, since $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function, using the Lebesgue dominated convergence theorem, we obtain that the transition probability defined by R is Feller, as well; therefore, we can use the fact (pointed out in Section 5.3 of [5]) that $\Gamma^R_{\mathrm{cp}} = \Gamma^R_{\mathrm{cpi}}$, where the superscript R indicates that we deal with sets of the

KBBY ergodic decomposition defined by R. Finally, using Proposition 6.2.5 and Theorem 6.3.4, both of Chapter 6 of [5], we obtain that $\Gamma_{\rm cp} = \Gamma_{\rm cpi}$.

For ease of reference, we restate in the next proposition part of Corollary 1.5, p. 52, of Chapter 2 of Engel and Nagel's monograph [1], adapted to our setting.

PROPOSITION 2.3. *Let $(P_t)_{t\in[0,+\infty)}$ be a Feller Markov transition function, and let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t\in[0,+\infty)}$. Assume that $(S_t)_{t\in[0,+\infty)}$, thought of as a semigroup of positive contractions of $C_b(X)$, is strongly continuous, and that the domain $D(A)$ of the generator A of $(S_t)_{t\in[0,+\infty)}$ is the entire space $C_b(X)$. Then $S_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ for every $t \in [0, +\infty)$, where the convergence of the series $\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ takes place in the norm topology of the Banach space of all bounded $C_b(X)$-valued operators on $C_b(X)$.*

Observation. It is the custom to denote by e^{tA} (or $\exp(tA)$) the series $\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$, $t \in [0, +\infty)$.

In the next theorem we obtain a characterization of the elements of the subset $\Gamma_{\rm cp}$ $(= \Gamma_{\rm cpi})$ of X defined by a Feller Markov transition function $(P_t)_{t\in[0,+\infty)}$ that satisfies the conditions of the last proposition.

THEOREM 2.4. *Let $(T_t)_{t\in[0,+\infty)}$ be a Feller jointly measurable regular Markov semigroup on $\mathcal{M}(X)$, assume that the transition function $(P_t)_{t\in[0,+\infty)}$ defined by $(T_t)_{t\in[0,+\infty)}$ satisfies the conditions of Proposition 2.3, let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t\in[0,+\infty)}$, and let A be the generator of $(S_t)_{t\in[0,+\infty)}$, where $(S_t)_{t\in[0,+\infty)}$ is thought of as a semigroup on $C_b(X)$. Finally, let $x \in X$.*

(a) *The following assertions are equivalent:*

(i) *$x \in \Gamma_{\rm cp}$ $(= \Gamma_{\rm cpi})$.*

(ii) *For every $f \in C_b(X)$, the limit $\lim_{t\to+\infty} \sum_{k=0}^{\infty} \frac{t^k A^k f(x)}{(k+1)!}$ exists and is a real number.*

(b) *If $x \in \Gamma_{\rm cp}$ (that is, if the equivalent assertions of (a) hold true), then $\langle f, \varepsilon_x \rangle = \lim_{t\to+\infty} \sum_{k=9}^{\infty} \frac{t^k A^k f(x)}{(k+1)!}$ for every $f \in C_b(X)$.*

Proof. (a) Let $x \in X$. Using the definition of Γ_{cp}, we obtain that $x \in \Gamma_{\mathrm{cp}}$ if and only if $\lim\limits_{t \to +\infty} \left\langle f, \frac{1}{t}\left(\text{P-}\int_0^t T_u \delta_x \, \mathrm{d}u\right)\right\rangle$ exists and is a real number for every $f \in C_b(X)$.

Note that all the results concerning pointwise integrals discussed in Subsection 3.3.1 of [11] hold true also in the more general case when X is Polish rather than a locally compact separable metric space; therefore, we can use Proposition 3.3.7 of [11] in order to obtain that, for every $t \in (0, +\infty)$ and for every $f \in C_b(X)$, we have

$$\left\langle f, \frac{1}{t}\left(\text{P-}\int_0^t T_u \delta_x \, \mathrm{d}u\right)\right\rangle = \frac{1}{t}\left\langle \text{P-}\int_0^t S_u f \, \mathrm{d}u, \delta_x \right\rangle = \frac{1}{t}\int_0^t S_u f(x) \, \mathrm{d}u.$$

Accordingly, $x \in \Gamma_{\mathrm{cp}}$ if and only if the limit $\lim\limits_{t \to +\infty} \frac{1}{t}\int_0^t S_u f(x) \, \mathrm{d}u$ exists and is a real number for every $f \in C_b(X)$.

Using Proposition 2.3, we obtain that the series $\sum\limits_{k=0}^{\infty} \frac{u^k}{k!} A^k f(x)$ converges conditionally to $S_u f(x)$ for every $u \in [0, +\infty)$ and $f \in C_b(X)$; thus, we further obtain that

$$\frac{1}{t}\int_0^t S_u f(x) \, \mathrm{d}u = \frac{1}{t}\int_0^t \sum_{k=0}^{\infty} \frac{u^k A^k f(x)}{k!} \, \mathrm{d}u \tag{2.2}$$

for every $t \in (0, +\infty)$ and $f \in C_b(X)$.

Since we assume that the domain of A is the entire space $C_b(X)$, using Corollary 1.5 of Chapter 2 of Engel and Nagel [1], we obtain that A is a $C_b(X)$-valued linear bounded operator on $C_b(X)$. Using Lemma 2.2, we obtain that the series $\sum\limits_{k=0}^{\infty} \int_0^t \frac{u^k A^k f(x)}{k!} \, \mathrm{d}u$ converges and

$$\int_0^t \sum_{k=0}^{\infty} \frac{u^k A^k f(x)}{k!} \, \mathrm{d}u = \sum_{k=0}^{\infty} \int_0^t \frac{u^k A^k f(x)}{k!} \, \mathrm{d}u \tag{2.3}$$

for every $t \in [0, +\infty)$ and $f \in C_b(X)$.

Using (2.3) in (2.2), we obtain that

$$\frac{1}{t}\int_0^t S_u f(x)\,\mathrm{d}u = \sum_{k=0}^{\infty}\frac{1}{t}\int_0^t \frac{u^k A^k f(x)}{k!}\,\mathrm{d}u = \sum_{k=0}^{\infty}\frac{1}{t}\left(\frac{u^{k+1}}{(k+1)!}\Bigg|_{u=0}^{u=t}\right)A^k f(x)$$

$$= \sum_{k=0}^{\infty}\frac{1}{t}\cdot\frac{t^{k+1}}{(k+1)!}A^k f(x) = \sum_{k=0}^{\infty}\frac{t^k A^k f(x)}{(k+1)!}$$

for every $t \in (0,+\infty)$ and $f \in C_b(X)$.

We conclude that $x \in \Gamma_{\mathrm{cp}}$ if and only if $\sum\limits_{k=0}^{\infty}\frac{t^k A^k f(x)}{(k+1)!}$ converges for all $f \in C_b(X)$.

(b) Assume that $x \in \Gamma_{\mathrm{cp}}$ and that $f \in C_b(X)$. Using the definition of ε_x and the proof of (a) above, we obtain that the limits that appear in the equalities below do exist, and the equalities hold true.

$$\langle f, \varepsilon_x\rangle = \lim_{t\to+\infty}\left\langle f, \frac{1}{t}\left(\text{P-}\int_0^t T_u\delta_x\,\mathrm{d}u\right)\right\rangle = \lim_{t\to+\infty}\frac{1}{t}\int_0^t S_u f(x)\,\mathrm{d}u$$

$$= \lim_{t\to+\infty}\sum_{k=0}^{\infty}\frac{t^k A^k f(x)}{(k+1)!}. \quad \square$$

Using Theorem 2.4, we will now obtain a characterization of the elements of Γ_{cpie} defined by a Feller regular Markov semigroup $(T_t)_{t\in[0,+\infty)}$ which satisfies the conditions of Theorem 2.4.

To this end, let $(T_t)_{t\in[0,+\infty)}$ be such a Feller regular Markov semigroup defined on a Polish space (X, d).

Recall (see Subsection 6.3.2 of [5], or the paper [6]) that the definition of Γ_{cpie} is: $\Gamma_{\mathrm{cpie}} = \{x \in \Gamma_{\mathrm{cpi}} \mid \varepsilon_x \text{ is ergodic for } (T_t)_{t\in[0,+\infty)}\}$.

Let $(P_t)_{t\in[0,+\infty)}$ be the transition function defined by $(T_t)_{t\in[0,+\infty)}$, let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$, and let A be the generator of the semigroup $(S_t)_{t\in[0,+\infty)}$, thought of as a semigroup of positive contractions of $C_b(X)$.

THEOREM 2.5. *Let $x \in \Gamma_{\mathrm{cpi}}$. Then $x \in \Gamma_{\mathrm{cpie}}$ if and only if*

$$\int_{\Gamma_{\mathrm{cpi}}}\left(\lim_{t\to+\infty}\sum_{k=0}^{\infty}\frac{t^k A^k f(y)}{(k+1)!} - \lim_{t\to+\infty}\sum_{k=0}^{\infty}\frac{t^k A^k f(x)}{(k+1)!}\right)^2 \mathrm{d}\varepsilon_x(y) = 0$$

for every $f \in C_b(X)$.

Proof. In order to prove the theorem, we will use the resolvent operator R of $(P_t)_{t\in[0,+\infty)}$ defined as follows: $R:\mathcal{M}(X)\to\mathcal{M}(X)$, $R\mu = \text{P-}\int\limits_0^{+\infty} \mathrm{e}^{-u} T_u\mu\,\mathrm{d}u$ for every $\mu\in\mathcal{M}(X)$ (see [6] or Section 6.2 of [5]).

It is easy to see that, for every $\mu\in\mathcal{M}(X)$, the pointwise integral defining $R\mu$ does exist. As pointed out before Proposition 2.3, R is a Feller regular Markov operator; therefore, R generates an ergodic decomposition of X. We will denote by Γ^R_{cp}, Γ^R_{cpi}, Γ^R_{cpie}, ε^R_x, $x\in\Gamma^R_{\mathrm{cp}}$, and f^*_R, $f\in \mathrm{BM}(X)$, the subsets of X, the Borel measures on X, and the real-valued functions on X that appear in the ergodic decomposition generated by R.

Using results of [6] (see also Section 6.3 of [5]), we obtain that $\Gamma_{\mathrm{cp}}=\Gamma^R_{\mathrm{cp}}$, $\Gamma_{\mathrm{cpi}}=\Gamma^R_{\mathrm{cpi}}$, $\Gamma_{\mathrm{cpie}}=\Gamma^R_{\mathrm{cpie}}$, $\varepsilon_x=\varepsilon^R_x$ for every $x\in\Gamma_{\mathrm{cp}}$, and $f^*=f^*_R$, where $f^*:X\to\mathbb{R}$ is defined by $f^*(x)=\begin{cases}\int\limits_X f(y)\,\mathrm{d}\varepsilon_x(y) & \text{if} \quad x\in\Gamma_{\mathrm{cp}}\\ 0 & \text{if} \quad x\notin\Gamma_{\mathrm{cp}}\end{cases}$ for every $f\in\mathrm{BM}(X)$.

From [7] (see also Section 5.3 of [5]), we obtain that $x\in\Gamma^R_{\mathrm{cpie}}$ if and only if

$$\int\limits_{\Gamma_{\mathrm{cpi}}} \left(f^*_R(y)-f^*_R(x)\right)^2\,\mathrm{d}\varepsilon^R_x(y)=0 \tag{2.4}$$

for all $f\in C_b(X)$ whenever $x\in\Gamma^R_{\mathrm{cpi}}$.

Using (b) of Theorem 2.4, we obtain that

$$f^*_R(x)=f^*(x)=\begin{cases}\lim\limits_{t\to+\infty}\sum\limits_{k=0}^{\infty}\frac{t^kA^kf(x)}{(k+1)!} & \text{if} \quad x\in\Gamma_{\mathrm{cp}}\\ 0 & \text{if} \quad x\notin\Gamma_{\mathrm{cp}}\end{cases} \tag{2.5}$$

for every $f\in C_b(X)$.

In view of (2.4) and (2.5), we obtain that the assertion of the theorem holds true. □

3. EXPONENTIAL ONE-PARAMETER CONVOLUTION SEMIGROUPS OF PROBABILITY MEASURES

A natural question concerning Theorem 2.4 and Theorem 2.5 of the previous section is: are there Polish spaces (X,d) and jointly measurable regular Markov semigroups $(T_t)_{t\in[0,+\infty)}$ on $\mathcal{M}(X)$ such that the transition function $(P_t)_{t\in[0,+\infty)}$ defined by $(T_t)_{t\in[0,+\infty)}$ satisfies the conditions of Theorem 2.4 and Theorem 2.5? Our goal in this section is to exhibit a family of such Markov semigroups. Moreover, we will use Theorem 2.4 and Theorem 2.5 in order to obtain interesting informatom about some of these semigroups. The semigroups $(T_t)_{t\in[0,+\infty)}$ (or, equivalently, the transition functions $(P_t)_{t\in[0,+\infty)}$ defined by $(T_t)_{t\in[0,+\infty)}$) that we will consider here are generated by exponential one-parameter convolution semigroups of probability measures.

In order to obtain the results that we have in mind, we need some preparation.

Let E be a Banach space and let $(x_t)_{t\in(0,+\infty)}$ be an E-valued function defined on $(0,+\infty)$ (note that, as explained in the introductory remarks in Appendix A of [11], we use here (as we generally do whenever convenient) subscript notation for functions).

As usual, we say that $(x_t)_{t\in(0,+\infty)}$ *converges as* $t \downarrow 0$ if there exists $x^* \in E$ such that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $\delta \in \mathbb{R}$, $\delta > 0$, such that $\|x_t - x^*\| < \varepsilon$ whenever $t \in (0,\delta)$. In this case, x^* is called the *limit of* $(x_t)_{t\in(0,+\infty)}$ *as* $t \downarrow 0$ *in the norm topology of* E, and we use the notation $\lim\limits_{t\downarrow 0} x_t$ or $\lim\limits_{\substack{t\to 0\\ t>0}} x_t$ for x^*; if we want to emphasize that the limit is in the norm topology of E, we write $\lim\limits_{t\downarrow 0}{}_{(E,\|\|)} x_t$ or $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(E,\|\|)} x_t$.

We say that $(x_t)_{t\in(0,+\infty)}$ has a *Cauchy behaviour as* $t \downarrow 0$ if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that $\|x_t - x_s\| < \varepsilon$ for every $t \in \mathbb{R}$, $0 < t \leq t_\varepsilon$, and every $s \in \mathbb{R}$, $0 < s \leq t_\varepsilon$.

Finally, we say that $(x_t)_{t\in(0,+\infty)}$ has a *sequentially uniformly Cauchy behaviour as* $t \downarrow 0$ if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that, for every sequence $(t_n)_{n\in\mathbb{N}}$ of elements of $(0,+\infty)$ that tends to 0, it follows that $\|x_{t_n} - x_{t_k}\| < \varepsilon$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $t_n \leq t_\varepsilon$ and $t_k \leq t_\varepsilon$.

LEMMA 3.1. (a) *If* $(x_t)_{t\in(0,+\infty)}$ *has a Cauchy behaviour as* $t \downarrow 0$*, then* $(x_t)_{t\in(0,+\infty)}$ *has a sequentially uniformly Cauchy behaviour as* $t \downarrow 0$*.* (b) *If* $(x_t)_{t\in(0,+\infty)}$ *has a sequentially uniformly Cauchy behaviour as* $t \downarrow 0$*, then* $(x_t)_{t\in(0,+\infty)}$ *converges as* $t \downarrow 0$ *in the norm topology of* E*.*

Proof. (a) Assume that $(x_t)_{t\in(0,+\infty)}$ has a Cauchy behaviour as $t \downarrow 0$, and let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Then there exists $t_\varepsilon \in (0,+\infty)$ such that $\|x_t - x_s\| < \varepsilon$ for every $t \in \mathbb{R}$, $0 < t \leq t_\varepsilon$, and every $s \in \mathbb{R}$, $0 < s \leq t_\varepsilon$.

We therefore obtain that for every sequence $(t_n)_{n\in\mathbb{N}}$ of elements of $(0,+\infty)$ that converges to zero, we have $\|x_{t_n} - x_{t_k}\| < \varepsilon$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $t_n \leq t_\varepsilon$ and $t_k \leq t_\varepsilon$.

Thus, $(x_t)_{t\in(0,+\infty)}$ has a sequentially uniformly Cauchy behaviour as $t \downarrow 0$.

(b) Assume that $(x_t)_{t\in(0,+\infty)}$ has a sequentially uniformly Cauchy behaviour as $t \downarrow 0$.

Since the metric defined by the norm of a Banach space is complete, it follows that $(x_{t_n})_{n\in\mathbb{N}}$ is a convergent sequence whenever $(t_n)_{n\in\mathbb{N}}$ is a sequence of elements of $(0,+\infty)$ that converges to zero. Using a well-known result concerning limits of functions, we obtain that, in order to prove that $(x_t)_{t\in(0,+\infty)}$ converges in the norm topology of E as $t\downarrow 0$, we only have to prove that given two sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ of elements of $(0,+\infty)$ such that both sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ converge to zero, the limits $\lim\limits_{n\to\infty} x_{t_n}$ and $\lim\limits_{k\to\infty} x_{s_k}$ (which, in view of our preceding discussion, do exist in E) are equal.

To this end, let $(t_n)_{n\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ be two sequences of elements of $(0,+\infty)$, assume that both sequences converge to zero, and set $y=\lim\limits_{n\to\infty} x_{t_n}$ and $z=\lim\limits_{k\to\infty} x_{s_k}$. In order to prove that $y=z$, we will prove that $\|y-z\|<\varepsilon$ for every $\varepsilon\in\mathbb{R}$, $\varepsilon>0$.

Thus, let $\varepsilon\in\mathbb{R}$, $\varepsilon>0$.

Since we assume that $(x_t)_{t\in(0,+\infty)}$ has a sequentially uniformly Cauchy behaviour as $t\downarrow 0$, we obtain that there exists $t_\varepsilon\in\mathbb{R}$, $t_\varepsilon>0$, such that, for every sequence $(u_l)_{l\in\mathbb{N}}$ of elements of $(0,+\infty)$ that converges to 0, it follows that $\left\|x_{u_{l_1}}-x_{u_{l_2}}\right\|<\dfrac{\varepsilon}{2}$ for every $l_1\in\mathbb{N}$ and $l_2\in\mathbb{N}$ such that $u_{l_1}\le t_\varepsilon$ and $u_{l_2}\le t_\varepsilon$.

Let $(v_l)_{l\in\mathbb{N}}$ be the sequence of elements of $(0,+\infty)$ defined as follows:

$$v_l=\begin{cases} t_n \text{ if } l=2n-1 & \text{for some } n\in\mathbb{N}\\ s_n \text{ if } l=2n & \text{for some } n\in\mathbb{N}\end{cases}$$

for every $l\in\mathbb{N}$.

Clearly, $(v_l)_{l\in\mathbb{N}}$ converges to zero.

Since both sequences $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ converge to zero, it follows that there exists $n_\varepsilon\in\mathbb{N}$ large enough such that $t_n\le t_\varepsilon$ and $s_n\le t_\varepsilon$ for every $n\in\mathbb{N}$, $n\ge n_\varepsilon$.

It follows that $\left\|x_{v_{l_1}}-x_{v_{l_2}}\right\|<\dfrac{\varepsilon}{2}$ for every $l_1\in\mathbb{N}$, $l_1\ge 2n_\varepsilon-1$, and $l_2\in\mathbb{N}$, $l_2\ge 2n_\varepsilon-1$. In particular, $\left\|x_{v_{2n-1}}-x_{v_{2n}}\right\|<\dfrac{\varepsilon}{2}$ for every $n\in\mathbb{N}$, $n\ge n_\varepsilon$.

We obtain that

$$\|y-z\|=\lim_{n\to\infty}\|x_{t_n}-x_{s_n}\|=\lim_{n\to\infty}\left\|x_{v_{2n-1}}-x_{v_{2n}}\right\|\le\frac{\varepsilon}{2}<\varepsilon.$$

We have therefore proved that $y=z$.

Accordingly, $(x_t)_{t\in(0,+\infty)}$ converges as $t\downarrow 0$ in the norm topology of E. $\square$

PROPOSITION 3.2. *Let $(x_t)_{t\in(0,+\infty)}$ be an E-valued function defined on $(0,+\infty)$. The following assertions are equivalent:*

(a) $\lim\limits_{t\downarrow 0}{}_{(E,\|\|)} x_t$ *does exist.*

(b) $(x_t)_{t\in(0,+\infty)}$ *has a Cauchy behaviour as* $t\downarrow 0$.

Proof. (a) $\Rightarrow$ (b). Assume that the limit of $(x_t)_{t\in(0,+\infty)}$ exists as $t \downarrow 0$ in the norm topology of E, and let x^* be this limit.

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since $x^* = \lim\limits_{t\downarrow 0}{}_{(E,\|\|)} x_t$, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$ such that $\|x_t - x^*\| < \dfrac{\varepsilon}{2}$ for every $t \in (0, t_\varepsilon]$. Accordingly,

$$\|x_t - x_s\| \leq \|x_t - x^*\| + \|x^* - x_s\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $t \in (0, t_\varepsilon]$ and $s \in (0, t_\varepsilon]$.

We have therefore proved that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that $\|x_t - x_s\| < \varepsilon$ for every $t \in (0, t_\varepsilon]$ and $s \in (0, t_\varepsilon]$. Thus, $(x_t)_{t\in(0,+\infty)}$ has a Cauchy behaviour as $t \downarrow 0$.

(b) $\Rightarrow$ (a). Using Lemma 3.1, we obtain that the implication holds true. $\square$

If E is a Banach space, we will use the notation $\mathcal{L}(E)$ for the Banach space of all linear bounded operators $Q : E \to E$, the norm on $\mathcal{L}(E)$ being the usual operator norm.

If $(F, \|\,\|)$ is a Banach space, $\mathbf{A}$ is a set of real numbers that contains the open interval $(0, +\infty)$, and $f : \mathbf{A} \to F$ is a function, and if the limit of $f(t)$ as $t \downarrow 0$ does exist in the norm topology of F, we will often use the notation $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(F\|\,\|)} f(t)$ (or $\lim\limits_{t\downarrow 0}{}_{(F,\|\,\|)} f(t)$) in order to emphasize that the limit is taken with respect to the norm on F. Of course, at times, F could be the space $\mathcal{L}(E)$ discussed in the previous paragraph.

Proposition 3.3. *Let (X, d) be a Polish space, let $(P_t)_{t\in[0,+\infty)}$ be a Feller Markov transition function on (X, d), and let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t\in[0,+\infty)}$. Let $\|\,\|_{\mathcal{L}(\mathcal{M}(X))}$ and $\mathrm{Id}_{\mathcal{M}(X)}$ be the operator norm on $\mathcal{L}(\mathcal{M}(X))$ and the identity operator on $\mathcal{M}(X)$, respectively. If $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(\mathcal{L}(\mathcal{M}(X)),\|\,\|_{\mathcal{L}(\mathcal{M}(X))})} \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(X)}}{t}$ does exist, then, for every $f \in C_b(X)$, the limit $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(C_b(X),\|\,\|)} \dfrac{S_t f - f}{t}$ does exist, as well, where, as usual, $\|\,\|$ is the uniform (sup) norm on $C_b(X)$.*

Proof. Clearly, the proposition is true whenever f is the constant zero function, $f \in C_b(X)$. Thus, we have to prove that $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(C_b(X),\|\,\|)} \dfrac{S_t f - f}{t}$ does exist for every $f \in C_b(X)$, $f \neq 0$.

To this end, let $f \in C_b(X)$, $f \neq 0$.

Set $g_t = \dfrac{S_t f - f}{t}$ and $Q_t = \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(X)}}{t}$ for every $t \in (0, +\infty)$.

We have to prove that $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(C_b(X),\|\,\|)} g_t$ does exist. Using Proposition 3.2, we obtain that it is enough to prove that $(g_t)_{t\in(0,+\infty)}$ has a Cauchy behaviour as $t\downarrow 0$ with respect to the sup norm on $C_b(X)$.

To this end, let $\varepsilon\in\mathbb{R}$, $\varepsilon>0$.

Since we assume that $\lim\limits_{\substack{t\to 0\\ t>0}}{}_{(\mathcal{L}(\mathcal{M}(X)),\|\,\|_{\mathcal{L}(\mathcal{M}(X))})} Q_t$ does exist, using Proposition 3.2 again, we obtain that $(Q_t)_{t\in(0,+\infty)}$ has a Cauchy behaviour as $t\downarrow 0$ with respect to the uniform norm operator topology of $\mathcal{L}(\mathcal{M}(X))$. Thus, there exists $t_\varepsilon\in\mathbb{R}$, $t_\varepsilon>0$, such that $\|Q_t-Q_u\|<\dfrac{\varepsilon}{2\|f\|}$ for every $t\in\mathbb{R}$, $0<t<t_\varepsilon$, and $u\in\mathbb{R}$, $0<u<t_\varepsilon$.

It follows that

$$|g_t(x)-g_u(x)|=\left|\frac{S_tf(x)-f(x)}{t}-\frac{S_uf(x)-f(x)}{u}\right|$$

$$=\left|\left\langle\frac{S_tf-f}{t},\delta_x\right\rangle-\left\langle\frac{S_uf-f}{u},\delta_x\right\rangle\right|=\left|\left\langle f,\frac{T_t\delta_x-\delta_x}{t}\right\rangle-\left\langle f,\frac{T_u\delta_x-\delta_x}{u}\right\rangle\right|$$

$$=|\langle f,(Q_t-Q_u)\delta_x\rangle|\le\|f\|\|Q_t-Q_u\|\|\delta_x\|<\|f\|\cdot\frac{\varepsilon}{2\|f\|}=\frac{\varepsilon}{2}$$

for every $x\in X$, $t\in\mathbb{R}$, $0<t\le t_\varepsilon$, and $u\in\mathbb{R}$, $0<u\le t_\varepsilon$.

Accordingly, $\|g_t-g_u\|\le\dfrac{\varepsilon}{2}<\varepsilon$ for every $t\in\mathbb{R}$, $0<t\le t_\varepsilon$ and $u\in\mathbb{R}$, $0<u\le t_\varepsilon$.

We obtain that $(g_t)_{t\in(0,+\infty)}$ has a Cauchy behaviour as $t\downarrow 0$ with respect to the sup norm on $C_b(X)$ because we have proved that for every $\varepsilon\in\mathbb{R}$, $\varepsilon>0$, there exists $t_\varepsilon\in\mathbb{R}$, $t_\varepsilon>0$, such that $\|g_t-g_u\|<\varepsilon$ for every $t\in(0,t_\varepsilon]$ and $u\in(0,t_\varepsilon]$. $\square$

We will now introduce the family of transition functions that, as mentioned at the beginning of this section, satisfy the conditions of Theorem 2.4 and Theorem 2.5.

To this end, let $(H,\cdot,d)$ be a locally compact separable metric semigroup, assume that H has a neutral element, and let $\mathbf{e}$ be the neutral element of H (thus, for the family of examples that we have in mind, the role of (X,d) is played by (H,d)).

As usual, we make the convention that $\nu^0=\delta_{\mathbf{e}}$ for every $\nu\in\mathcal{M}(H)$ (see the discussion preceding Proposition B.2.8 of Appendix B of [11], and we use the notation $\nu^k=\underbrace{\nu*\nu*\cdots*\nu}_{k\text{ times}}$ whenever $\nu\in\mathcal{M}(H)$, where $*$ is the operation of convolution on $\mathcal{M}(H)$.

Let $\mu \in \mathcal{M}(H)$ be a probability measure and set

$$\mu_t = \mathrm{e}^{-t} \exp(t\mu) = \mathrm{e}^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^k$$

for every $t \in [0, +\infty)$. As discussed in Section B.3 of Appendix B of [11], the family $(\mu_t)_{t \in [0,+\infty)}$ is a one-parameter convolution semigroup of probability measures; we call $(\mu_t)_{t \in [0,+\infty)}$ the *exponential one-parameter convolution semigroup of probability measures defined by* μ.

For every $t \in [0, +\infty)$, let P_t and (S_t, T_t) be the transition probability and the Markov pair defined by μ_t, respectively (for details, see Example 1.1.16 and the discussion preceding Proposition 2.2.10, both of [11]).

By Proposition 2.2.10 of [11], $(P_t)_{t \in [0,+\infty)}$ is a transition function and $((S_t, T_t))_{t \in [0,+\infty)}$ is the family of Markov pairs defined by $(P_t)_{t \in [0,+\infty)}$. Moreover, using Example 1.1.16 of [11], we obtain that $(P_t)_{t \in [0,+\infty)}$ is a Feller transition function.

Our goal is to prove that $(P_t)_{t \in [0,+\infty)}$ satisfies the conditions of Theorem 2.4 and Theorem 2.5. To this end, we need some preparation.

We start with a lemma in which we extend the calculus formula

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{e}^{-t} \mathrm{e}^{ta}) = a - 1, \quad a \in \mathbb{R},$$

to the case when a is a probability measure in $\mathcal{M}(H)$.

In the proof of the lemma, we will deal with products of series of real numbers and series of elements of a Banach algebra with unit (in our case, the Banach algebra is $\mathcal{M}(H)$ and the unit of $\mathcal{M}(H)$ is $\delta_{\mathbf{e}}$, where $\mathbf{e}$ is the neutral element of H). Even though this type of "mixed" products was not discussed explicitly in [11], we can use the results of Section B.2 of Appendix B of [11] to study the products. More precisely, taking into consideration that, for every $r \in \mathbb{R}$ and $\mu \in \mathcal{M}(H)$, the product $r\mu$ can be thought of as the product $(r\delta_{\mathbf{e}}) * \mu$, we obtain that, given a series $\sum_{n=0}^{\infty} r_n$ of real numbers and another series $\sum_{n=0}^{\infty} \mu_n$ of elements of $\mathcal{M}(H)$, for the purpose of multiplying the two series, we can think of the series $\sum_{n=0}^{\infty} r_n$ as the series $\sum_{n=0}^{\infty} (r_n \delta_{\mathbf{e}})$ of elements of $\mathcal{M}(H)$; thus, we can apply various results concerning the multiplication of two series of elements of $\mathcal{M}(H)$ to the two series $\sum_{n=0}^{\infty} r_n$ and $\sum_{n=0}^{\infty} \mu_n$ (for instance, we can use Theorem B.2.7 of Appendix B of [11] applied to $\sum_{n=0}^{\infty} (r_n \delta_{\mathbf{e}})$ and $\sum_{n=0}^{\infty} \mu_n$

in order to obtain that $\left(\sum\limits_{n=0}^{\infty} r_n\right)\left(\sum\limits_{n=0}^{\infty} \mu_n\right) = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{k=0}^{n} r_k\mu_{n-k}\right))$.

LEMMA 3.4. *If $\mu\in\mathcal{M}(H)$ is a probability measure, then $\lim\limits_{\substack{t\to 0\\ t>0}}\dfrac{\mathrm{e}^{-t}\exp(t\mu)-\delta_{\mathbf{e}}}{t}$ exists and is equal to $\mu-\delta_{\mathbf{e}}$, where the limit is taken in the norm topology of $\mathcal{M}(H)$.*

Proof. In view of the comments made before the lemma concerning products of series of real numbers and series of elements of $\mathcal{M}(H)$, we obtain that all the series that appear in the following equalities do converge, the limits that appear do exist, and the equalities hold true for $t>0$:

$$\frac{\mathrm{e}^{-t}\exp(t\mu)-\delta_{\mathbf{e}}}{t} = \frac{\left(\sum\limits_{k=0}^{\infty}\frac{(-t)^k}{k!}\right)\left(\sum\limits_{k=0}^{\infty}\frac{t^k}{k!}\mu^k\right)-\delta_{\mathbf{e}}}{t}$$

$$= \lim_{n\to+\infty}\frac{\left(\sum\limits_{k=0}^{n}\frac{(-t)^k}{k!}\right)\left(\sum\limits_{k=0}^{n}\frac{t^k}{k!}\mu^k\right)-\delta_{\mathbf{e}}}{t}.$$

Since

$$\left(\sum_{k=0}^{n}\frac{(-t)^k}{k!}\right)\left(\sum_{k=0}^{n}\frac{t^k}{k!}\mu^k\right)-\delta_{\mathbf{e}}$$

$$= \left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots+\frac{(-1)^n t^n}{n!}\right)$$

$$\times\left(\delta_{\mathbf{e}}+t\mu+\frac{t^2}{2!}\mu^2+\frac{t^3}{3!}\mu^3+\cdots+\frac{t^n}{n!}\mu^n\right)-\delta_{\mathbf{e}}$$

$$= \delta_{\mathbf{e}}-t\delta_{\mathbf{e}}+t\mu+\left(\sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^j}{j!}\cdot\frac{t^{k-j}}{(k-j)!}\mu^{k-j}\right)-\delta_{\mathbf{e}}$$

$$= t\left(-\delta_{\mathbf{e}}+\mu+\sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right)$$

for every $t\in\mathbb{R}$, $t>0$, and every $n\in\mathbb{N}\cup\{0\}$, it follows that

$$\frac{\mathrm{e}^{-t}\exp(t\mu)-\delta_{\mathbf{e}}}{t} = \lim_{n\to+\infty}\frac{t\left(-\delta_{\mathbf{e}}+\mu+\sum\limits_{k=2}^{n}\sum\limits_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right)}{t} \tag{3.1}$$

for every $t\in(0,+\infty)$.

Since $\lim\limits_{n\to+\infty}\left(\delta_{\mathbf{e}} - t\delta_{\mathbf{e}} + t\mu + \sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^k}{j!(k-j)!}\mu^{k-j}\right)$ does exist (the limit is equal to $e^{-t}\exp(t\mu)$), it follows that, under the assumption that $t \neq 0$,

$$\lim_{n\to+\infty}\left(-\delta_{\mathbf{e}} + \mu + \sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right)$$

does exist, as well, and

$$\lim_{n\to+\infty} t\left(-\delta_{\mathbf{e}} + \mu + \sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right)$$

$$= t\lim_{n\to+\infty}\left(-\delta_{\mathbf{e}} + \mu + \sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right). \tag{3.2}$$

Using (3.2), we obtain that (3.1) becomes

$$\frac{\mathrm{e}^{-t}\exp(t\mu) - \delta_{\mathbf{e}}}{t} = \lim_{n\to+\infty}\left(-\delta_{\mathbf{e}} + \mu + \sum_{k=2}^{n}\sum_{j=0}^{k}(-1)^j\frac{t^{k-1}}{j!(k-j)!}\mu^{k-j}\right) \tag{3.3}$$

for every $t \in (0,+\infty)$.

Let $\eta : \mathbb{N}\cup\{0\} \to \mathcal{M}(H)$ be defined as follows:

$$\eta(n) = \begin{cases} 0 & \text{if} \quad n = 0 \text{ or } 1 \\ \sum\limits_{k=2}^{n}\sum\limits_{j=0}^{k}(-1)^j\dfrac{t^{k-2}}{j!(k-j)!}\mu^{k-j} & \text{if} \quad n \geq 2 \end{cases}.$$

Now note that, using (3.3), we obtain that $\lim\limits_{n\to+\infty}(-\delta_{\mathbf{e}} + \mu + t\eta(n))$ exists (and is equal to $\dfrac{\mathrm{e}^{-t}\exp(t\mu) - \delta_{\mathbf{e}}}{t}$) for every $t \in (0,+\infty)$. Therefore, we further obtain that, for $t \in (0,+\infty)$, the limit $\lim\limits_{n\to+\infty}\eta(n)$ does exist. Set $\nu = \lim\limits_{n\to+\infty}\eta(n)$.

It follows that $\lim\limits_{\substack{t\to0\\t>0}}\dfrac{\mathrm{e}^{-t}\exp(t\mu) - \delta_{\mathbf{e}}}{t}$ does exist and is equal to $-\delta_{\mathbf{e}} + \mu$ because $-\delta_{\mathbf{e}} + \mu + \sum\limits_{k=2}^{n}\sum\limits_{j=0}^{k}(-1)^j\dfrac{t^{k-1}}{j!(k-j)!}\mu^{k-j} = -\delta_{\mathbf{e}} + \mu + t\eta(n)$ for every $n \in \mathbb{N}$, $n \geq 2$, and $t \in (0,+\infty)$, and because $\lim\limits_{\substack{t\to0\\t>0}}\lim\limits_{n\to+\infty}(-\delta_{\mathbf{e}} + \mu + t\eta(n))$ does exist and is equal to $-\delta_{\mathbf{e}} + \mu$. $\square$

THEOREM 3.5. *Let $\mu \in \mathcal{M}(H)$ be a probability measure, let $(\mu_t)_{t\in[0,+\infty)}$ be the exponential one-parameter convolution semigroup of probability measures defined by μ, let $(P_t)_{t\in[0,+\infty)}$ be the (Feller) transition function defined by $(\mu_t)_{t\in[0,+\infty)}$, and let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t\in[0,+\infty)}$. Then $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t}$ does exist in the norm topology of $\mathcal{L}(\mathcal{M}(H))$.*

Proof. Let $T : \mathcal{M}(H) \to \mathcal{M}(H)$ be the convolution operator defined by μ; thus, T acts as follows: $T\nu = \mu * \nu$ for every $\nu \in \mathcal{M}(H)$ (for details on convolution operators, see Example 1.1.16 of [11] and the paper [9]).

In order to prove the theorem, we will actually prove a bit more; namely, we will show that $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t}$ does exist and is equal to $T - \mathrm{Id}_{\mathcal{M}(H)}$ in the norm topology of $\mathcal{L}(\mathcal{M}(H))$. Thus, it is enough to prove that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $t_\varepsilon \in (0, +\infty)$ such that $\left\| \left(\dfrac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t} - \big(T - \mathrm{Id}_{\mathcal{M}(H)}\big) \right) \nu \right\| < \dfrac{\varepsilon}{2}$ for every $t \in (0, t_\varepsilon)$ and every probability measure $\nu \in \mathcal{M}(H)$.

To this end, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Using Lemma 3.4, we obtain that there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that $\left\| \dfrac{\mathrm{e}^{-t} \exp(t\mu) - \delta_{\mathbf{e}}}{t} - (\mu - \delta_{\mathbf{e}}) \right\| < \dfrac{\varepsilon}{2}$ for every $t \in \mathbb{R}$, $t \in (0, t_\varepsilon)$.

Taking into consideration that $\mathcal{M}(H)$ is a Banach algebra (see Section B.2 of Appendix B in [11]), we obtain that

$$\left\| \left(\frac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t} - (T - \mathrm{Id}_{\mathcal{M}(H)}) \right) \nu \right\| = \left\| \frac{T_t\nu - \nu}{t} - (T\nu - \nu) \right\|$$

$$= \left\| \frac{\mu_t * \nu - \nu}{t} - (\mu * \nu - \nu) \right\| = \left\| \left(\frac{\mu_t - \delta_{\mathbf{e}}}{t} - (\mu - \delta_{\mathbf{e}}) \right) * \nu \right\|$$

$$\leq \left\| \frac{\mu_t - \delta_{\mathbf{e}}}{t} - (\mu - \delta_{\mathbf{e}}) \right\| \|\nu\| = \left\| \frac{\mathrm{e}^{-t} \exp(t\mu) - \delta_{\mathbf{e}}}{t} - (\mu - \delta_{\mathbf{e}}) \right\| < \frac{\varepsilon}{2}$$

for every $t \in (0, t_\varepsilon)$ and every probability measure $\nu \in \mathcal{M}(H)$. $\square$

In view of our discussion so far, we are now in a position to prove that, in our setting (described after Proposition 3.3), a transition function defined by an exponential one-parameter convolution semigroup of probabilities generated by a probability measure satisfies the conditions of Theorem 2.4 and Theorem 2.5. We discuss the details in the next theorem.

THEOREM 3.6. *Let $\mu \in \mathcal{M}(H)$, $\mu \geq 0$, $\|\mu\| = 1$, let $(\mu_t)_{t\in[0,+\infty)}$ be the exponential one-parameter convolution semigroup of probability measures defined by μ, and let $(P_t)_{t\in[0,+\infty)}$ and $((S_t, T_t))_{t\in[0,+\infty)}$ be the Feller transition function and the family of Markov-Feller pairs defined by $(\mu_t)_{t\in[0,+\infty)}$, respectively. Then:*

(a) *The semigroup $(S_t)_{t\in[0,+\infty)}$ thought of as a semigroup of positive contractions of $C_b(H)$ is strongly continuous and has a generator, say A.*

(b) *The domain $D(A)$ of A is the entire space $C_b(H)$.*

Proof. Note that it is enough to prove that, for every $f \in C_b(H)$, the limit $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{S_t f - f}{t}$ exists in the norm topology of $C_b(H)$, because if the limits $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{S_t f - f}{t}$, $f \in C_b(H)$, do exist, then, for every $f \in C_b(H)$, the map $t \mapsto S_t f$, $t \in [0, +\infty)$, is continuous, and the domain $D(A)$ of the generator A of $(S_t)_{t\in[0,+\infty)}$ is the space $C_b(H)$, so both (a) and (b) of the theorem are true.

Thus, let P_μ and (S_μ, T_μ) be the (Feller) transition probability and the Markov-Feller pair defined by μ, respectively (for details on the definition and the various properties of P_μ and (S_μ, T_μ), see [9] and Example 1.1.16 of [11]).

Clearly, the proof of the theorem will be completed as soon as we show that $S_\mu f - f$ is the limit, in the norm topology of $C_b(H)$, of the $C_b(H)$-valued function $t \mapsto \dfrac{S_t f - f}{t}$, $t \in (0, +\infty)$, as $t \to 0$, $t > 0$, for every $f \in C_b(H)$.

To this end, let $f \in C_b(H)$, and note that we may (and therefore we do) assume that $f \neq 0$.

Now let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Using the proof of Theorem 3.5, we obtain that $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t}$ does exist and is equal to $T_\mu - \mathrm{Id}_{\mathcal{M}(H)}$; therefore, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that $\left\| \dfrac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t} - (T_\mu - \mathrm{Id}_{\mathcal{M}(H)}) \right\|_{\mathcal{L}(\mathcal{M}(H))} < \dfrac{\varepsilon}{2\|f\|}$ for every $t \in (0, t_\varepsilon)$, where $\| \; \|_{\mathcal{L}(\mathcal{M}(H))}$ is the norm of the Banach space $\mathcal{L}(\mathcal{M}(H))$ (the uniform operator norm).

In view of the definition of t_ε, we obtain that

$$\left\| \left(\frac{T_t - \mathrm{Id}_{\mathcal{M}(H)}}{t} \right)(\nu) - (T_\mu - \mathrm{Id}_{\mathcal{M}(H)})(\nu) \right\| < \frac{\varepsilon}{2\|f\|}$$

for every $t \in (0, t_\varepsilon)$ and for every probability measure $\nu \in \mathcal{M}(H)$. In particular, for every $x \in H$, if we set $\nu = \delta_x$, we obtain that

$$\left\| \frac{T_t \delta_x - \delta_x}{t} - (T_\mu \delta_x - \delta_x) \right\| < \frac{\varepsilon}{2\|f\|}$$

for every $t \in (0, t_\varepsilon)$.

It follows that

$$\left|\frac{S_t f(x) - f(x)}{t} - (S_\mu f(x) - f(x))\right| = \left|\left\langle \frac{S_t f - f}{t} - (S_\mu f - f), \delta_x \right\rangle\right|$$
$$= \left|\left\langle f, \frac{T_t\delta_x - \delta_x}{t} - (T_\mu \delta_x - \delta_x)\right\rangle\right|$$
$$\le \|f\| \cdot \left\|\frac{T_t\delta_x - \delta_x}{t} - (T_\mu \delta_x - \delta_x)\right\| < \|f\| \cdot \frac{\varepsilon}{2\|f\|} = \frac{\varepsilon}{2}$$

for every $x \in H$ and $t \in (0, t_\varepsilon)$.

Thus, we obtain that $\sup\limits_{x\in H} \left|\frac{S_t f(x) - f(x)}{t} - (S_\mu f(x) - f(x))\right| \le \frac{\varepsilon}{2} < \varepsilon$ for every $t \in (0, t_\varepsilon)$.

Accordingly,

$$\left\|\frac{S_t f - f}{t} - (S_\mu f - f)\right\| \le \frac{\varepsilon}{2} < \varepsilon \tag{3.4}$$

for every $t \in (0, t_\varepsilon)$.

Since for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $t_\varepsilon \in \mathbb{R}$, $t_\varepsilon > 0$, such that (3.4) holds true for every $t \in (0, t_\varepsilon)$, it follows that $\lim\limits_{\substack{t\to 0\\ t>0}} \frac{S_t f - f}{t}$ does exist in $C_b(H)$ and is equal to $S_\mu f - f$. $\square$

Note that in Theorem 3.6 we extend and strengthen (i) of part A of Theorem 4.2.8 (the Hunt representation theorem) on pp. 268–269 of Heyer's monograph [3] (or of any of the following results: Theorem 4.1.14, pp. 257–258 of [3], Theorem 4.1.16, p. 259 of [3], and Theorem 4.2.1, p. 260 of [3]) in the case of the exponential one-parameter convolution semigroup $(\mu_t)_{t\in[0,+\infty)}$ of probabilities defined by a probability measure. Indeed, Theorem 3.6 is an extension of the above-mentioned results of Heyer's monograph [3] because we do not assume in Theorem 3.6 that H is a Lie group (if H is actually a Lie group, it is obvious that $(\mu_t)_{t\in[0,+\infty)}$ satisfies the conditions of the above-mentioned results of [3]). Also, Theoreem 3.6 is a strengthening of the results mentioned above of Heyer's monograph [3] for the one-parameter semigroup $(\mu_t)_{t\in[0,+\infty)}$ because if $(S_t)_{t\in[0,+\infty)}$ is the semigroup of positive contractions of $C_b(H)$ considered in Theorem 3.6, then in the theorem we prove that the domain of the generator of $(S_t)_{t\in[0,+\infty)}$ is the entire space $C_b(H)$.

We will now discuss an application of the results obtained so far in the case in which we deal with a certain type of exponential one-parameter convolution semigroup of probabilities defined by a probability measure, and we assume that H (the semigroup on which the probability measures that make up the convolution semigroup are defined) is a compact metric group.

THEOREM 3.7. *Let $(H,\cdot,d)$ be a compact metric group, and let $\mathbf{e}$ be the neutral element of H. Let λ be the Haar probability measure on H, and let $\mu \in \mathcal{M}(H)$ be a probability measure such that $\bigcup_{n=0}^{\infty}(supp\ (\mu^n)) = H$. As in Example 1.1.16 of [11], let (S_μ, T_μ) be the Markov-Feller pair defined by μ. The following two assertions hold true for every real-valued continuous bounded function f defined on H:*

(a) *The series $\sum_{k=0}^{\infty} \frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}$ converges for every $t \in [0,+\infty)$ and $x \in H$, where $C(H)$ $(= C_0(H) = C_b(H))$ is the Banach space of all real-valued bounded continuous functions defined on H and $\mathrm{Id}_{C(H)}$ is, of course, the identity operator on $C(H)$.*

(b) *The limit*

$$\lim_{t\to+\infty} \sum_{k=0}^{\infty} \frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}$$

(or $\lim_{t\to+\infty} \sum_{k=0}^{\infty} \frac{t^k((\mu-\delta_{\mathbf{e}})^k * f)(x)}{(k+1)!}$, where we use the notation $\nu * f$ for the function $\nu * f : H \to \mathbb{R}$ defined by $(\nu * f)(x) = \int_H f(zx)\,\mathrm{d}\nu(z)$ for every $\nu \in \mathcal{M}(H)$) exists for every $x \in H$, is independent of x, and is equal to $\langle f, \lambda\rangle$ $(= \int_H f(y)\,\mathrm{d}\lambda(y))$.

Proof. (a) Let $x \in H$, $t \in [0,+\infty)$, and $f \in C(H)$.

In order to prove that the series $\sum_{k=0}^{\infty} \frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}$ converges, we note that the series converges absolutely; that is, the series $\sum_{k=0}^{\infty} \left|\frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}\right|$ converges. Indeed,

$$\sum_{k=0}^{\infty} \left|\frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}\right| \le \sum_{k=0}^{\infty} \frac{t^k \left\|\left(S_\mu - \mathrm{Id}_{C(H)}\right)^k f\right\|}{(k+1)!}$$

$$\le \sum_{k=0}^{\infty} \frac{t^k \left\|(S_\mu - \mathrm{Id}_{C(H)})^k\right\|_{\mathcal{L}(C(H))} \|f\|}{(k+1)!} \le \sum_{k=0}^{\infty} \frac{t^k \left\|S_\mu - \mathrm{Id}_{C(H)}\right\|_{\mathcal{L}(C(H))}^k \|f\|}{(k+1)!}$$

$$\le \sum_{k=0}^{\infty} \frac{t^k \left\|S_\mu - \mathrm{Id}_{C(H)}\right\|_{\mathcal{L}(C(H))}^k \|f\|}{k!} = \mathrm{e}^{t\|S_\mu - \mathrm{Id}_{C(H)}\|_{\mathcal{L}(C(H))}}\|f\| < +\infty,$$

where $\| \|$ is the uniform (sup) norm of $C(H)$, of course.

Thus, $\displaystyle\sum_{k=0}^{\infty}\left|\frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}\right|$ is a convergent series.

(b) Let $(\mu_t)_{t\in[0,+\infty)}$ be the exponential one-parameter convolution semigroup of probability measures defined by μ, let $(P_t)_{t\in[0,+\infty)}$ be the (Feller) transition function defined by $(\mu_t)_{t\in[0,+\infty)}$, and let $((S_t,T_t))_{t\in[0,+\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t\in[0,+\infty)}$.

By Theorem 3.6, the semigroup $(S_t)_{t\in[0,+\infty)}$, thought of as a semigroup of positive contractions of $C(H)$, has a generator, that we denote by A, and the domain of A is the entire space $C(H)$.

Let P_μ be the transition probability generated by μ, and let (S_μ, T_μ) be the Markov-Feller pair generated by P_μ (for the definitions of P_μ and of (S_μ, T_μ), see Example 1.1.16 of [11], or the paper [9]).

Using the proof of Theorem 3.6, we obtain that $A = S_\mu - \mathrm{Id}_{C(H)}$.

Since H is compact, the sequence $(\mu^n)_{n\in\mathbb{N}}$ is obviously tight; therefore, using Proposition 4.2 of [9], we obtain that μ is an equicontinuous measure.

Using Proposition 2.2.15 of Subsection 2.2.3 of [11], we obtain that $(P_t)_{t\in[0,+\infty)}$ is a $C_0(H)$ $(= C(H))$-equicontinuous transition function.

Since H is a group, it follows that H has left zeroids (actually, any element of H is a left zeroid because $H = \bigcap\limits_{a\in H} Ha$; for a discussion of zeroids in semigroups, see Section A.1 of Appendix A of [11]); since $\bigcup\limits_{n=0}^{\infty}(\mathrm{supp}\ (\mu^n)) = H$, it follows that all the orbits of $(P_t)_{t\in[0,+\infty)}$ are dense in H. Taking into consideration that the Haar probability measure λ is an invariant probability of $(P_t)_{t\in[0,+\infty)}$, using Example 7.2.15 of [11], we obtain that $(P_t)_{t\in[0,+\infty)}$ is uniquely ergodic and λ is the unique invariant probability of $(P_t)_{t\in[0,+\infty)}$.

Using again Example 7.2.15 of [11], we obtain that $\Gamma_{\mathrm{cp}} = \Gamma_{\mathrm{cpie}} = H$.

In view of Theorem 2.4, we obtain that $\displaystyle\lim_{t\to+\infty}\sum_{k=0}^{\infty}\frac{t^k(S_\mu - \mathrm{Id}_{C(H)})^k f(x)}{(k+1)!}$ exists for every $x \in H$ and every $f \in C(H)$, and is equal to $\langle f, \lambda\rangle$ (so, the limit is independent of x). $\square$

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