PRACTICAL EXPONENTIAL STABILITY OF NONLINEAR TIME-VARYING CASCADE SYSTEMS

DJAMILA BELDJERD, LYNDA OUDJEDI and MOUSSADEK REMILI

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In this paper, we give some sufficient conditions which guarantee practical uniform exponential stability of nonlinear time-varying cascade systems using the Lyapunov functional method. In this way, we extend some existing results under more generalized assumptions.

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1. INTRODUCTION

In this paper, we study the practical stability of nonlinear time-varying system of the form

$$\dot{x}_1 = f_1(t, x_1) + g(t, x)x_2, \tag{1.1}$$

$$\dot{x}_2 = f_2(t, x_2), \tag{1.2}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, and $x := \text{col}(x_1, x_2)$. The function f_1, f_2 and g are continuous, locally Lipschitz in x uniformly in t, and f_1 is continuously differentiable in both arguments.

For related works in the autonomous case, see the papers [17, 18] by Sontag and further bibliography cited therein. See also [6, 12–15, 19]. In particular, Sontag showed that the input to state stability (ISS) is closed under composition. It is also worth noticing that Jiang *et al.* [5, 8, 9] generalized the concept of the ISS to the concept of input to state to practical stability (ISPS).

Recently, Chaillet and Loria in [2] and [3] studied the uniform semi-global practical asymptotic stability of the cascade system under some hypotheses. In [1] Benabdallah *et al.* investigated global practical uniform exponential stability of such dynamical systems by using a known result by Corless which appeared in [4]. Some good results related to the subject have been obtained, see [1–19].

The purpose of this paper is to establish sufficient conditions for the practical exponential stability of a class of nonlinear nonautonomous systems. In the spirit of a result of [4], we develop the practical exponential stability with more general assumptions. We obtain a new theorem with more general comparable conditions which will allow us to generalize some results by Benabdallah *et al.* [1].

This paper is organized as follows. First in Section 2, we give some definitions and results about practical uniform exponential stability. Then, in Section 3, after giving some sufficient conditions to guarantee that a nonlinear time-varying is practically uniformly exponentially stable system:

$$\dot{x}_1 = f_1(t, x_1), \tag{1.3}$$

we introduce suitable conditions on function $g(t, x)$ and we show that if both systems (1.2) and (1.3) are practically uniformly exponentially stable, then (1.1)–(1.2) is practically uniformly exponentially stable.

2. PRELIMINARIES

We consider the following system

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{2.1}$$

where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. We denote by

$$B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}, \text{ and } B^r = \{x \in \mathbb{R}^n : \|x\| \geq r\}.$$

2.1. Exponential Convergence to a Ball

Definition 2.1. The system (2.1) is (globally, uniformly) exponentially convergent to B_r (or B_r is globally uniformly exponentially stable) iff there exists $a > 0$ with the property that, for any initial conditions $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, there exists $C(x_0) \geq 0$ such that, if $x(.) : \mathbb{R}_+ \to \mathbb{R}^n$ is any solution of (2.1) with $x(t_0) = x_0$, then

$$\|x(t)\| \leq r + C(x_0)exp[-a(t - t_0)] \text{ for all } t \geq t_0. \tag{2.2}$$

System (2.1) is globally practically uniformly exponentially stable (G.U.P.A.S.) if there exists $r > 0$ such that B_r is globally uniformly exponentially stable.

Note that (2.2) implies that

$$\|x(t)\| \leq r + C(x_0) \text{ for all } t \geq t_0.$$

Hence, the solutions of (2.1) are bounded and can be extended indefinitely. In the above definition, a is called an exponential rate of convergence; the number r is called an asymptotic (norm). If the system (2.1) satisfies the conditions of the above definition with $\lim_{x_0 \to 0} C(x_0) = 0$, then the system is said to be (globally, uniformly) exponentially stable to within B_r. If, in addition, $r = 0$, then the system is exponentially stable about zero.

2.2. **Comparison principle**

Quite often when we study the equation $\dot{x} = f(t, x)$ we need to compute bounds on the solution $x(t)$ without computing the solution it self. The comparaison lemma is one tool that can be used toward that goal. It applies to a situation where the derivative of scalar differentiable function $V(t)$ satisfies inequality of the form $\dot{V}(t) \leq f(t, V(t))$ for all t in certain interval.

LEMMA 2.2. *Let $V(t)$ a continuous function whose derivative $\dot{V}(t)$ satisfies the differential inequality*

$$\dot{V}(t) \leq -a(t)V(t) + b(t),$$

where a and b are continuous functions. Then

$$V(t) \leq \mathrm{e}^{\sigma(t)} V(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\sigma(s)} b(s) \mathrm{d}s,$$

where $\sigma(t) = -\int_{t_0}^{t} a(s)\mathrm{d}s$.

3. **MAIN RESULTS**

3.1. **Practical exponential stability of nonautonomous systems**

We present in this section our contribution. Our first theorem gives sufficient conditions for convergence exponential of B_ρ. We start this section by giving a result from [4] on the exponential stability of (2.1), with the existence of a uniform Lyapunov function.

Condition 3.1. There exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and scalars $c_1,\ c_2 > 0$ which satisfy

$$c_1 \left\| x \right\|^2 \leq V(t, x) \leq c_2 \left\| x \right\|^2,$$

for all $x \in \mathbb{R}^n$, such that, for some scalars $c_3 > 0$ and $k > 0$

$$V(t, x) > k \Longrightarrow \dot{V}(t, x) \leq -2c_3(V(t, x) - k).$$

Theorem 3.2 ([4]). *Suppose that the system (2.1) satisfies Condition 3.1. Then (2.1) is exponentially convergent to B_ρ with rate c_3, where*

$$c(x_0) = \begin{cases} 0 & \text{if } V(t_0, x_0) \leq k, \\ \sqrt{\dfrac{V(t_0, x_0) - k}{c_1}} & \text{if } V(t_0, x_0) > k, \end{cases}$$

with $\rho = \sqrt{\dfrac{k}{c_1}}$. Also, (2.1) is exponentially stable to within B_ρ.

In the theorem below, we give sufficient conditions for the exponential stability of (2.1) with a more general Lyapunov-like function.

Condition 3.3. There exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and scalars $c_1, c_2, c_3, p, q, r, k > 0$ which satisfy

$$c_1 \|x\|^p \leq V(t, x) \leq c_2 \|x\|^q, \tag{3.1}$$

$$\dot{V}(t, x) \leq -c_3 p(\|x\|^r - k), \tag{3.2}$$

for all $x \in W = \begin{cases} B_\delta \text{ if } p > q \text{ where } \delta = \left(\dfrac{c_2}{c_1}\right)^{\frac{1}{p-q}}, \\ B^\eta \text{ if } p < q \text{ where } \eta = \left(\dfrac{c_1}{c_2}\right)^{\frac{1}{p-q}}. \end{cases}$

Theorem 3.4. *Suppose that the system (2.1) satisfies condition 3.3. Then, (2.1) is exponentially convergent to B_ρ, where*

$$\rho = \begin{cases} \left(\dfrac{c_2}{c_1}\right)^{\frac{1}{p-q}} & \text{if } p > q, \\ \sqrt[p]{\dfrac{kc_2}{c_1 \eta^{r-q}}} & \text{if } p < q \leq r, \\ \sqrt[p]{\dfrac{max\{c_2 (k)^{\frac{q}{r}}, V(t_0, x_0)\}}{c_1}} & \text{if } p < q \text{ and } q > r. \end{cases}$$

Proof. We first prove that the solutions are bounded. Then, we prove the exponentially stability of B_ρ.

a) Boundedness of solutions. We distinguish three cases of the behavior of $\dot{V}$.

(1) If $\dot{V}(t) \leq 0$, since V is a positive definite function, then V is a decreasing function. Hence, V is necessarily bounded and from (3.1) we have

$$\|x(t)\| \leq \sqrt[p]{\frac{V(t_0, x_0)}{c_1}}.$$

(2) If $\dot{V}(t) \geq 0$, from Condition 3.3 it follows that

$$\|x(t)\| \leq \sqrt[r]{k}.$$

(3) If $\dot{V}$ is oscillatory, there exists $(t_n)_{n\geq 0}$, $t_n \geq 0$ and $\lim_{n\to+\infty} t_n = +\infty$ such that $\dot{V}(t_n) = 0$. Without loss of generality we assume that for $t \in [t_n; t_{n+1}]$ we have $\dot{V}(t) \geq 0$ which implies that $\|x(t)\| \leq \sqrt[r]{k}$. For $t \in [t_{n+1}; t_{n+2}]$ we have $\dot{V}(t) \leq 0$ implies that $V(t) \leq V(t_{n+1})$. Hence,

$$\|x(t)\| \leq \sqrt[r]{k}.$$

b) Practical exponential stability of solutions. If $p > q$, from (3.1) we get

$$\|x\|^q \left[c_1 \|x\|^{p-q} - c_2\right] \leq 0.$$

Hence,

$$\|x\| \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-q}} = \delta,$$

witch implies that

$$\|x(t)\| \leq \rho + c(x_0)\mathrm{e}^{-\alpha(t-t_0)},$$

with $\rho = \delta$ and $c(x_0) = 0$.

If $p < q$, from (3.1) we have

$$\|x\|^p \left[c_1 - c_2 \|x\|^{q-p}\right] \leq 0 \Longrightarrow \|x\| \geq \left(\frac{c_1}{c_2}\right)^{\frac{1}{q-p}} := \eta.$$

Using (3.2), we treat two cases:

1) For $r \geq q$: we have

$$\dot{V}(t,x) \leq -c_3 p(\|x\|^q \|x\|^{r-q} - k), \ \text{ for } \|x\| \geq \eta,$$

then

$$\begin{aligned} \dot{V}(t,x) &\leq -c_3 p(\eta^{r-q} \|x\|^q - k) \\ &\leq -\frac{c_3 \eta^{r-q} p}{c_2}(V(t,x) - K), \end{aligned}$$

such that $K = \dfrac{c_2 k}{\eta^{r-q}}$. We conclude by Theorem 3.2 that

$$\|x(t)\| \leq \rho + c(x_0)\mathrm{e}^{-\alpha(t-t_0)},$$

where $\rho = \sqrt[p]{\dfrac{K}{c_1}}$, $\alpha = \dfrac{c_3 \eta^{r-q}}{c_2}$, and $c(x_0) = \sqrt[p]{\dfrac{(V(t_0,x_0) - K)}{c_1}}$.

2) For $r < q$: we have

$$V(t,x) \leq \max\{c_2\,(k)^{\frac{q}{r}}\,, V(t_0,x_0)\},$$

then

$$\|x(t)\| \leq \rho + c(x_0)e^{-\alpha(t-t_0)},$$

with $\rho = \sqrt[p]{\dfrac{\max\{c_2\,(k)^{\frac{q}{r}}\,, V(t_0,x_0)\}}{c_1}}$ and $c(x_0) = 0.$ □

THEOREM 3.5. *Suppose that the system (2.1) satisfies condition 3.3 with $p = q = r$ and $W = \mathbb{R}^n$. Then, (2.1) is globally exponentially convergent to $B\rho$, where*

$$\rho = \sqrt[p]{\frac{kc_2}{c_1}}.$$

Proof. Our proof starts with the observation that using condition 3.3 we get

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \\ &\leq -c_3 p(\|x\|^r - k) \\ &\leq -\frac{c_3 p}{c_2}[V(t,x) - kc_2]. \end{aligned}$$

We conclude by Theorem 3.2 that

$$V(t,x) \leq kc_2 + (V(t_0,x_0) - kc_2)\mathrm{e}^{-\frac{c_3}{c_2}p(t-t_0)}.$$

Since for all $a,\ b > 0$ and $p \geq 1$,

$$(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}},$$

then

$$\|x(t)\| \leq \rho + c(x_0\mathrm{e}^{-\alpha(t-t_0)},$$

with $\rho = \sqrt[p]{\dfrac{kc_2}{c_1}}$, $\alpha = \dfrac{c_3}{c_2}$, and

$$c(x_0) = \begin{cases} 0 & \text{if } V(t_0,x_0) \leq kc_2, \\ \sqrt[p]{\dfrac{(V(t_0,x_0) - kc_2)}{c_1}} & \text{if } V(t_0,x_0) > kc_2. \end{cases}$$ □

3.2. **Practical exponential stability of cascade systems**

In this section, we give sufficient conditions that guarantee the practical uniform exponential stability of system (1.1)–(1.2). We start this section by giving a result from [1] on the practical exponential stability of nonlinear time-varying cascade systems, with the existence of a uniform Lyapunov function.

THEOREM 3.6 ([1]). *If assumptions (H1) and (H2) below hold and the interconnection term is bounded, i.e., there exists a constant $M > 0$ such that $\|g(t,x)\| \leq M$ for all (t,x), then system (1.1)–(1.2) is practically globally uniformly exponentially stable.*

H1) There exists a continuously differentiable function V_1 and some positive numbers $c_1,\ c_2,\ c_3,\ c_4$, and k_1 such that

$$\begin{aligned} c_1 \|x_1\|^2 &\leq V_1(t,x_1) \leq c_2 \|x_1\|^2, \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\leq -c_3 V_1(t,x_1) + k_1, \\ \left\| \frac{\partial V_1}{\partial x_1} \right\| &\leq c_4 \|x_1\|. \end{aligned}$$

H2) There exists a continuously differentiable function V_2 and some positive numbers $b_1,\ b_2,\ b_3$, and k_2 such that

$$\begin{aligned} b_1 \|x_2\|^2 &\leq V_2(t,x_2) \leq b_2 \|x_2\|^2, \\ \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2) &\leq -b_3 V_2(t,x_2) + k_2. \end{aligned}$$

We propose in this part to state theorems generalizing Theorem 3.6. Indeed, in Theorem 3.6 we assume that the term of interconnection verifies $\|g(t,x)\| \leq M$, there are therefore the upper boundedness of $g(t,x)$, whereas in the following theorem we suppose that the perturbation term $g(t,x)$ satisfies the linear growth bound

$$\|g(t,x)\| \leq \varepsilon \|x\| + M, \qquad \text{for all } t \geq 0, \tag{3.3}$$

where M is a positive constant and $\varepsilon > 0$.

THEOREM 3.7. *If the assumptions (H1) and (H2) hold and the interconnection term is bounded, i.e. there exists a constants $M,\ \varepsilon > 0$ such that $\|g(t,x)\| \leq \varepsilon \|x\| + M$ for all (t,x), then system (1.1)–(1.2) is practically uniformly exponentially stable.*

Proof. The time derivative of $V_1(t,x)$ along the trajectories of (1.1)–(1.2) is

$$\dot{V}_1(t,x_1) = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} g(t,x) x_2$$

Using assumption (H1) and (3.3) we obtain

$$\dot{V}_1(t,x_1) \leq -c_3 V_1(t,x_1) + k_1 + c_4 \|x_1\| \|x_2\| (\varepsilon \|x\| + M).$$

From assumption (H2) $\dot{x}_2 = f_2(t, x_2)$ is practically exponentially stable by Theorem 3.2. Hence, there exists λ such that $\|x_2\| \leq \lambda$. Using $\|x\| \leq \|x_1\| + \|x_2\|$ we obtain

$$\begin{aligned}\dot{V}_1(t, x_1) &\leq -c_3c_1 \|x_1\|^2 + k_1 + c_4\lambda\varepsilon \|x_1\|^2 + c_4\lambda(\varepsilon\lambda + M) \|x_1\| \\ &\leq -(c_3c_1 - c_4\lambda\varepsilon) \|x_1\|^2 + c_4\lambda(\varepsilon\lambda + M) \|x_1\| + k_1.\end{aligned}$$

ε is chosen such that $\mu_1 = c_3c_1 - c_4\lambda\varepsilon > 0$. Therefore, one obtains

$$\begin{aligned}\dot{V}_1(t, x_1) &\leq -\mu_1 \|x_1\|^2 + c_4\lambda(\varepsilon\lambda + M) \|x_1\| + k_1 \\ &\leq -\mu_1 \|x_1\|^2 + \mu_1\theta \|x_1\|^2 - \mu_1\theta \|x_1\|^2 + c_4\lambda(\varepsilon\lambda + M) \|x_1\| + k_1 \\ &\leq -\mu_1(1-\theta) \|x_1\|^2 - \mu_1\theta \|x_1\|^2 + c_4\lambda(\varepsilon\lambda + M) \|x_1\| + k_1,\end{aligned}$$

where $0 < \theta < 1$. If $\|x_1\| \geq \dfrac{c_4\lambda(\varepsilon\lambda + M)}{\mu_1\theta}$, then

$$\dot{V}_1(t, x_1) \leq -\frac{\mu_1(1-\theta)}{c_1} V_1(t, x_1) + k_1, \quad \forall \|x_1\| \geq \frac{c_4\lambda(\varepsilon\lambda + M)}{\mu_1\theta}.$$

Setting $W(t, x_1, x_2) = V_1(t, x_1) + \alpha V_2(t, x_2)$ where α is a positive constant. The derivative of W along the trajectories of system (1.1)–(1.2) is

$$\begin{aligned}\dot{W}(t) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1) + \frac{\partial V_1}{\partial x_1} g(t, x)x_2 + \alpha\left(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}\frac{\partial x_2}{\partial t}\right) \\ &\leq -\frac{\mu_1(1-\theta)}{c_1} V_1(t, x_1) + k_1 + \alpha(-b_3V_2(t, x_2) + k_2) \\ &\leq -\frac{\mu_1(1-\theta)}{c_1} V_1(t, x_1) - \alpha b_3V_2(t, x_2) + k_1 + \alpha k_2 \\ &\leq -\min\left(\frac{\mu_1(1-\theta)}{c_1}, b_3\right) W(t) + k_1 + \alpha k_2.\end{aligned}$$

Let $\mu = \min\left(\dfrac{\mu_1(1-\theta)}{c_1}, b_3\right)$, it follows that

$$\dot{W}(t) \leq -\mu W(t) + k_1 + \alpha k_2.$$

Hence, by Lemma 2.2, system (1.1)–(1.2) is practically uniformly exponentially stable. □

Remark 3.8. Note that Theorem 3.6 is a special case of Theorem 3.7 when $\varepsilon = 0$.

For the proof of Theorem 3.10 below we will use the next lemma.

LEMMA 3.9. *Let V be a positive definite and continuously differentiable function defined such that*

$$\dot{V}(t) \leq -\alpha V(t) + \beta \sqrt[s]{V(t)} + k,$$

where α, β, k are positives constants, and $s > 1$. Then V is bounded.

Proof. Take

$$f(V) = -\alpha V + \beta \sqrt[s]{V} + k.$$

There are three possibilities for the behavior of $\dot{V}(t)$.

Case 1) If $\dot{V}(t) \leq 0$, since V is a positive definite function, then V is a decreasing function. Hence, V is necessarily bounded.

Case 2) If $\dot{V}(t) \geq 0$, in this case $f(V) \geq 0$ and $f'(V) = \dfrac{-s\alpha V^{1-\frac{1}{s}} + \beta}{sV^{1-\frac{1}{s}}}$. It is easy to see that

$$f'(\overline{V}) = 0 \text{ and } f(\overline{V}) = \frac{s-1}{\alpha^{\frac{1}{s-1}}}\left(\frac{\beta}{s}\right)^{\frac{s}{s-1}} + k > 0,$$

where $\overline{V} = (\dfrac{\beta}{s\alpha})^{\frac{s}{s-1}}$ and $f'(V) < 0$ for $V(t) > \overline{V}$ and $\lim\limits_{V\to+\infty} f(V) = -\infty$. Thus, there exists $\xi > \overline{V}$ such that $f(\xi) = 0$. Consequently

$$f(V) > 0 \quad \text{for all} \quad V(t) < \xi.$$

Hence, V is bounded.

Case 3) If $\dot{V}$ is oscillatory. There exists the sequence $(t_n)_{n\geq 0}$ such that $t_n \geq 0$, and $\lim\limits_{n\to+\infty} t_n = +\infty$ *with* $\dot{V}(t_n) = 0$, $\forall n$. Without loss of generality, we suppose that on $[t_n; t_{n+1}] : \dot{V}(t) \geq 0$, from **case 2** there exists finite constant $\xi_n > 0$ such that $V(t) \leq \xi_n$ for all $t \in [t_{n+1}; t_{n+2}]$.

If $t \in [t_{n+1}; t_{n+2}] : \dot{V}(t) \leq 0$ and $V(t) \leq V(t_{n+1}) \leq \xi_n$ so $V(t) \leq \xi_n$ for all $t \in [t_n; t_{n+2}]$, consequently , $V(t) \leq \sup_{n\geq 0} \xi_n$, for all $t \geq t_0$. □

THEOREM 3.10. *If assumptions (H3) and (H4) below hold and the interconnection term is bounded, i.e., there exists a constant $M > 0$ such that $\|g(t,x)\| \leq M$ for all (t,x) , then system (1.1)–(1.2) is practically uniformly exponentially stable.*

H3) There exists a continuously differentiable function V_1 and some positive numbers c_1, c_2, c_3, c_4, k_1, and $p, q, r > 1$ such that

$$\begin{aligned} c_1 \|x_1\|^p &\leq V_1(t,x_1) \leq c_2 \|x_1\|^q, \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\leq -c_3 \|x_1\|^r + k_1, \\ \left\| \frac{\partial V_1}{\partial x_1} \right\| &\leq c_4 \|x_1\|. \end{aligned}$$

H4) There exists a continuously differentiable function V_2 and some positive numbers $b_1,\ b_2,\ b_3,\ k_2,$ and $p,\ q,\ r > 1$ such that

$$b_1 \|x_2\|^p \leq V_2(t,x_2) \leq b_2 \|x_2\|^q,$$

$$\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial x_2}f_2(t,x_2) \quad \leq \quad -b_3\left\|x_2\right\|^r+k_2.$$

Proof. a) Boundedness of V_1.

Case 1) If $p=q=r$

$$\begin{aligned}\dot{V}_1(t,x_1) &= \frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f_1(t,x_1)+\frac{\partial V_1}{\partial x_1}g(t,x)x_2\\ &\leq -c_3\left\|x_1\right\|^r+k_1+M\left\|\frac{\partial V_1}{\partial x_1}\right\|\left\|x_2\right\|\\ &\leq -c_3\left\|x_1\right\|^r+k_1+c_4M\left\|x_1\right\|\left\|x_2\right\|.\end{aligned}$$

From assumption (H4) we have that $\dot{x}_2=f_2(t,x_2)$ is practically exponentially stable. Hence, by Theorem 3.5, there exists $\lambda>0$ such that $\|x_2\|\leq\lambda$. Then

$$\dot{V}_1(t,x_1)\leq-\frac{c_3}{c_2}V_1(t,x_1)+\frac{c_4M\lambda}{\sqrt[r]{c_1}}\sqrt[r]{V_1(t,x_1)}+k_1.$$

Take $f(V_1)=-\alpha V_1+\beta\sqrt[r]{V_1}+k_1$ with $\alpha=\frac{c_3}{c_2}$, and $\beta=\frac{c_4M\alpha}{\sqrt[r]{c_1}}$. We conclude by Lemma 3.9 $(s=r)$ that V_1 is bounded.

Case 2) If $p>q$ V_1 is bounded (see the proof of Theorem 3.4).

Case 3) If $p<q$, we have

$$\|x_1\|>\left(\frac{c_2}{c_1}\right)^{\frac{1}{q-p}}=\eta,$$

and

$$\begin{aligned}\dot{V}_1(t,x_1) &= \frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f_1(t,x_1)+\frac{\partial V_1}{\partial x_1}g(t,x)x_2\\ &\leq -c_3\left\|x_1\right\|^r+k_1+M\left\|\frac{\partial V_1}{\partial x_1}\right\|\left\|x_2\right\|\\ &\leq -c_3\left\|x_1\right\|^r+k_1+c_4M\left\|x_1\right\|\left\|x_2\right\|.\end{aligned}$$

Since for all $\xi>0$, $\|x_1\|\|x_2\|\leq(\frac{\|x_1\|^2}{2\xi}+2\xi\|x_2\|^2)$, we get

$$\dot{V}_1(t,x_1)\leq-c_3\left\|x_1\right\|^r+\frac{c_4M}{2\xi}\left\|x_1\right\|^2+\frac{c_4M\xi}{2}\left\|x_2\right\|^2+k_1.$$

We discuss two cases.

1) For $r\geq q$: we have

$$\begin{aligned}\dot{V}_1(t,x_1) &\leq -c_3\eta^{r-q}\left\|x_1\right\|^q+\frac{c_4M}{2\xi\eta^{p-2}}\left\|x_1\right\|^p+\frac{c_4M\xi}{2}\lambda^2+k_1\\ &\leq -\frac{c_3\eta^{r-q}}{c_2}V_1(t,x_1)+\frac{c_4M}{2c_1\xi\eta^{p-2}}V_1(t,x_1)+\frac{c_4M\xi}{2}\lambda^2+k_1\end{aligned}$$

$$\leq \quad -(\frac{c_3\eta^{r-q}}{c_2} - \frac{c_4M}{2c_1\xi\eta^{p-2}})V_1(t,x_1) + \frac{c_4M\xi}{2}\lambda^2 + k_1.$$

We choose ξ such that

$$\frac{c_3\eta^{r-q}}{c_2} - \frac{c_4M}{2c_1\xi\eta^{p-2}} > 0.$$

It follows that

$$\dot{V}_1(t,x_1) \leq -\beta_1 V_1(t,x_1) + K_1,$$

where

$$\xi = \frac{c_2c_4M}{c_1c_3\eta^{r-q+p-2}}, \quad \beta_1 = \frac{c_3\eta^{r-q}}{2c_2}, \quad K_1 = \frac{c_4M\xi}{2}\lambda^2 + k_1.$$

We conclude by Lemma 2.2, that V_1 is bounded.

2) For $r < q$: we have

$$\begin{aligned}\dot{V}_1(t,x_1) &\leq -c_3\|x_1\|^r + k_1 + c_4M\|x_1\|\|x_2\| \\ &\leq -c_3\|x_1\|^r + \lambda c_4M\|x_1\| + k_1 \\ &\leq -c_3\|x_1\|^r + \frac{\lambda c_4M}{\eta^{r-1}}\|x_1\|^r + k_1 \\ &\leq -\beta_2\|x_1\|^r + k_1,\end{aligned}$$

where $\beta_2 = c_3 - \dfrac{\lambda c_4M}{\eta^{r-1}}$ and M is chosen such that $\beta_2 > 0$. Hence, by Theorem 3.4, V_1 is bounded.

b) Practical uniform exponential stability of system (1.1)–(1.2).

Set $W(t,x_1,x_2) = V_1(t,x_1) + \alpha V_2(t,x_2)$ where α is a positive constant. The derivative of W along the trajectories of system (1.1)–(1.2) is

$$\begin{aligned}\dot{W}(t) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1) + \frac{\partial V_1}{\partial x_1}g(t,x)x_2 + \alpha(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}f_2(t,x_2)) \\ &\leq -c_3\|x_1\|^r + k_1 + c_4M\|x_1\|\|x_2\| + \alpha(-b_3\|x_2\|^r + k_2) \\ &\leq -\beta_3\|x_1\|^r + k_1 - \alpha b_3\|x_2\|^r + \alpha k_2 \\ &\leq -\mu_1 V_1^{\frac{r}{q}}(t,x_1) + k_1 - \alpha\mu_2 V_2^{\frac{r}{q}}(t,x_2) + \alpha k_2,\end{aligned}$$

where $\beta_3 = min(\beta_1,\beta_2,c_3)$, $\mu_1 = \dfrac{\beta_3}{c_2^{\frac{r}{q}}}$ and $\mu_2 = \dfrac{b_3}{b_2^{\frac{r}{q}}}$. We remark that

$$\begin{aligned}\dot{W}(t) \leq &- \mu_1\, V_1(t,x_1) - \alpha\mu_2\, V_2(t,x_2) + \mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) \\ &+ \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + k_1 + \alpha k_2.\end{aligned}$$

Let $\mu = \min(\mu_1,\mu_2)$, we obtain

$$\dot{W}(t) \leq -\mu W(t) + \mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + k_1 + \alpha k_2.$$

The boundedness of V_1 and V_2 implies that there exists $k_3 > 0$ such that
$$\mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) \ + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) \leq k_3,$$
thus,
$$\dot{W}(t) \leq -\mu W(t) + k_1 + \alpha k_2 + k_3.$$
By Lemma 2.2, system (1.1)–(1.2) is practically uniformly exponentially stable. □

Remark 3.11. We can give a different proof of the boundedness of V_1 for $p = q = r > 1$. Indeed
$$\begin{aligned}
\dot{V}_1(t,x_1) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} g(t,x)x_2 \\
&\leq -c_3 \|x_1\|^r + k_1 + c_4 M \|x_1\| \|x_2\| \\
&\leq -c_3 \|x_1\|^r + c_4 M\lambda \|x_1\| + k_1 \\
&\leq -c_3 \|x_1\|^r + \theta c_3 \|x_1\|^r - \theta c_3 \|x_1\|^r + c_4 M\lambda \|x_1\| + k_1 \\
&\leq -c_3(1-\theta) \|x_1\|^r + k_1, \ \forall \|x_1\| > \sqrt[r-1]{\frac{\lambda c_4 M}{c_3\theta}},
\end{aligned}$$
where $0 < \theta < 1$. We conclude by Theorem 3.5, that V_1 is bounded.

The example below illustrates our results.

Example. Consider the system

$$\left\{ \begin{aligned} \dot{x}_1 &= -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{7}{4}}}{1+x_1^2}e^{-x_1^2} + \frac{1}{1+t^2}x_2, \\ \dot{x}_2 &= -x_2^{\frac{3}{2}} + 2e^{-x_2^{\frac{1}{4}}}. \end{aligned} \right. \tag{3.4}$$

In this case,
$$\begin{aligned}
f_1(t,x_1) &= -\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{7}{4}}}{1+x_1^2}e^{-x_1^2}, \\
f_2(t,x_2) &= -x_2^{\frac{3}{2}} + 2e^{-x_2^{\frac{1}{4}}}, \\
g(t,x) &= \frac{1}{1+t^2}.
\end{aligned}$$

Set $V_1(t,x_1) = x_1^{\frac{5}{4}}$ and $V_2(t,x_2) = x_2^{\frac{5}{4}}$.

Verification of assumption H3). We have
$$\begin{aligned}
\|x_1\|^{\frac{5}{4}} &\leq V_1(t,x_1) \leq \|x_1\|^{\frac{3}{2}}, \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) &\leq \frac{5}{4}x_1^{\frac{1}{4}}(-\frac{1}{4}x_1^{\frac{3}{2}} + \frac{x_1^{\frac{7}{4}}}{1+x_1^2}e^{-x_1^2}) \\
&\leq -\frac{5}{16}x_1^{\frac{7}{4}} + \frac{5}{4} \leq -\frac{5}{16}\|x_1\|^{\frac{7}{4}} + \frac{5}{4}.
\end{aligned}$$

With $p=\frac{5}{4}$, $q=\frac{3}{2}$, $r=\frac{7}{4}$, $c_1=1$, $c_2=1$, $c_3=\frac{5}{16}$, $c_4=\frac{5}{4}$, $k_1=\frac{5}{4}$ and $\|x_1\|\geq 1=\eta$.

Verification of assumption H4). We have

$$\begin{aligned}
\|x_2\|^{\frac{5}{4}} &\leq V_2(t,x_2)\leq \|x_2\|^{\frac{3}{2}}\\
\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial x_2}f_2(t,x_1) &\leq \frac{5}{4}x_2^{\frac{1}{4}}(-x_2^{\frac{3}{2}}+2e^{-x_2^{\frac{1}{4}}})\\
&\leq -\frac{5}{4}x_2^{\frac{7}{4}}+\frac{5}{2}\\
&\leq -\frac{5}{4}\|x_2\|^{\frac{7}{4}}+\frac{5}{2},
\end{aligned}$$

with $p=\frac{5}{4}$, $q=\frac{3}{2}$, $r=\frac{7}{4}$, $b_1=1$, $b_2=1$, $b_3=\frac{5}{4}, c_4=\frac{5}{4}$, $k_2=\frac{5}{2}$ and $\|x_1\|\geq 1=\eta$. Therefore, we can apply Theorem 3.10 to prove that system (3.4) is practically uniformly exponentially stable.

THEOREM 3.12. *If assumptions (H3) and (H4) hold and the interconnection term is bounded, i.e., there exists a constants $M,\varepsilon>0$ such that $\|g(t,x)\|\leq\varepsilon\|x\|+M$ for all (t,x), then system (1.1)–(1.2) is practically uniformly exponentially stable.*

Proof. a) Boundedness of V_1:

1) If $p=q=r>2$, we have

$$\begin{aligned}
\dot V_1(t,x_1) &= \frac{\partial V_1}{\partial t}+\frac{\partial V_1}{\partial x_1}f_1(t,x_1)+\frac{\partial V_1}{\partial x_1}g(t,x)x_2\\
&\leq -c_3\|x_1\|^r+k_1+c_4\|x_1\|\|x_2\|(\varepsilon\|x\|+M)\\
&\leq -c_3\|x_1\|^r+k_1+c_4\|x_1\|\|x_2\|(\varepsilon\|x_1\|+\varepsilon\|x_2\|+M)\\
&\leq -c_3\|x_1\|^r+k_1+c_4\|x_1\|\|x_2\|(\varepsilon\|x_1\|+\varepsilon\lambda+M)\\
&\leq -c_3\|x_1\|^r+k_1+c_4(\varepsilon\lambda+M)\|x_1\|\|x_2\|+c_4\varepsilon\lambda\|x_1\|^2\\
&\leq -c_3\|x_1\|^r+\frac{c_4(\varepsilon\lambda+M)}{2}\|x_1\|^2+c_4\varepsilon\lambda\|x_1\|^2+A,
\end{aligned}$$

where $A=\frac{c_4(\varepsilon\lambda+M)}{2}\lambda^2+k_1$. For $0<\theta<1$, we obtain

$$\begin{aligned}
\dot V_1(t,x_1) &\leq -c_3\|x_1\|^r+c_3\theta\|x_1\|^r-c_3\theta\|x_1\|^r+\frac{3c_4\varepsilon\lambda+c_4M}{2}\|x_1\|^2+A\\
&\leq -c_3(1-\theta)\|x_1\|^r-c_3\theta\|x_1\|^r+\frac{3c_4\varepsilon\lambda+c_4M}{2}\|x_1\|^2+A,
\end{aligned}$$

so

$$\dot V_1(t,x_1)\leq -c_3(1-\theta)\|x_1\|^r+A,\quad \forall\|x_1\|>\sqrt[r-2]{\frac{3c_4\varepsilon\lambda+c_4M}{2c_3\theta}}.$$

From Theorem 3.5 V_1 is bounded.

2) If $p < q$, we have

$$\begin{aligned}\dot{V}_1(t,x_1) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} g(t,x)x_2 \\ &\leq -c_3\|x_1\|^r + k_1 + c_4\|x_1\|\|x_2\|(\varepsilon\|x\| + M) \\ &\leq -c_3\|x_1\|^r + k_1 + c_4(\varepsilon\lambda + M)\|x_1\|\|x_2\| + c_4\varepsilon\lambda\|x_1\|^2 \\ &\leq -c_3\|x_1\|^r + k_1 + \frac{c_4(\varepsilon\lambda + M)}{2}\|x_1\|^2 + \frac{c_4(\varepsilon\lambda + M)}{2}\|x_2\|^2 + c_4\varepsilon\lambda\|x_1\|^2,\end{aligned}$$

for $r \geq q \geq 2$, we have

$$\dot{V}_1(t,x_1) \leq -(c_3 - \frac{c_4(\varepsilon\lambda + M)}{2\eta^{r-2}} - \frac{c_4\varepsilon\lambda}{\eta^{r-2}})\|x_1\|^r + A.$$

If ε is chosen such that

$$\beta_1 = c_3 - \frac{c_4(3\varepsilon\lambda + M)}{2\eta^{r-2}} > 0,$$

then by Theorem 3.4, V_1 is bounded.

b) Practical uniform exponential stability of system (1.1)–(1.2):

Set $W(t,x_1,x_2) = V_1(t,x_1) + \alpha V_2(t,x_2)$ where α is a positive constant. The derivative of W along the trajectories of system (1.1)–(1.2) is

$$\begin{aligned}\dot{W}(t) &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1) + \frac{\partial V_1}{\partial x_1} g(t,x)x_2 + \alpha(\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2)) \\ &\leq -c_3\|x_1\|^r + k_1 + c_4(\varepsilon\|x\| + M)\|x_1\|\|x_2\| + \alpha(-b_3\|x_2\|^r + k_2) \\ &\leq -\beta\|x_1\|^r + A - \alpha b_3\|x_2\|^r + k_2\alpha \\ &\leq -\mu_1 V_1^{\frac{r}{q}}(t,x_1) - \alpha\mu_2 V_2^{\frac{r}{q}}(t,x_2) + B,\end{aligned}$$

with $\beta = \min(\beta_1, c_3)$, $\mu_1 = \frac{\beta}{c_2^{\frac{r}{q}}}$, $\mu_2 = \frac{b_3}{b_2^{\frac{r}{q}}}$ and $B = A + \alpha k_2$. By the same procedure of the previous theorem we obtain

$$\dot{W}(t) \leq -\mu W + \mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + B.$$

The boundedness of V_1 and V_2 implies that there exists a finite constant C such that

$$\mu_1(V_1(t,x_1) - V_1^{\frac{r}{q}}(t,x_1)) + \alpha\mu_2(V_2(t,x_2) - V_2^{\frac{r}{q}}(t,x_2)) + B \leq C.$$

Hence,

$$\dot{W}(t) \leq -\mu W(t) + C.$$

By Lemma 2.2, system (1.1)–(1.2) is practically uniformly exponentially stable. □

Example. Consider the system

$$(3.5)\qquad \begin{cases} \dot{x}_1 = -\frac{1}{9}x_1^{\frac{5}{2}} + \frac{4x_1^{\frac{15}{8}}e^{-\frac{1}{2}t}}{9(1+x_1^2)} + (\frac{10^{-3}xe^{-x^2}}{1+t^2} + \frac{1}{5})x_2 \\ \dot{x}_2 = -x_2^{\frac{5}{2}} + \frac{17}{9}e^{-x_2^{\frac{1}{8}}} \end{cases}$$

In this case,

$$\begin{aligned} f_1(t,x_1) &= -\frac{1}{9}x_1^{\frac{5}{2}} + \frac{4x_1^{\frac{15}{8}}e^{-\frac{1}{2}t}}{9(1+x_1^2)}, \\ f_2(t,x_2) &= -x_2^{\frac{5}{2}} + \frac{17}{9}e^{-x_2^{\frac{1}{8}}}, \\ g(t,x) &= \frac{10^{-3}xe^{-x^2}}{1+t^2} + \frac{1}{5}. \end{aligned}$$

Set $V_1(t,x_1) = x_1^{\frac{9}{8}}$ and $V_2(t,x_2) = x_2^{\frac{9}{8}}$.
Verification of assumption H3): We have

$$\begin{aligned} \|x_1\|^{\frac{9}{8}} &\le V_1(t,x_1) \le \|x_1\|^2 \\ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}f_1(t,x_1) &\le \frac{9}{8}x_1^{\frac{1}{8}}(-\frac{1}{9}x_1^{\frac{5}{2}} + \frac{4x_1^{\frac{15}{8}}e^{-\frac{1}{2}t}}{9(1+x_1^2)}) \\ &\le -\frac{1}{8}x_1^{\frac{21}{8}} + \frac{1}{2} \le -\frac{1}{8}\|x_1\|^{\frac{21}{8}} + \frac{1}{2}. \end{aligned}$$

With $p = \frac{9}{8}$, $q = 2$, $r = \frac{21}{8}$, $c_1 = 1$, $c_2 = 1$, $c_3 = \frac{1}{8}, c_4 = \frac{9}{8}$, $k_1 = \frac{1}{2}$ and $\|x_1\| \ge 1 = \eta$.
Verification of assumption H4): We have

$$\begin{aligned} \|x_2\|^{\frac{9}{8}} &\le V_2(t,x_2) \le \|x_2\|^2 \\ \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}f_2(t,x_1) &\le \frac{9}{8}x_2^{\frac{1}{8}}(-x_2^{\frac{5}{2}} + \frac{17}{9}e^{-x_2^{\frac{1}{8}}}) \\ &\le -\frac{9}{8}x_2^{\frac{21}{8}} + \frac{17}{8} \le -\frac{9}{8}\|x_2\|^{\frac{21}{8}} + \frac{17}{8}. \end{aligned}$$

With $p = \frac{9}{8}$, $q = 2$, $r = \frac{21}{8}$, $b_1 = 1$, $b_2 = 1$, $b_3 = \frac{9}{8}$, $k_2 = \frac{17}{8}$ and $\|x_2\| \ge 1 = \eta$.

We have also $\|g(t,x)\| \le \varepsilon \parallel x \parallel + M$ where $M = \frac{1}{5} < \frac{2c_3\eta^{r-2}}{c_4} = \frac{2}{9}$ and $\varepsilon = 10^{-3}\mathrm{e}^{-2}$.

Therefore, we can apply Theorem 3.12 to prove that system (3.5) is practically uniformly exponentially stable.

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University of Oran
Department of Mathematics
31000 Oran, Algeria
dj.beldjerd@gmail.com
oudjedi@yahoo.fr
remilimous@gmail.com