SPECTRAL PROPERTIES OF DIRAC-TYPE OPERATORS DEFINED ON FOLIATED MANIFOLDS

VLADIMIR SLESAR

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In this expository paper we overview the main results regarding the lower bound for the spectrum of the basic Dirac operator; for some of them we indicate alternative proofs.

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1. INTRODUCTION

On a closed differential manifold endowed with a foliated structure and a bundle-like metric tensor field (*i.e.* the manifold can be locally described as a Riemannian submersion [17]), the natural differential operators play an important role in the study of the geometry of the underlying manifold.

The relevant features of the so called *basic* Laplacian which acts on the de Rham complex of *basic* differential forms (or *transversal* Laplacian, if one consider general differential forms instead of basic forms), has been studied in the last period of time [3, 5, 12, 15, 19, 21].

Concerning the Dirac-type operators, the *transversal Dirac operator* for Riemannian foliations was introduced in [8]. It is used to define the *basic Dirac operator*, which is a symmetric, essentially self-adjoint and transversally elliptic operator [8]. We emphasize the fact that a Weitzenböck-Lichnerowicz formula can be obtained, and in the case when the *mean curvature form* of the Riemannian foliation is basic-harmonic, Bochner techniques can be implemented and vanishing results can be obtained [4].

For the study of the spectral properties of the basic Dirac operator it turns out that an important aspect is the way the mean curvature form varies when the bundle-like metric is changed; first of all we refer to [2, 6, 13]. In [6], using a Hodge-type decomposition theorem from [2], the author show that any bundle-like metric can be transformed such that the new bundle-like metric have basic mean curvature form; in fact the transformation leaves invariant the transversal metric and the basic component of the mean curvature form of the initial metric. Furthermore, as an application of stochastic flows in theory of Riemannian foliations, in [13] the author constructs a *dilation* of the metric which turn it into a metric with basic-harmonic mean curvature.

In [9], the authors proved the invariance of the spectrum of the basic Dirac operator with respect to a special class of transformations of the bundle-like metric; more exactly, the metric on M can be changed in any way that leaves the transverse metric on the normal bundle intact. Using this result one can derive a method for studying the spectrum of such Dirac-type operator [9]; that is, one may assume the bundle like metric to be chosen so that the mean curvature is basic-harmonic, the result being therefore pulled back in the general case using [6] and [13]. As an application, the authors finally obtained the eigenvalues estimate for arbitrary Riemannian foliation with bundle-like metric. The lower bound of eigenvalues is known to be influenced by the existence of additional geometric objects (see *e.g.* [14]), so an inconvenience of the above method is that these geometric objects may not be invariant with respect to the above bundle-like metric changes.

Within [20] we show that a classical approach can be also taken, using a Weitzenböck-Lichnerowicz formula for the basic Dirac operator which is different from [4, 8]. As a result, the dilation procedure from [13] is not needed anymore and we use only the convenient metric change from [6] which leaves invariant the basic Dirac operator; others additional geometric objects also remain invariant, including the corresponding eigenspinors associated to the basic Dirac operator [20].

In this paper we establish some results from [20] using a more direct approach. The second section contains the definitions and the main features of the geometric objects we use throughout the paper.

2. THE TRANSVERSAL AND BASIC DIRAC OPERATORS

We are going to consider in what follows a smooth, closed Riemannian manifold $(M, g, \mathcal{F})$ endowed with a foliation $\mathcal{F}$ such that the metric g is bundle-like [17]; the dimension of M will be denoted by n. We also denote by $T\mathcal{F}$ the leafwise distribution tangent to leaves, while $Q = T\mathcal{F}^{\perp} \simeq TM/T\mathcal{F}$ will be the transversal distribution. Let us assume $\dim T\mathcal{F} = p$, $\dim Q = q$, so $p + q = n$.

As a consequence, the tangent and the cotangent vector bundles associated with M split as follows

$$\begin{aligned} TM &= Q \oplus T\mathcal{F}, \\ TM^* &= Q^* \oplus T\mathcal{F}^*. \end{aligned}$$

The canonical projection operators will be denoted by π_Q and $\pi_{T\mathcal{F}}$, respectively.

Throughout this paper we will use local vector fields $\{E_i, F_a\}$ defined on a neighborhood of an arbitrary point $x \in M$, so that they determine an orthonormal basis at any point where they are defined, $\{E_i\}$ spanning the distribution Q and $\{F_a\}$ spanning the distribution $T\mathcal{F}$.

A standard linear connection employed for the study of the basic geometry of our Riemannian foliated manifold is the *Bott connection* (see *e.g.* [21]); it is a metric and torsion-free connection. If we denote by ∇^g the canonical Levi-Civita connection, then on the transversal distribution Q we can define the connection ∇ by the following relations

$$\begin{cases} \nabla_U X := \pi_Q\left([U, X]\right), \\ \nabla_Y X := \pi_Q\left(\nabla^g_Y X\right), \end{cases}$$

for any smooth sections $U \in \Gamma(T\mathcal{F})$, X, $Y \in \Gamma(Q)$. In particular we can associate to ∇ the transversal scalar curvature Scal^{∇}.

We restrict the classical de Rham complex of differential forms $\Omega(M)$ to the complex of basic differential forms, defined as

$$\Omega_b(M) := \{\omega \in \Omega(M) \mid \iota_U \omega = 0,\ \mathcal{L}_U \omega = 0\},$$

where U is again an arbitrary leafwise vector field, $\mathcal{L}$ being the Lie derivative along U, while ι stands for interior product. Considering now the de Rham exterior derivative d, it is possible to define the basic operator $d_b := d_{|\Omega_b(M)}$ (see *e.g.* [2]). Let us notice that basic de Rham complex is defined independent of the metric structure g.

An example of differential form which is not necessarily basic is represented by the *mean curvature form*. In order to define it, we first of all set $k^\sharp := \pi_Q\left(\sum_a \nabla^g_{F_a} F_a\right)$ to be the *mean curvature vector field* associated with the distribution $T\mathcal{F}$, $\sharp$ being the *musical* isomorphism. Then k will be the *mean curvature form* which is subject to the condition $k(U) = g(k^\sharp, U)$, for any vector field U.

Considering [2], we have the orthogonal decomposition

$$\Omega(M) = \Omega_b(M) \bigoplus \Omega_b(M)^\perp,$$

with respect to the $C^\infty-$Frechet topology. So, on any Riemannian foliation the mean curvature form can be decomposed as the sum

$$k = k_b + k_o,$$

where $k_b \in \Omega_b(M)$ is the *basic* component of the mean curvature, k_o being the orthogonal complement. From now on we denote $\tau := k_b^\sharp$.

Using the above notations, at any point x on M we consider the Clifford algebra $Cl(Q_x)$ which, with respect to the orthonormal basis $\{E_i\}$ is generated by 1 and the vectors $\{E_i\}$ over the complex field, being subject to the relations $E_i \cdot E_j + E_j \cdot E_i = -2\delta_{ij}$, $1 \leq i, j \leq q$, where dot stands for Clifford multiplication. The resulting bundle $Cl(Q)$ of Clifford algebras will be called the *Clifford bundle* over M, associated with Q. Let us also consider a vector bundle E over M and suppose we have a smooth bundle action

$$\Gamma\left(Cl(Q)\right) \otimes \Gamma\left(E\right) \longrightarrow \Gamma\left(E\right),$$

denoted also with Clifford multiplication such that

$$(u \cdot v) \cdot s = u \cdot (v \cdot s),$$

for $u, v \in \Gamma\left(Cl(Q)\right)$, $s \in \Gamma\left(E\right)$.

As a result, E becomes a bundle of Clifford modules (see *e.g.* [18]).

If a Clifford bundle E is endowed with a connection ∇^E, then ∇^E is said to be *compatible* with the Clifford action and the Levi-Civita connection ∇ if

$$\nabla^E_U\left(u \cdot s\right) = \left(\nabla_U u\right) \cdot s + u \nabla^E_U s,$$

for any $U \in \Gamma(TM)$, $u \in \Gamma\left(Cl(Q)\right), s \in \Gamma(E)$, extending canonically the connection ∇ to $\Gamma\left(Cl(Q)\right)$.

In what follows, we assume the existence of a hermitian structure $(\cdot \mid \cdot)$ on E such that $(X \cdot s_1 \mid s_2) = -\left(s_1 \mid X \cdot s_2\right)$, for any $X \in \Gamma\left(Q\right)$, s_1, $s_2 \in \Gamma\left(E\right)$, and a metric connection ∇^E, compatible with $Cl(Q)$ action and the connection ∇.

On the above *transverse Dirac bundle* over M, in accordance with [8], we introduce now the *transverse Dirac operator*,

$$D_{tr} := \sum_i E_i \cdot \nabla^E_{E_i}, .$$

If we take its restriction to the *basic* (or holonomy invariant) *sections*

$$\Gamma_b\left(E\right) := \left\{ s \in \Gamma_b\left(E\right) \mid \nabla^E_U s = 0, \text{ for any } U \in \Gamma\left(T\mathcal{F}\right) \right\}, \tag{1}$$

and add a term related to the basic component of the mean curvature form, then we obtain the *basic Dirac operator* [8, 9]

$$\begin{aligned} D_b \quad &:= \quad \sum_i E_i \cdot \nabla^E_{E_i} - \frac{1}{2}\tau \\ &= \quad \sum_i E_i \cdot \left(\nabla^E_{E_i} - \frac{1}{2}\left\langle E_i, \tau \right\rangle \right) \end{aligned}$$

Remark 2.1. The basic Dirac operator is elliptic in the directions of the distribution Q and essentially self-adjoint with respect to the inner product canonically associated with the closed Riemannian manifold [8].

From now on we consider the *modified connection* on the space of basic sections $\Gamma_b(E)$ [20]

$$\bar{\nabla}^E_X s := \nabla^E_X s - \frac{1}{2}\langle X, \tau\rangle s,$$

for any $X \in \Gamma(TM)$ and $s \in \Gamma_b(E)$, $\langle\cdot,\cdot\rangle$ being the scalar product in TM. We get

$$D_b = \sum_i E_i \cdot \bar{\nabla}^E_{E_i}.$$

We must observe that the modified connection is not a metric connection.

Let us now consider the case of a Riemannian foliation with basic mean curvature; in this setting

$$\tau = \pi_Q\left(\sum_a \nabla^g_{F_a} F_a\right). \tag{2}$$

We introduce the *transverse divergence* of a vector field $v \in \Gamma(Q)$ using the other musical isomorphism $\flat$ and the interior product ι (see e. g. [21]):

$$\begin{aligned} \operatorname{div}^{\nabla} v &:= \sum_i \langle \nabla_{E_i} v, E_i\rangle \\ &= \sum_i \iota_{E_i} \nabla_{E_i} v^{\flat}, \end{aligned}$$

where $\{E_i\}$ is again a transversal orthonormal basis.

Several useful calculus properties of the modified connection are listed below.

Proposition 2.2. *The formal adjoint operator of the modified connection can be computed as*

$$\left(\bar{\nabla}^E_{E_i}\right)^* = -\bar{\nabla}^E_{E_i} - \operatorname{div}^{\nabla} E_i. \tag{3}$$

Proof. We adapt the classical computation to our specific framework, employing also (2) and the Green theorem [16]

$$\begin{aligned} & \int_M \left(\left(-\bar{\nabla}^E_{E_i} - \operatorname{div}^{\nabla} E_i\right) s_1 \mid s_2\right) - \int_M \left(s_1 \mid \bar{\nabla}^E_{E_i} s_2\right) \\ = & \int_M -E_i\left(s_1 \mid s_2\right) + \int_M \left(-\sum_a \left(\nabla^g_{F_a} E_i \mid F_a\right) - \operatorname{div}^{\nabla} E_i\right)\left(s_1 \mid s_2\right) \\ = & \int_M -E_i\left(s_1 \mid s_2\right) - \int_M \operatorname{div} E_i\left(s_1 \mid s_2\right) \\ = & -\int_M \operatorname{div}\left(\left(s_1 \mid s_2\right) E_i\right) \\ = & \ 0, \end{aligned}$$

and the conclusion follows. $\square$

PROPOSITION 2.3. *Considering now the corresponding curvature operators, we have the equality* $\bar{R}^E_{X,Y} = R^E_{X,Y}$.

Proof. Using the classical definition of the curvature operator, we start with the relation

$$\bar{R}^E_{X,Y} = \bar{\nabla}^E_X \bar{\nabla}^E_Y - \bar{\nabla}^E_Y \bar{\nabla}^E_X - \bar{\nabla}^E_{[X,Y]}. \tag{4}$$

Furthermore, we get

$$\begin{aligned}
\bar{\nabla}^E_X \bar{\nabla}^E_Y &= \left(\nabla^E_X - \frac{1}{2}\langle X, \tau\rangle\right)\left(\nabla^E_Y - \frac{1}{2}\langle Y, \tau\rangle\right) \\
&= \nabla^E_X \nabla^E_Y - \frac{1}{2}\langle X, \tau\rangle \nabla^E_Y - \frac{1}{2}\langle \nabla_X Y, \tau\rangle \nabla^E_X \\
&\quad -\frac{1}{2}\langle Y, \nabla_X \tau\rangle \nabla^E_X - \frac{1}{2}\langle Y, \tau\rangle \nabla^E_X + \frac{1}{4}\langle X, \tau\rangle \langle Y, \tau\rangle .
\end{aligned}$$

We also have the corresponding relation for the second term of (4); for the third term we have

$$\bar{\nabla}^E_{[X,Y]} = \nabla^E_{[X,Y]} - \frac{1}{2}\langle [X, Y], \tau\rangle .$$

Now, first of all let us emphasize that k_b is a closed 1-form [2], so

$$\langle Y, \nabla_X \tau\rangle = \langle X, \nabla_Y \tau\rangle .$$

Also, the Bott connection is torsion-free, so

$$\langle \nabla_X Y - \nabla_X Y - [X, Y], \tau\rangle = 0.$$

Summing up, we observe that all terms containing τ vanish and we obtain the above relation. $\square$

3. THE LOWER EIGENVALUE ESTIMATE

The lower eigenvalue estimate is obtained using a new *transverse Weitzenböck-Lichnerowicz formula.* In [20] the formula is obtained using the Laplacian of the modified connection; in the following we present a straightforward approach.

We begin by considering the case of a Riemannian foliation with basic mean curvature; we get successively

$$\begin{aligned}
D_b^2 &= \left(\sum_i E_i \cdot \bar{\nabla}^E_{E_i}\right)\left(\sum_j E_j \cdot \bar{\nabla}^E_{E_j}\right) \\
&= \sum_{i,j} E_i \cdot \nabla_{E_i} E_j \cdot \bar{\nabla}^E_{E_j} + \sum_{i,j} E_i \cdot E_j \cdot \bar{\nabla}^E_{E_i}\left(\bar{\nabla}^E_{E_j}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i,j} E_i \cdot E_j \cdot \bar{\nabla}^E_{\nabla_{E_i} E_j} + \sum_{i,j} E_i \cdot E_j \cdot \bar{\nabla}^E_{E_i} \left(\bar{\nabla}^E_{E_j} \right) \\
&= \sum_i \left(-\bar{\nabla}^E_{E_i} s_1 - \operatorname{div}^\nabla E_i \right) \bar{\nabla}^E_{E_i} \\
&\quad + \sum_{i<j} E_i \cdot E_j \cdot \left(\bar{\nabla}^E_{E_i} \bar{\nabla}^E_{E_j} - \bar{\nabla}^E_{E_i} \bar{\nabla}^E_{E_j} - \bar{\nabla}^E_{\left(\nabla_{E_i} E_j - \nabla_{E_j} E_i \right)} \right).
\end{aligned}
$$

Considering the above propositions and the fact that D_b is defined only on basic spinors using (1), the formula finally becomes

$$
D_b^2 = \sum_i \left(\bar{\nabla}^E_{E_i} \right)^* \bar{\nabla}^E_{E_j} + \sum_{i<j} E_i \cdot E_j \cdot R^E_{E_i, E_j}.
$$

Now, integrating over the closed manifolds M, we end up with [20]

$$
\|D_b s\|^2 = \sum_i \left\| \bar{\nabla}^E_{E_i} s \right\|^2 + \int_M (\mathcal{R} s \mid s), \tag{5}
$$

for any $s \in \Gamma_b(E)$, with $\mathcal{R} := \sum_{i<j} E_i \cdot E_j \cdot R^E_{E_i, E_j}$.

From now on we consider a more specific setting, namely we assume that the foliation $\mathcal{F}$ is transversally oriented and has a transverse spin structure. This means that there exists a principal $Spin(q)$-bundle $\tilde{P}$ which is a double sheeted covering of the transversal principal $SO(q)$-bundle of oriented orthonormal frames P, such that the restriction to each fiber induces the covering projection $Spin(q) \to SO(q)$; such a foliation is called *spin foliation* [9]. Similar to the classical case [18], if we denote by Δ_q the spin irreducible representation associated with Q, then one can construct the *foliated spinor bundle* $S := \tilde{P} \times_{Spin(q)} \Delta_q$. The hermitian metric on S is now induced from the transverse metric. Also, the lifting of the Riemannian connection on P can be used to introduce canonically a connection on S, which will be denoted also by ∇. In this particular setting the advantage comes from the simplification of the curvature term, as the twisted curvature term vanishes, and we obtain $\mathcal{R} = \frac{1}{4}\mathrm{Scal}^\nabla$ [18]. As a consequence, the formula (5) becomes

$$
\|D_b s\|^2 = \left\| \bar{\nabla} s \right\|^2 + \frac{1}{4} \int_M \mathrm{Scal}^\nabla |s|^2. \tag{6}
$$

In what follows we show that, once we assume the mean curvature form to be basic, then we can prove directly the proper version of the lower eigenvalues estimate for the spectrum of D_b.

Our approach is in fact very close to the original one due to Friedrich [7]. For a real basic function f we define

$$\bar{\nabla}^f_X s := \bar{\nabla}_X s + fX \cdot s.$$

After calculations, using the Weitzenböck-Lichnerowicz formula (6), we get

$$\begin{aligned}
(D_b - f)^2 &= \left(\sum_i E_i \cdot \bar{\nabla}_{E_i} - f\right)\left(\sum_j E_j \cdot \bar{\nabla}_{E_j} - f\right) \\
&= \sum_{i,j} E_i \cdot \nabla_{E_i} E_j \cdot \bar{\nabla}_{E_j} + \sum_{i,j} E_i \cdot E_j \cdot \bar{\nabla}_{E_i}\left(\bar{\nabla}_{E_j}\right) \\
&\quad - fD_b - \sum_i E_i(f) E_i - fD_b + f^2 \\
&= \bar{\nabla}^*_{E_i} \bar{\nabla}_{E_i} - 2fD_b - \sum_i E_i(f) E_i + f^2 + \frac{1}{4}\mathrm{Scal}^{\nabla}.
\end{aligned}$$

On the other hand, using (3) we get successively

$$\begin{aligned}
\sum_k \bar{\nabla}^{f*}_{E_i} \bar{\nabla}^f_{E_i} &= \sum_i \left(-\bar{\nabla}_{E_i} - \mathrm{div}^{\nabla}(E_i) - fE_i\right)\left(\bar{\nabla}_{E_i} + fE_i\right) \\
&= \sum_i \bar{\nabla}^*_{E_i} \bar{\nabla}_{E_i} - \sum_i E_i(f) E_i - \sum_i f \nabla_{E_i} E_i - fD_b \\
&\quad - \sum_i f \mathrm{div}^{\nabla}(E_i) E_i - fD_b + qf^2 \\
&= \sum_i \bar{\nabla}^*_{E_i} \bar{\nabla}_{E_i} - 2fD_b - \sum_i E_i(f) E_i + qf^2,
\end{aligned}$$

In the above calculations we use the equality

$$\begin{aligned}
\mathrm{div}^{\nabla}(E_i) E_i &= \sum_{i,j} \left(\nabla_{E_j} E_i \mid E_j\right) E_i \\
&= -\sum_j \nabla_{E_j} E_j.
\end{aligned}$$

Now, integrating over the closed manifold M and arguing as in the proof of (5), we get the relation:

$$\int_M |D_b s - fs|^2 = \int_M \left|\bar{\nabla}^f s\right|^2 + \frac{1}{4}\int_M \mathrm{Scal}^{\nabla} |s|^2 + \int_M (1-q) f^2 |s|^2.$$

From here, taking $f := \frac{\lambda}{q}$, and s to be the eigenspinor corresponding to the eigenvalue λ, we obtain

$$\int_M \left(\frac{q-1}{q}\lambda^2 - \frac{1}{4}\mathrm{Scal}^{\nabla} \right) |s|^2 = \int_M \left| \bar{\nabla}^f s \right|^2 ,$$

and the lower eigenvalue estimate follows:

$$\lambda^2 \geq \frac{q}{4(q-1)} \mathrm{Scal}_0^{\nabla} ,$$

where $\mathrm{Scal}_0^{\nabla} := \min_{x \in M} \mathrm{Scal}_x^{\nabla}$.

As pointed out in the introductory section, in order to obtain the general case [9] we use now only [6], and the dilation procedure of the leafwise part of the metric of [13], based on stochastic flows is not needed in this context.

Let us consider now the setting of a Riemannian spin foliation with a basic mean curvature. The above estimate can be refined in the presence of a non-trivial basic 1-form θ of constant length which is transversally harmonic with respect to the transversal Dirac operator D_{tr} [20]; for the particular case of basic-harmonic mean curvature see [11]. Similar to the classical case [14] we define a *twistor-like* operator with respect to the modified connection $\bar{\nabla}$ and the basic Dirac operator D_b:

$$T_X s = \bar{\nabla}_X s + \frac{1}{q-1} X \cdot D_b s - \frac{1}{q-1} \langle X, \theta \rangle \theta \cdot D_b s - \langle X, \theta \rangle \bar{\nabla}_\theta s ,$$

for any $s \in \Gamma_b(S)$, $X \in \Gamma(Q)$. Now, if s is the eigenspinor corresponding to the eigenvalue λ, after calculation we get [20]

$$\begin{aligned} \int_M \left(\frac{q-2}{q-1}\lambda^2 |s|^2 - \frac{1}{4}\mathrm{Scal}^{\nabla} |s|^2 \right) &= \int_M |Ts|^2 + \int_M \frac{q-3}{q-1} \left|\bar{\nabla}_\theta s\right|^2 \\ &\quad + \int_M \frac{1}{2(q-1)} \left(|D_b(\theta \cdot s)|^2 - |\theta \cdot D_b s|^2 \right), \end{aligned}$$

We use Rayleigh inequality [11]

$$\int_M |D_b(\theta \cdot s)|^2 \geq \int_M |\theta \cdot D_b s|^2 ,$$

and the result is obtained.

THEOREM 3.1 ([20]). *Let $(M, g, \mathcal{F})$ be a Riemannian foliation admitting a non-trivial basic 1-form of constant length which is harmonic with respect to the transversal Dirac operator, and let λ be the first eigenvalue of the basic Dirac operator. Then*

$$\lambda^2 \geq \frac{q-1}{4(q-2)} \mathrm{Scal}_0^{\nabla} ,$$

where $\mathrm{Scal}_0^{\nabla} = \min_{x \in M} \mathrm{Scal}_x^{\nabla}$.

Corollary 3.2 ([20]). *Finally, in the particular case when the 1-form θ is parallel, then the 1-form is transversally harmonic, and we get the generalization of the main result from [1] in the setting of a Riemannian foliation.*

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University of Craiova,
Department of Mathematics,
Al.I. Cuza Nr. 13,
Craiova, RO-200585,
Romania
vlslesar@central.ucv.ro