

BIHARMONIC IMMERSIONS INTO SPHERES AND ELLIPSOIDS

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This paper is focused on biharmonic immersions into spheres and ellipsoids: we shall discuss generalities, geometric properties of solutions and rigidity conditions which force a biharmonic submanifold of a sphere to be CMC. We shall also describe recent developments concerning the existence of biharmonic curves and, more generally, hypersurfaces, into Euclidean ellipsoids.

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1. INTRODUCTION

Harmonic maps (see [17] for an introduction to this topic) are critical points of the *energy* functional

$$(1.1) \quad E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g ,$$

where $\varphi : (M, g) \rightarrow (N, h)$ is a smooth map between two Riemannian manifolds M and N . In analytical terms, the condition of harmonicity is equivalent to the fact that the map φ is a solution of the Euler-Lagrange equation associated to the energy functional (1.1), *i.e.*

$$(1.2) \quad \text{trace}_g (\nabla d\varphi) = 0 .$$

The left member of (1.2) is a vector field along the map φ , or, equivalently, a section of the pull-back bundle $\varphi^{-1}(TN)$: it is called the *tension field* and denoted by $\tau(\varphi)$. A related topic of growing interest deals with the study of the so-called *biharmonic maps*: these maps, which provide a natural generalisation of harmonic maps, are the critical points of the bienergy functional (as suggested by Eells–Lemaire [17])

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g .$$

In [27] G. Jiang derived the first variation and the second variation formulas for the bienergy. In particular, he showed that the Euler-Lagrange equation associated to $E_2(\varphi)$ is

$$(1.3) \quad \tau_2(\varphi) = -J(\tau(\varphi)) = -\Delta\tau(\varphi) - \text{trace } R^N(d\varphi, \tau(\varphi))d\varphi = 0 ,$$

where J denotes (formally) the Jacobi operator of φ , Δ is the rough Laplacian on sections of $\varphi^{-1}(TN)$ that, for a local orthonormal frame $\{e_i\}_{i=1}^m$ on M , is defined by

$$(1.4) \quad \Delta = - \sum_{i=1}^m \{ \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi - \nabla_{\nabla_{e_i}^M e_i}^\varphi \} ,$$

and

$$R^N(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

is the curvature operator on (N, h) . We point out that (1.3) is a *fourth order* semi-linear elliptic system of differential equations. We also note that any harmonic map is an absolute minimum of the bienergy, and so it is trivially biharmonic. Therefore, a general working plan is to study the existence of biharmonic maps which are not harmonic: these shall be referred to as *proper biharmonic maps*. We refer to [31] for existence results and general properties of biharmonic maps.

An immersed submanifold into a Riemannian manifold (N, h) is called a *biharmonic submanifold* if the immersion is a biharmonic map. In a purely geometric context, B.-Y. Chen [11] defined biharmonic submanifolds $M \subset \mathbb{R}^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = (\Delta H_1, \dots, \Delta H_n) = 0$, where $H = (H_1, \dots, H_n)$ is the mean curvature vector as seen in \mathbb{R}^n and Δ is the Beltrami-Laplace operator on M . It is important to point out that, if we apply the definition of biharmonic maps to immersions into the Euclidean space, we recover Chen's notion of biharmonic submanifolds. In this sense, work on biharmonic immersions into non-flat Riemannian manifolds can be regarded in the spirit of a generalization of Chen's biharmonic submanifolds.

In their famous paper [18], Eells and Sampson obtained existence, in every homotopy class of mappings between compact manifolds, of an energy minimizing harmonic map, under the assumption that the Riemannian curvature tensor of the target manifold N is nonpositive (the compactness assumption can be weakened and replaced by some technical conditions which ensure that solutions of the associated non-linear heat equation

$$\tau(\varphi_t) = \frac{\partial \varphi_t}{\partial t}$$

remain bounded as $t \rightarrow +\infty$). By contrast, in the case of mappings into manifolds of positive curvature, solutions are unstable and so they are much more difficult to obtain and understand: in this order of ideas, the most important and studied case is $N = \mathbb{S}^n$ (see [19] for an overview and methods).

In a way which reflects the above cited energy minimizing property of harmonic maps into negatively curved manifolds, a general result of Jiang [27] tells us that a compact, orientable, biharmonic submanifold M into a manifold N such that $\text{Riem}^N \leq 0$ is necessarily minimal. Moreover, C. Oniciuc, in [35], proved that also CMC biharmonic isometric immersions into a manifold N with $\text{Riem}^N \leq 0$ are necessarily minimal. In fact, it is still open the Chen's conjecture: biharmonic submanifolds into a non-positive constant sectional curvature manifold are minimal. The Chen's conjecture was generalized in [6] for biharmonic submanifolds into a Riemannian manifold with non-positive sectional curvature, although Y. Ou and L. Tang found in [37] a counterexample. These facts have pushed research towards the investigation of biharmonic submanifolds of the Euclidean sphere: in Section 3 below, we shall review the main results in this area. A further step is the study of biharmonic submanifolds into Euclidean ellipsoids (for harmonic maps, this was carried out in [20]), because these manifolds are geometrically rich and interestingly do not have constant sectional curvature: in Section 2, we give, in particular, a complete classification of proper biharmonic curves into 3-dimensional ellipsoids while, in Section 4 we shall describe proper biharmonic submanifolds of dimension ≥ 2 .

2. BIHARMONIC CURVES INTO SPHERES AND ELLIPSOIDS

Biharmonic curves $\gamma : I \subset \mathbb{R} \rightarrow (N, h)$ of a Riemannian manifold are the solutions of the fourth order differential equation

$$(2.1) \quad \nabla_{\gamma'}^3 \gamma' - R(\gamma', \nabla_{\gamma'} \gamma') \gamma' = 0,$$

where ∇ is the Levi-Civita connection on (N, h) and R is its curvature operator. These curves arise from a variational problem and are a natural generalisation of geodesics. In the last decade biharmonic curves have been extensively studied and classified in several spaces by analytical inspection of (2.1) (see, for example, [6, 8, 9, 7, 25, 16, 21, 22, 32]). Although much work has been done, the full understanding of biharmonic curves in a surface of the Euclidean three-dimensional space is far from being achieved. As yet, we have a clear picture of biharmonic curves in a surface only in very few cases: one of them is when the surface is invariant by the action of a one parameter group of isometries of the ambient space. For example, in [8] it was proved that a biharmonic curve on a

surface of revolution in the Euclidean space (invariant by the action of $SO(2)$) must be a parallel, that is an orbit of the action of the group on the surface. This property was then generalized to invariant surfaces in a 3-dimensional manifold (see [32]).

The main obstacle in trying to describe and classify biharmonic curves in a surface by analytical methods is that (2.1) is a fourth order differential equation which, in general, is very hard to tackle. In the first part of this Section we present some general notions concerning biharmonic curves. The second part of the Section is devoted to the study of biharmonic curves into spheres and ellipsoids, where purely algebraic methods can be applied. Material and results of this Section can be found in [7, 8, 9, 33].

Let now $\gamma : I \rightarrow (N, h)$ be a curve parametrized by arc length, from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. In this case, putting $T = \gamma'$, the tension field becomes $\tau(\gamma) = \nabla_T T$ and the biharmonicity equation (1.3) reduces to

$$(2.2) \quad \nabla_T^3 T - R(T, \nabla_T T)T = 0.$$

In order to describe geometrically equation (2.2), let us recall the definition of the Frenet frame (for the purposes of this paper, the case $\dim N = n = 2$ would suffice, but we insert the general definition because the reader may find it useful in other contexts):

Definition 2.1. The Frenet frame (see [28]) $\{F_i\}_{i=1, \dots, n}$ associated to a curve $\gamma : I \subset \mathbb{R} \rightarrow (N^n, h)$, parametrized by arc length, is the orthonormalisation of the $(n + 1)$ -uple $\{\nabla_{\frac{\partial}{\partial t}}^{(k)} d\gamma(\frac{\partial}{\partial t})\}_{k=0, \dots, n}$ described by:

$$\begin{aligned} F_1 &= d\gamma\left(\frac{\partial}{\partial t}\right), \\ \nabla_{\frac{\partial}{\partial t}}^\gamma F_1 &= k_1 F_2, \\ \nabla_{\frac{\partial}{\partial t}}^\gamma F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, \dots, n-1, \\ \nabla_{\frac{\partial}{\partial t}}^\gamma F_n &= -k_{n-1} F_{n-1}, \end{aligned}$$

where the functions $\{k_1, k_2, \dots, k_{n-1}\}$ are called the *curvatures* of γ and ∇^γ is the Levi-Civita connection on the pull-back bundle $\gamma^{-1}(TN)$. Note that $F_1 = T = \gamma'$ is the unit tangent vector field along the curve.

Using the Frenet frame, the biharmonic equation (2.2) reduces to a differential system involving the curvatures of γ and if we look for proper biharmonic solutions, that is for biharmonic curves with $k_1 \neq 0$, we have

PROPOSITION 2.2 ([7]). *Let $\gamma : I \subset \mathbb{R} \rightarrow (N^n, h)$ ($n \geq 2$) be a curve parametrized by arc length from an open interval of \mathbb{R} into an n -dimensional Riemannian manifold (N^n, h) . Then γ is proper biharmonic if and only if:*

$$(2.3) \quad \begin{cases} k_1 = \text{constant} \neq 0 \\ k_1^2 + k_2^2 = R(F_1, F_2, F_1, F_2) \\ k_2' = -R(F_1, F_2, F_1, F_3) \\ k_2 k_3 = -R(F_1, F_2, F_1, F_4) \\ R(F_1, F_2, F_1, F_j) = 0, \quad j = 5, \dots, n. \end{cases}$$

As a special case of (2.3), if $\gamma : I \subset \mathbb{R} \rightarrow (N^2, h)$ is a curve into a surface, then γ is proper biharmonic if and only if

$$(2.4) \quad \begin{cases} k_1 = \text{constant} \neq 0 \\ k_1^2 = K, \end{cases}$$

where K is the Gaussian curvature of the surface (N^2, h) . Here we propose a scheme to classify biharmonic curves into an ellipsoid in the three-dimensional Euclidean space by using purely algebraic methods.

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable function: we shall assume that, for all $p \in N^2 = F^{-1}(0)$, $(\text{grad } F)(p) \neq 0$, so that N^2 is a regular surface in \mathbb{R}^3 . If we denote by C_{HF} the cofactor matrix of the Hessian HF of F , the Gaussian curvature of the surface N^2 is given by (see, for example, [23])

$$(2.5) \quad K = \frac{(\text{grad } F)(C_{HF})(\text{grad } F)^\top}{\|\text{grad } F\|^4}.$$

Let now $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ be two differentiable functions such that $F^{-1}(0)$ and $G^{-1}(0)$ are, as above, two regular surfaces in \mathbb{R}^3 , and also assume that at all points $p \in F^{-1}(0) \cap G^{-1}(0)$ the gradients $\text{grad } F$ and $\text{grad } G$ are linearly independent. Then $F^{-1}(0) \cap G^{-1}(0)$ defines the trace of a regular curve in \mathbb{R}^3 that, locally, can be parametrized by arc length as $\gamma(s) = (x(s), y(s), z(s))$, $s \in (a, b)$. The unit tangent vector to γ is then

$$\gamma'(s) = \frac{d\gamma}{ds} = T = \frac{\text{grad } F \wedge \text{grad } G}{\|\text{grad } F\| \|\text{grad } G\|}.$$

The curve γ can be seen as a curve both of $F^{-1}(0)$ and $G^{-1}(0)$. For each point $p = \gamma(s)$, $s \in (a, b)$, we denote by $k_n^F(p)$ (respectively $k_n^G(p)$) the normal curvature at p of the surface $F^{-1}(0)$ (respectively $G^{-1}(0)$) in the direction of T .

The curvature $k(s)$ of the curve $\gamma : (a, b) \rightarrow \mathbb{R}^3$ can be computed in terms of the normal curvatures $k_n^F(p)$ and $k_n^G(p)$, $p = \gamma(s)$, as

$$(2.6) \quad k^2 = \frac{1}{\sin^2 \vartheta} \left((k_n^F)^2 + (k_n^G)^2 - 2(k_n^F)(k_n^G) \cos \vartheta \right),$$

where ϑ is the angle between $(\text{grad } F)(p)$ and $(\text{grad } G)(p)$, that is

$$\cos \vartheta = \frac{\langle \text{grad } F, \text{grad } G \rangle}{\|\text{grad } F\| \|\text{grad } G\|} .$$

The proof of (2.6) is immediate. In fact, $k(s)$ is the norm of $\gamma''(s) = d^2\gamma/ds^2$ which is normal to T . Thus,

$$\gamma'' = \alpha \frac{\text{grad } F}{\|\text{grad } F\|} + \beta \frac{\text{grad } G}{\|\text{grad } G\|}$$

for some functions $\alpha, \beta : (a, b) \rightarrow \mathbb{R}$ which, recalling that

$$k_n^F = \left\langle \gamma'', \frac{\text{grad } F}{\|\text{grad } F\|} \right\rangle, \quad k_n^G = \left\langle \gamma'', \frac{\text{grad } G}{\|\text{grad } G\|} \right\rangle ,$$

can be expressed by:

$$\alpha = \frac{k_n^F - k_n^G \cos \vartheta}{\sin^2 \vartheta}, \quad \beta = \frac{k_n^G - k_n^F \cos \vartheta}{\sin^2 \vartheta} .$$

Finally, looking at $\gamma(s)$ as a curve in the surface $F^{-1}(0)$, at a point $p = \gamma(s)$ the geodesic curvature $k_1(s)$, the normal curvature $k_n^F(p)$ and the curvature $k(s)$ are related by the formula

$$(2.7) \quad k^2 = k_1^2 + (k_n^F)^2 .$$

Thus, combining (2.6) and (2.7), we have the following proposition:

PROPOSITION 2.3. *Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ be two differentiable functions such that $F^{-1}(0)$ and $G^{-1}(0)$ are two regular surfaces in \mathbb{R}^3 . Assume that at all points $p \in F^{-1}(0) \cap G^{-1}(0)$ the gradients $\text{grad } F$ and $\text{grad } G$ are linearly independent. Then the geodesic curvature k_1 of the curve $\gamma : (a, b) \rightarrow F^{-1}(0) \subset \mathbb{R}^3$, with $\gamma(s) \in F^{-1}(0) \cap G^{-1}(0)$, for all $s \in (a, b)$, is given by*

$$(2.8) \quad k_1^2 = \frac{(\cos \vartheta k_n^F - k_n^G)^2}{\sin^2 \vartheta} .$$

The main point here is that, in order to compute the geodesic curvature of the curve γ defined as in Proposition 2.3, there is no need to parametrize the intersection curve, because (2.8) can be explicitly written in terms of $\text{grad } F$, $\text{grad } G$ and the Hessian matrices HF and HG . Now we apply this machinery to the study of biharmonic curves into an ellipsoid in the Euclidean space \mathbb{R}^3 . With respect to an adapted orthonormal frame of \mathbb{R}^3 , we can describe an ellipsoid in \mathbb{R}^3 by means of $\mathcal{Q} = F^{-1}(0)$, where

$$(2.9) \quad F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \quad a, b, c > 0 .$$

According to (2.4), the Gauss curvature of the surface along a proper biharmonic curve must be a positive constant. If we compute the Gauss curvature of the ellipsoid \mathcal{Q} , by means of (2.5), we find

$$(2.10) \quad K = \frac{1}{a^2 b^2 c^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^2}.$$

If $a = b = c$, then the quadric is a sphere and the proper biharmonic curves are the circles of radius $\sqrt{2}a/2$, a result proved in [8]. In all the other cases, combining (2.4) and (2.10), we conclude that, if there exists a proper biharmonic curve, then it must be the intersection of the given ellipsoid \mathcal{Q} with another ellipsoid of the type

$$(2.11) \quad \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = d^2,$$

where $d \in \mathbb{R}$.

THEOREM 2.4. *Let \mathcal{Q} be an ellipsoid as above which is not a sphere, and assume $a \geq b > c$ (the case $a > b \geq c$ is similar). Then \mathcal{Q} admits a proper biharmonic curve if and only if*

$$(2.12) \quad a = b.$$

Moreover, if (2.12) holds, the biharmonic curve is the intersection of the ellipsoid \mathcal{Q} with the ellipsoid (2.11) with $d^2 = 1/(ac)$.

Proof. (Outline). As we proved above, if there exists a proper biharmonic curve γ , it must be the intersection of \mathcal{Q} with an ellipsoid (2.11), *i.e.*

$$(2.13) \quad \gamma : \begin{cases} F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ G(x, y, z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - d^2 = 0. \end{cases}$$

Then, using (2.8), we can compute the geodesic curvature of the intersection curve γ as a curve of the quadric \mathcal{Q} . A long, but straightforward, computation yields:

$$(2.14) \quad k_1^2 = \frac{1}{d^2} \frac{\left[d^2 \lambda_4 - \left(\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6} \right) \lambda_6 \right]^2}{\lambda_8^2 \left[d^2 \left(\frac{x^2}{a^8} + \frac{y^2}{b^8} + \frac{z^2}{c^8} \right) - \left(\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6} \right)^2 \right]},$$

where

$$\lambda_n = a^n y^2 z^2 (b^2 - c^2)^2 + b^n x^2 z^2 (a^2 - c^2)^2 + c^n x^2 y^2 (a^2 - b^2)^2.$$

Now, since \mathcal{Q} is not a sphere, we recall our hypothesis $a \geq b > c$ and also note that the curve γ is a real curve with infinity points if and only if $d^2c^2 - 1 > 0$. Under these hypotheses, a tedious computation shows that the condition that the curve γ is proper biharmonic, that is (2.4), becomes

$$\frac{1 - a^2c^2d^4}{a^4c^2d^4(c^2d^2 - 1)} = 0,$$

from which the desired result follows. \square

Remark 2.5. In [33], the methods of the previous theorem are used to obtain a complete classification of biharmonic curves into non-degenerate quadrics in the Euclidean 3-dimensional space, and this represents an instance where the understanding of biharmonic curves is satisfactory.

3. BIHARMONIC IMMERSIONS IN \mathbb{S}^N

The key ingredient in the study of biharmonic submanifolds is the splitting of the bitension field with respect to its normal and tangent components. In the case when the ambient space is the unit Euclidean sphere we have the following characterization.

THEOREM 3.1 ([12, 35, 38]). *An immersion $\varphi : M^m \rightarrow \mathbb{S}^n$ is biharmonic if and only if*

$$(3.1) \quad \begin{cases} \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) - mH = 0, \\ 2\text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot) + \frac{m}{2} \text{grad } |H|^2 = 0, \end{cases}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $|H|$ the mean curvature function, ∇^\perp and Δ^\perp the connection and the Laplacian in the normal bundle of φ , respectively.

In the codimension one case, denoting by $A = A_\eta$ the shape operator with respect to a (local) unit section η in the normal bundle and putting $f = (\text{trace } A)/m$, the above result reduces to the following.

COROLLARY 3.2 ([5, 35]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be an orientable hypersurface. Then φ is biharmonic if and only if*

$$(3.2) \quad \begin{cases} \text{(i)} & \Delta f = (m - |A|^2)f, \\ \text{(ii)} & A(\text{grad } f) = -\frac{m}{2}f \text{grad } f. \end{cases}$$

A special class of immersions in \mathbb{S}^n consists of the parallel mean curvature immersions (PMC), that is immersions such that $\nabla^\perp H = 0$. For this class of immersions Theorem 3.1 reads as follows.

COROLLARY 3.3 ([2]). *Let $\varphi : M^m \rightarrow \mathbb{S}^n$ be a PMC immersion. Then φ is biharmonic if and only if*

$$(3.3) \quad \text{trace } B(A_H(\cdot), \cdot) = mH,$$

or equivalently,

$$(3.4) \quad \begin{cases} \langle A_H, A_\xi \rangle = 0, & \forall \xi \in C(NM) \text{ with } \xi \perp H, \\ |A_H|^2 = m|H|^2, \end{cases}$$

where NM denotes the normal bundle of M in \mathbb{S}^n .

We now list the main examples of proper biharmonic immersions in \mathbb{S}^n .

B1. The canonical inclusion of the small hypersphere

$$\mathbb{S}^{n-1}(1/\sqrt{2}) = \left\{ (x, 1/\sqrt{2}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, |x|^2 = 1/2 \right\} \subset \mathbb{S}^n.$$

B2. The canonical inclusion of the standard (extrinsic) products of spheres

$$\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) = \left\{ (x, y) \in \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}, |x|^2 = |y|^2 = 1/2 \right\} \subset \mathbb{S}^n,$$

$n_1 + n_2 - 1$ and $n_1 \neq n_2$.

B3. The maps $\varphi = \iota \circ \phi : M \rightarrow \mathbb{S}^n$, where $\phi : M \rightarrow \mathbb{S}^{n-1}(1/\sqrt{2})$ is a minimal immersion, and $\iota : \mathbb{S}^{n-1}(1/\sqrt{2}) \rightarrow \mathbb{S}^n$ denotes the canonical inclusion.

B4. The maps $\varphi = \iota \circ (\phi_1 \times \phi_2) : M_1 \times M_2 \rightarrow \mathbb{S}^n$, where $\phi_i : M_i^{m_i} \rightarrow \mathbb{S}^{n_i}(1/\sqrt{2})$, $0 < m_i \leq n_i$, $i = 1, 2$, are minimal immersions, $m_1 \neq m_2$, $n_1 + n_2 - 1$, and $\iota : \mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \rightarrow \mathbb{S}^n$ denotes the canonical inclusion.

Remark 3.4. (i) The proper biharmonic immersions of class **B3** are pseudo-umbilical, i.e. $A_H = |H|^2 \text{Id}$, have parallel mean curvature vector field and mean curvature $|H| = 1$. Clearly, $\nabla A_H = 0$.

(ii) The proper biharmonic immersions of class **B4** are no longer pseudo-umbilical, but still have parallel mean curvature vector field and their mean curvature is $|H| = |m_1 - m_2|/m \in (0, 1)$, where $m = m_1 + m_2$. Moreover, $\nabla A_H = 0$ and the principal curvatures in the direction of H , i.e. the eigenvalues of A_H , are constant on M and given by $\lambda_1 = \dots = \lambda_{m_1} = (m_1 - m_2)/m$, $\lambda_{m_1+1} = \dots = \lambda_{m_1+m_2} = -(m_1 - m_2)/m$. Specific B4 examples were given by W. Zhang in [42] and generalized in [1, 41].

When a biharmonic immersion has constant mean curvature (CMC) the following bound for $|H|$ holds.

THEOREM 3.5 ([36]). *Let $\varphi : M \rightarrow \mathbb{S}^n$ be a CMC proper biharmonic immersion. Then $|H| \in (0, 1]$, and $|H| = 1$ if and only if φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$, that is φ is **B3**.*

Let us look in detail at the case of CMC proper biharmonic hypersurfaces in \mathbb{S}^{m+1} :

THEOREM 3.6 ([1, 2]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a CMC proper biharmonic hypersurface. Then*

- (i) $|A|^2 = m$;
- (ii) *the scalar curvature s is constant and positive, $s = m^2(1 + |H|^2) - 2m$;*
- (iii) *for $m > 2$, $|H| \in (0, (m-2)/m) \cup \{1\}$. Moreover, $|H| = 1$ if and only if $\varphi(M)$ is an open subset of the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$, and $|H| = (m-2)/m$ if and only if $\varphi(M)$ is an open subset of the standard product $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Remark 3.7. In the minimal case the condition $|A|^2 = m$ is exhaustive. In fact a minimal hypersurface in \mathbb{S}^{m+1} with $|A|^2 = m$ is a minimal standard product of spheres (see [15, 29]). We point out that the full classification of CMC hypersurfaces in \mathbb{S}^{m+1} with $|A|^2 = m$, therefore biharmonic, is not known.

COROLLARY 3.8. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a complete proper biharmonic hypersurface.*

- (i) *If $|H| = 1$, then $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and φ is an embedding.*
- (ii) *If $|H| = (m-2)/m$, $m > 2$, then $\varphi(M) = \mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ and the universal cover of M is $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{R}$.*

In the following, we shall no longer assume that the biharmonic hypersurfaces have constant mean curvature, and we shall split our study in three cases. In Case 1 we shall study the proper biharmonic hypersurfaces with respect to the number of their distinct principal curvatures, in Case 2 we shall study them with respect to $|A|^2$ and $|H|^2$, and in Case 3 the study will be done with respect to the sectional and Ricci curvatures of the hypersurface.

3.1. Case 1

Obviously, if $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ is an umbilical proper biharmonic hypersurface in \mathbb{S}^{m+1} , then $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

When the hypersurface has at most two or exactly three distinct principal curvatures everywhere we obtain the following rigidity results.

THEOREM 3.9 ([1]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a hypersurface. Assume that φ is proper biharmonic with at most two distinct principal curvatures everywhere. Then φ is CMC and $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$. Moreover, if M is complete, then either $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and φ is an embedding,*

or $\varphi(M) = \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$ and φ is an embedding when $m_1 \geq 2$ and $m_2 \geq 2$.

THEOREM 3.10 ([1]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$, $m \geq 3$, be a proper biharmonic hypersurface. The following statements are equivalent:*

- (i) φ is quasi-umbilical,
- (ii) φ is conformally flat,
- (iii) $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

It is well known that, if $m \geq 4$, a hypersurface $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ is quasi-umbilical if and only if it is conformally flat. From Theorem 3.10 we see that under the biharmonicity hypothesis the equivalence remains true when $m = 3$.

THEOREM 3.11 ([3]). *There exist no compact CMC proper biharmonic hypersurfaces $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ with three distinct principal curvatures everywhere.*

In particular, in the low dimensional cases, Theorem 3.9, Theorem 3.11 and a result of S. Chang (see [10]) imply the following.

THEOREM 3.12 ([6, 3]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper biharmonic hypersurface.*

- (i) *If $m = 2$, then $\varphi(M)$ is an open part of $\mathbb{S}^2(1/\sqrt{2}) \subset \mathbb{S}^3$.*
- (ii) *If $m = 3$ and M is compact, then φ is CMC and $\varphi(M) = \mathbb{S}^3(1/\sqrt{2})$ or $\varphi(M) = \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

We recall that an orientable hypersurface $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ is said to be *isoparametric* if it has constant principal curvatures or, equivalently, the number ℓ of distinct principal curvatures $k_1 > k_2 > \dots > k_\ell$ is constant on M and the k_i 's are constant. The distinct principal curvatures have constant multiplicities m_1, \dots, m_ℓ , $m = m_1 + m_2 + \dots + m_\ell$.

In [26], T. Ichiyama, J.I. Inoguchi and H. Urakawa classified the proper biharmonic isoparametric hypersurfaces in spheres.

THEOREM 3.13 ([26]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be an orientable isoparametric hypersurface. If φ is proper biharmonic, then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

3.2. Case 2

The starting point is the following result that in the present form was proved in [4], although point (a) is originally taken from [13].

THEOREM 3.14 ([4, 13]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface and assume that φ is proper biharmonic.*

- (a) if $|A|^2 \leq m$ then φ is CMC and $|A|^2 = m$;
- (b) if $|A|^2 \geq m$ then φ is CMC and $|A|^2 = m$.

Remark 3.15. It is worth pointing out that the statement (a) of Theorem 3.14 is similar in the minimal case: if $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ is a minimal hypersurface with $|A|^2 \leq m$, then either $|A| = 0$ or $|A|^2 = m$ (see [40]). By way of contrast, an analog of the statement (b) in Theorem 3.14 is not true in the minimal case. In fact, it was proved in [39] that if a minimal hypersurface $\varphi : M^3 \rightarrow \mathbb{S}^4$ has $|A|^2 > 3$, then $|A|^2 \geq 6$. But, if the compact minimal hypersurface of \mathbb{S}^{m+1} with $|A|^2 \geq m$ has at most two distinct principal curvatures, then $|A|^2 = m$ (see [24]); and we believe that any proper biharmonic hypersurface in \mathbb{S}^{m+1} has at most two principal curvatures everywhere.

Obviously, from Theorem 3.14 we get the following result.

PROPOSITION 3.16. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper biharmonic and $|A|^2$ is constant, then φ is CMC and $|A|^2 = m$.*

The next result is a direct consequence of Theorem 3.14.

PROPOSITION 3.17. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper biharmonic and $|H|^2 \geq 4(m - 1)/(m(m + 8))$, then φ is CMC. Moreover,*

- (i) if $m \in \{2, 3\}$, then $\varphi(M)$ is a small hypersphere $\mathbb{S}^m(1/\sqrt{2})$;
- (ii) if $m = 4$, then $\varphi(M)$ is a small hypersphere $\mathbb{S}^4(1/\sqrt{2})$ or a standard product of spheres $\mathbb{S}^3(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

For the non-compact case we have the following.

PROPOSITION 3.18. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$, $m > 2$, be a non-compact hypersurface. Assume that M is complete and has non-negative Ricci curvature. If φ is proper biharmonic, $|A|^2$ is constant and $|A|^2 \geq m$, then φ is CMC and $|A|^2 = m$. In this case $|H|^2 \leq ((m - 2)/m)^2$.*

COROLLARY 3.19. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a non-compact hypersurface. Assume that M is complete and has non-negative Ricci curvature. If φ is proper biharmonic, $|A|^2$ is constant and $|H|^2 \geq 4(m - 1)/(m(m + 8))$, then φ is CMC and $|A|^2 = m$. In this case, $m \geq 4$ and $|H|^2 \leq ((m - 2)/m)^2$.*

PROPOSITION 3.20. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a non-compact hypersurface. Assume that M is complete and has non-negative Ricci curvature. If φ is proper biharmonic, $|A|^2$ is constant, $|A|^2 \leq m$ and H is nowhere zero, then φ is CMC and $|A|^2 = m$.*

Proof. As H is nowhere zero we consider $\eta = H/|H|$ a global unit section in the normal bundle. Then, on M ,

$$(3.5) \quad \Delta f = (m - |A|^2)f,$$

where $f = |H| > 0$. As $m - |A|^2 \geq 0$ by a classical result (see, for example, [30, p. 2]) we conclude that $m = |A|^2$ and therefore f is constant. \square

3.3. Case 3

We first present another result of J.H. Chen in [13].

THEOREM 3.21 ([13]). *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper biharmonic, M has non-negative sectional curvature and $m \leq 10$, then φ is CMC and $\varphi(M)$ is either $\mathbb{S}^m(1/\sqrt{2})$, or $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Then in [4] the following result was proved.

THEOREM 3.22. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$, $m \geq 3$, be a hypersurface. Assume that M has non-negative sectional curvature and for all $p \in M$ there exists $X_p \in T_pM$, $|X_p| = 1$, such that $\text{Ricci}(X_p, X_p) = 0$. If φ is proper biharmonic, then $\varphi(M)$ is an open part of $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Remark 3.23. If $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$, $m \geq 3$, is a compact hypersurface, then the conclusion of Theorem 3.22 holds replacing the hypothesis on the Ricci curvature with the requirement that the first fundamental group is infinite. In fact, the full classification of compact hypersurfaces in \mathbb{S}^{m+1} with non-negative sectional curvature and infinite first fundamental group was given in [14].

4. BIHARMONIC SUBMANIFOLDS INTO ELLIPSOIDS

We study biharmonic submanifolds into Euclidean ellipsoids $Q^{p+q+1}(c, d)$ defined as follows:

$$Q^{p+q+1}(c, d) = \left\{ (x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^n : \frac{|x|^2}{c^2} + \frac{|y|^2}{d^2} = 1 \right\},$$

where c, d are fixed positive constants. The symmetry of $Q^{p+q+1}(c, d)$ makes it natural to look for biharmonic generalized Clifford's tori. More precisely, we shall be interested in isometric immersions of the following type:

$$(4.1) \quad \varphi : \quad \mathbb{S}^p(a) \quad \times \quad \mathbb{S}^q(b) \quad \longrightarrow \quad Q^{p+q+1}(c, d)$$

$$(x_1, \dots, x_{p+1}, \quad y_1, \dots, y_{q+1}) \longmapsto (x_1, \dots, x_{p+1}, y_1, \dots, y_{q+1}),$$

where i denotes the inclusion and the radii a, b must satisfy the following condition:

$$(4.2) \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1.$$

In this context, we have the following result, proved in [34]:

THEOREM 4.1. *Let $\varphi : \mathbb{S}^p(a) \times \mathbb{S}^q(b) \rightarrow Q^{p+q+1}(c, d)$ be an isometric immersion as in (4.1). If*

$$(4.3) \quad a^2 = c^2 \frac{p}{p+q} ; \quad b^2 = d^2 \frac{q}{p+q}$$

then the immersion is minimal. If (4.3) does not hold and

$$(4.4) \quad a^2 = c^2 \frac{c}{c+d} ; \quad b^2 = d^2 \frac{d}{c+d} ,$$

then the immersion is proper biharmonic.

Remark 4.2. We observe that, interestingly, if $c = p$ and $d = q$, then we have generalized minimal Clifford's tori, but we do not have proper biharmonic submanifolds of the type (4.1).

We also point out that, according to Theorem 4.1, the ellipsoid $Q^3(c, d)$ ($p = q = 1, c \neq d$) admits a proper biharmonic torus, while in \mathbb{S}^3 there exists no genus 1 proper biharmonic submanifold, as we pointed out in Section 3.

Proof. We briefly outline the method of proof: essentially, we work by using coordinates in \mathbb{R}^n , suitably restricted to the ellipsoid or to the torus, according to necessity. It is necessary to proceed to a non-trivial direct computation of the tension and the bitension field. In order to describe the final result, we need to introduce the following vector fields:

$$(4.5) \quad \eta^Q = \frac{\eta_1^Q}{|\eta_1^Q|} ,$$

where

$$\eta_1^Q = \left(\frac{1}{c^2} x_1, \dots, \frac{1}{c^2} x_{p+1}, \frac{1}{d^2} y_1, \dots, \frac{1}{d^2} y_{q+1} \right) ,$$

which represents a unit normal vector field on the ellipsoid $Q^{p+q+1}(c, d)$. And also

$$(4.6) \quad \eta^T = \frac{\eta_1^T}{|\eta_1^T|} ,$$

where

$$(4.7) \quad \eta_1^T = \left(\frac{c^2}{a^2} x_1, \dots, \frac{c^2}{a^2} x_{p+1}, -\frac{d^2}{b^2} y_1, \dots, -\frac{d^2}{b^2} y_{q+1} \right) ,$$

which is a unit normal vector on the torus T viewed as a submanifold of the ellipsoid $Q^{p+q+1}(c, d)$. A first step leads us to the expression of the tension field:

$$(4.8) \quad \tau = \lambda \eta_1^T ,$$

where

$$\lambda = - \left[\frac{c^4}{a^2} + \frac{d^4}{b^2} \right]^{-1} \left[\frac{p c^2}{a^2} - \frac{q d^2}{b^2} \right] .$$

In particular, using (4.2), it is now immediate to conclude that (4.3) is equivalent to the minimality of the immersion.

Next, one proceeds to the computation of the bitension field τ_2 , the main difficulty being to work out the expression of the rough Laplacian (1.4). The result is:

$$(4.9) \quad \tau_2 = - \left[\mu \eta_1^T + \text{trace } R^Q(d i, \tau) d i \right] ,$$

where, using the fact that $|\eta_1^T|^2$ is constant on T , we have defined

$$(4.10) \quad \mu = \frac{\lambda}{|\eta_1^T|^2} \left[\frac{p c^4}{a^4} + \frac{q d^4}{b^4} \right]$$

(note also that, if (4.3) does not hold, then $\lambda \neq 0$, so that $\mu \neq 0$ and the immersion is not minimal).

In order to end the proof, one has to investigate for which values (if any) of a, b the bitension τ_2 vanishes. In order to overcome the difficulties arising from the presence of the curvature tensor in (4.9), that can be done in an efficient way by studying the vanishing of normal and tangential components separately. In particular, one proves that the normal component of τ_2 is identically zero if and only if (4.4) holds, while the tangential part of τ_2 vanishes for all values of a and b . \square

We also point out that, by using similar methods, it is possible to study biharmonic submanifolds into Euclidean ellipsoids of revolution $Q^{p+1}(c, d)$ defined as follows:

$$Q^{p+1}(c, d) = \left\{ (x, y) \in \mathbb{R}^{p+1} \times \mathbb{R} = \mathbb{R}^n : \frac{|x|^2}{c^2} + \frac{y^2}{d^2} = 1 \right\} ,$$

where c, d are fixed positive constants. In this case, the symmetry of $Q^{p+1}(c, d)$ makes it natural to look for biharmonic hyperspheres, that is isometric immersions of the following type:

$$(4.11) \quad \begin{aligned} \varphi : \quad \mathbb{S}^p(a) \quad \times \quad \{b\} \quad \longrightarrow \quad Q^{p+1}(c, d) \\ (x_1, \dots, x_{p+1} \quad , \quad b) \quad \longmapsto \quad (x_1, \dots, x_{p+1}, b) , \end{aligned}$$

where i denotes the inclusion and the constants a, b must again satisfy the condition

$$(4.12) \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$$

(note that a is a radius, so it is positive, while the only request on b is: $|b| < d$).

In this context, we have the following result (see [34]):

THEOREM 4.3. *Let $\varphi : \mathbb{S}^p(a) \times \{b\} \rightarrow Q^{p+1}(c, d)$ be an isometric immersion as in (4.11). If*

$$(4.13) \quad a^2 = c^2 ; \quad b = 0$$

then the immersion is minimal (this is the case of the equator hypersphere). If

$$(4.14) \quad a = c \sqrt{\frac{c}{c+d}} ; \quad b = \pm d \sqrt{\frac{d}{c+d}} ,$$

then the immersion is proper biharmonic.

We conclude by saying that the results of this Section enable us to derive the existence of a new, large family of proper biharmonic immersions into ellipsoids. That is a consequence of the following *composition properties*, which extend a result of [5] for the case of immersions into spheres:

THEOREM 4.4. *Let $\varphi : \mathbb{S}^p(a) \rightarrow Q^{p+1}(c, d)$ be a proper biharmonic immersion as in Theorem 4.3, and let $\psi : M^m \rightarrow \mathbb{S}^p(a)$ be a minimal immersion. Then $\varphi \circ \psi : M^m \rightarrow Q^{p+1}(c, d)$ is a proper biharmonic immersion.*

THEOREM 4.5. *Let $\varphi : \mathbb{S}^p(a) \times \mathbb{S}^q(b) \rightarrow Q^{p+q+1}(c, d)$ be a proper biharmonic immersion as in Theorem 4.1, and let $\psi_1 : M_1^{m_1} \rightarrow \mathbb{S}^p(a)$, $\psi_2 : M_2^{m_2} \rightarrow \mathbb{S}^q(b)$ be two minimal immersions. Then $\varphi \circ (\psi_1 \times \psi_2) : M_1^{m_1} \times M_2^{m_2} \rightarrow Q^{p+q+1}(c, d)$ is a proper biharmonic immersion.*

Remark 4.6. We point out that all biharmonic submanifolds into ellipsoids, constructed using the previous composition properties, have parallel mean curvature vector field.

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