ON STRONGLY CONTINUOUS ONE-PARAMETER GROUPS OF AUTOMORPHISMS

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We prove structure theorems for strongly continuous one-parameter groups formed by surjective linear isometries of spaces of bounded N-linear functionals over strictly convex complex Banach spaces. Complete description is given in the case of Hilbert-equivalent norms on the basis of probability arguments. As a consequence, we classify the strongly continuous one-parameter automorphism groups of all infinite-dimensional Cartan factors of Jordan theory.

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1. INTRODUCTION, MAIN RESULTS

In 2008, in a finally unpublished draft version of [2], F. Botelho and J. Jamison launched the following conjecture:

CONJECTURE 1.1. *Let $\mathbf{X_1}, \mathbf{X}_2$ be complex Banach spaces. Suppose $t \mapsto U_k^t$ $(k=1,2)$ are maps $\mathbb{R} \to \mathcal{U}(X_k) := \{\text{surjective linear isometries of } X_k\}$ with the one-parameter group property $U_1^{t+s} \otimes U_2^{t+s} = [U_1^t \otimes U_2^t][U_1^s \otimes U_2^s]$ $(t, s \in \mathbb{R})$ and such that for every fixed bounded bilinear functional $\phi : \mathbf{X}_1 \times \mathbf{X}_2 \to \mathbb{C}$ and for every couple $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{X}_1 \times \mathbf{X}_2$, the function $t \mapsto \phi\big(U_1^t \mathbf{x}_1, U_2^t \mathbf{x}_2\big)$ is continuous. Then both $t \mapsto U_1^t$ and $t \mapsto U_2^t$ are strongly continuous one-parameter groups.*[1]

Actually, they even outlined a proof for 1.1 under certain additional hypothesis, implying a requirement on the representations of the elementary forms which turned out to be contradictory due to the fact that we have $(\kappa_1 \mathbf{x}_1) \otimes (\kappa_2 \mathbf{x}_2) = \mathbf{x}_1 \otimes \mathbf{x}_2$ whenever $\kappa_1 \kappa_2 = 1$. Using a completely different probabilistic approach, in course of the proof of [10, Theorem 1.1] we have shown the following slightly generalized version of the conjecture for the case with Hilbert spaces:

[1] That is $U_k^{t+s} = U_k^t U_k^s$ $(t, s \in \mathbb{R},\ k = 1, 2)$ and the Banach space valued functions $t \mapsto U_k^t \mathbf{x}_k$ are norm-continuous for every fixed couple $\mathbf{x}_1, \mathbf{x}_2$.

Theorem 1.2. *Let* $\mathbf{H}^{(1)},\dots,\mathbf{H}^{(N)}$ *be Hilbert spaces,* $\big[\mathbf{U}(t) : t\in\mathbb{R}\big]$ *be a one-parameter group where* $\mathbf{U}(t) := U_{1,t}\otimes\cdots\otimes U_{N,t}$ *with unitary operators* $U_{k,t}\in\mathcal{U}(\mathbf{H}^{(k)})$ *having the following continuity property:* $t\mapsto\prod_{k=1}^N\langle U_{k,t}\mathbf{x}_k|\mathbf{h}_k\rangle$ *is continuous for every fixed* $\mathbf{x}_1,\mathbf{h}_1\in\mathbf{H}^{(1)},\dots,\mathbf{x}_N,\mathbf{h}_N\in\mathbf{H}^{(N)}$. *Then we can find functions* $\kappa_1,\dots,\kappa_N:\mathbb{R}\to\mathbb{T}(:=\{\kappa\in\mathbb{C} : |\kappa|=1\})$ *with* $\prod\limits_{k=1}^{N}\kappa_k\equiv 1$ *such that the families* $\big[\kappa_k(t)U_{k,t} : t\in\mathbb{R}\big]$ *are strongly continuous one-parameter groups. Thus, by Stone's classical theorem* [8, 12] *there are possibly unbounded self-adjoint operators* $A_k : \mathrm{dom}(A_k)\to\mathbf{H}^{(k)}$ *defined on dense linear submanifolds such that* $\mathbf{U}(t)=\big[\exp(itA_1)\big]\otimes\cdots\otimes\big[\exp(itA_N)\big]$ $(t\in\mathbb{R})$.

At first sight our arguments in [10] rely heavily upon the Hilbert space structure. In this paper we are going to investigate how far can we get rid of the scalar product. In Section 2 we revise the arguments on adjusted componentwise continuity of functions of type $t\mapsto\mathbf{x}_{1,t}\otimes\cdots\otimes\mathbf{x}_{N,t}$. The new results extend to the setting of uniformly convex spaces and we get the following conclusion.

Proposition 1.3. *Let* $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$ *be uniformly convex Banach spaces. Assume* $\big[\mathbf{U}(t) : t\in\mathbb{R}\big]$ *is a one-parameter group where* $\mathbf{U}(t)=U_{1,t}\otimes\cdots\otimes U_{N,t}$ *with invertible bounded linear operators* $U_{k,t}\in\mathcal{L}(\mathbf{X}^{(k)})$ *such that the functions* $t\mapsto\big\|\mathbf{U}(t)(\mathbf{x}^{(1)}\otimes\cdots\otimes\mathbf{x}^{(N)})\big\|$ *are all continuous. Then, for each index* $k=1,\dots,N$ *there exist multipliers* $\rho_k,\widetilde{\rho}_k:\mathbb{R}\to\mathbb{C}$ *such that,*

$$\begin{array}{ll}\big[\rho_k(t)U_{k,t} :\ t\in\mathbb{R}\big] & \textit{is a strongly continuous family,}\\ \big[\widetilde{\rho}_k(t)U_{k,t} :\ t\in\mathbb{R}\big] & \textit{is a one-parameter group.}\end{array}$$

Actually, we establish 1.3 a bit more generally: for spaces where weak convergence implies norm convergence. Also we shall see that the multipliers can be chosen to be bounded both from above and below from 0 (that is $0<\inf|\rho_k|,\inf|\widetilde{\rho}_k|$ and $\sup|\rho_k|,\sup|\widetilde{\rho}_k|<\infty$). In particular, the conditions in 1.3 hold if the operators $U_{k,t}$ are all isometries, and in this case we can choose even $|\rho_k|=|\widetilde{\rho}_k|\equiv 1$.

Unfortunately the later considerations in ([3], Sections 4, 5) leading to probabilistic arguments seem to be too closely related with Hilbert space structure. Minor modifications seem possible when replacing L^2-estimates with L^p-type estimates resulting in perhaps technically interesting facts with not much farther generality from the original Hilbert setting of [3]. We do not proceed into this direction.

A perhaps rather aesthetical generalization with equivalent general Banach norms in the Hilbert spaces instead of the ones defined by inner products can be developed due to the amenability of the additive group of $\mathbb{R}$.

Theorem 1.4. *In the setting of Theorem* 1.2, *let* $|||\cdot|||_1,\dots,|||\cdot|||_N$ *denote equivalent Banach-norms on the respective spaces* $\mathbf{H}^{(k)}$ *and let each operator* $U_{k,t}$ *be a surjective* $|||\cdot|||_k$*-isometry (instead of being* $\langle\cdot|\cdot\rangle_k$*-unitary as supposed originally). Then there are equivalent inner products* $\langle\langle\cdot\,|\,\cdot\rangle\rangle_k$ *on the respective spaces* $\mathbf{H}^{(k)}$ *such that the conclusions of Theorem* 1.3 *hold with suitable possibly unbounded* $\langle\langle\cdot\,|\,\cdot\rangle\rangle_k$*-self-adjoint operators* A_k.

The above result is an immediate consequence of the following fact.

Proposition 1.5. *Any linear automorphism of a bounded circular domain* $\mathbf{D}$ *in a Hilbert space* $\mathbf{H}$ *is necessarily a scalar type operator as being a preserver of some equivalent inner product. Given a strongly continuous one-parameter group* $\mathcal{U} := \big[U^t:\ t\in\mathbb{R}\big]$ *of linear automorphisms of* $\mathbf{D}$, *or more generally a bounded strongly continuous one-parameter subgroup of* $\mathcal{L}(\mathbf{H})$, *there exists an equivalent* $\mathcal{U}$*-invariant inner product on* $\mathbf{H}$.

Though it is likely that Proposition 1.5 appeared already in the literature of Banach space geometry, as far we have not found proper references even after an inquiry with several colleagues. Therefore we devote the short Appendix (Section 4) to its proof relying upon Banach limits. In [10] we emphasized the natural connection of Theorem 1.2 with the Jordan theory of bounded symmetric domains during the Introduction of [10]. One of the aims of this note is to work out the first step in this direction: in Section 3, with slight modifications of the proofs in [10], we obtain the full description of the unbounded JB*-derivations of the infinite-dimensional Cartan factors [5, 6] of Types 1, 2, 3.

Recall that, by a classical representation theorem by F. Riesz [8], each Type 1 Cartan factor is isometrically isomorphic to some $\mathcal{L}(\mathbf{H}^{(1)},\mathbf{H}^{(2)})$ with suitable Hilbert spaces $\mathbf{H}^{(1)},\mathbf{H}^{(2)}$. Theorem 1.2 furnishes the complete description of their unbounded JB*-derivations as operations of the form $X\mapsto A_1X+XA_2$ with suitable possibly unbounded self-adjoint operators $A_k\in\mathcal{L}(\mathbf{H}^{(k)})$. The next theorem provides the full description of the unbounded JB*-derivations of the Cartan factors of types 2 and 3 which can be represented without loss of generality in the form $\mathcal{F}_2$ resp. $\mathcal{F}_3$ below.

Theorem 1.6. *Let* $\mathbf{H}$ *be a Hilbert space with a fixed orthonormed basis* $\mathbf{E}=\{\mathbf{e}_j:\ j\in J\}$ *and let* $X\mapsto X^{\mathrm{T}},\overline{X}$ *denote the operator transposition resp. conjugation associated to* $\mathbf{E}$.[2] *Assume* $\big[\mathbf{U}(t):\ t\in\mathbb{R}\big]$ *resp.* $\big[\mathbf{V}(t):\ t\in\mathbb{R}\big]$ *are one-parameter groups of surjective linear isometries of* $\mathcal{F}_2:=\{X\in\mathcal{L}(\mathbf{H}):\ X=X^{\mathrm{T}}\}$ *resp.* $\mathcal{F}_3:=\{Z\in\mathcal{L}(\mathbf{H}):\ Z=-Z^{\mathrm{T}}\}$ *such that all the functions*

[2] The matrices of X^{T} resp. $\overline{X}$ are the transpose resp. conjugate of the matrix of X with respect to $\mathbf{E}$: $\langle X^{\mathrm{T}}\mathbf{e}_j|\mathbf{e}_\ell\rangle=\langle X\mathbf{e}_\ell|\mathbf{e}_j\rangle$ and $\langle\overline{X}\mathbf{e}_j|\mathbf{e}_\ell\rangle=\overline{\langle X\mathbf{e}_j|\mathbf{e}_\ell\rangle}$ for all $j,\ell\in J$.

$t \mapsto \langle [\mathbf{U}(t)X]\mathbf{x}|\mathbf{y}\rangle$ resp. $t \mapsto \langle [\mathbf{V}(t)Z]\mathbf{x}|\mathbf{y}\rangle$ are continuous for all fixed $\mathbf{x},\mathbf{y}\in\mathbf{H}$ and $X\in\mathcal{F}_2$ resp. $Z\in\mathcal{F}_3$. Then

$$\begin{aligned}\mathbf{U}(t)X &= [\exp(itA)]X[\exp(it\overline{A})] \quad (X\in\mathcal{F}_2,\ t\in\mathbb{R}),\\ \mathbf{V}(t)Z &= [\exp(itB)]Z[\exp(it\overline{B})] \quad (Z\in\mathcal{F}_3,\ t\in\mathbb{R})\ .\end{aligned}$$

Thus, the possibly unbounded JB-derivations of $\mathcal{F}_3,\mathcal{F}_3$ are of the form $X\mapsto AX+X\overline{A}$ resp. $Z\mapsto BZ+Z\overline{B}$ for some possibly unbounded self-adjoint operators A,B on $\mathbf{H}$.*

Since the surjective linear isometries of a spin factor (Cartan factor of Type 4) are automatically unitary with respect to the underlying Hilbert space scalar product. and since the exceptional Cartan factors (of Type 5, 6) are finite dimensional (16 resp. 27 dimensions), Theorem 1.6 completes the description of the possibly unbounded derivations of any Cartan factor. One may intend to apply this result to get the description of the possibly unbounded JB*-derivations for all JB*-triples[3] via the Gelfand-Neimark type theorem by Friedmen-Russo [3]. This latter asserts that any JB*-triple be embedded into a suitable ℓ^∞-direct sum of Cartan factors as a weak*-dense norm-closed submanifold. We finish by raising the following two related open problems:

Problems 1.7. a) Describe the pointwise continuous (resp. weakly continuous) one-parameter groups of surjective linear isometries for any JB*-triple.

b) Describe the pointwise weak*-continuous one-parameter groups of surjective linear isometries for all JBW*-triples (JB*-triples with predual).

c) (Non-linear issue of 1.7). Describe the pointwise continuous (resp. pointwise weakly-, resp. weak*-continuous in the JBW*-case) one-parameter groups of holomorphic automorphisms of the unit ball of any JB*-triple. In particular, in the case of the unit ball of a Hilbert space, are they all of the form $\exp\left(\left[a-\langle\mathbf{x}|\mathbf{a}\rangle\mathbf{x}+iA\mathbf{x}\right]\frac{\partial}{\partial\mathbf{x}}\right)$ with suitable vector $\mathbf{a}$ and a possibly unbounded self-adjoint operator A?

2. ADJUSTED CONTINUITY OF TENSOR MAPS

Henceforth, throughout the whole paper, let $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$ be reflexive Banach spaces with duals denoted by $[\mathbf{X}^{(k)}]^*,\dots,[\mathbf{X}^{(N)}]^*$ where weak convergence along with convergence in norm entails norm convergence. That is, by assumption, for each fixed index $k=1,\dots,N$ and for any net $\left[\mathbf{x}_j:\ j\in J\right]$ in

[3] Banach spaces with holomorphically symmetric unit ball. They were introduced and axiomatized algebraically by means of a three-variable product in 1983 by Kaup [7]. For an elementary introduction see *e.g.* [6].

$\mathbf{X}^{(k)}$ we have

(2.1) $\quad \|\mathbf{x}_j-\mathbf{x}\| \to 0$ whenever $\|\mathbf{x}_j\| \to \|\mathbf{x}\|$ and $\langle \mathbf{x}_j-\mathbf{x}, \phi\rangle \to 0 \quad (\phi \in [\mathbf{X}^{(k)}]^*)$.

Notice [1] that *uniformly convex* spaces are reflexive satisfying (2.1), and a Banach space is uniformly convex if and only if its dual is uniformly smooth (and vice versa, via reflexivity).

As usually, for the natural coupling between $\mathbf{X}^{(k)}$ and $[\mathbf{X}^{(k)}]^*$ we write $\langle \mathbf{x}, \phi\rangle \big(:= \phi(\mathbf{x})\big)$. Given any family $\mathbf{x}^{(k)} \in \mathbf{X}^{(k)}, \phi^{(k)} \in [\mathbf{X}^{(k)}]^*$ $(k=1,\dots,N)$, $\mathbf{x}^{(1)} \otimes \cdots \otimes \mathbf{x}^{(N)}$ resp. $\phi^{(1)} \otimes \cdots \otimes \phi^{(N)}$ will denote the elementary N-linear functionals $\big(\psi^{(1)},\dots,\psi^{(N)}\big) \mapsto \prod_{k=1}^N \langle \mathbf{x}^{(k)}, \psi^{(k)}\rangle$ resp. $\big(\mathbf{y}^{(1)},\dots,\mathbf{y}^{(N)}\big) \mapsto \prod_{k=1}^N \langle \mathbf{y}^{(k)}, \phi^{(k)}\rangle$. Remark that

$$
\begin{aligned}
&0 \neq \mathbf{x}^{(1)} \otimes \cdots \otimes \mathbf{x}^{(N)} = \mathbf{y}^{(1)} \otimes \cdots \otimes \mathbf{y}^{(N)} \ \text{ if and only if} \\
&\mathbf{x}^{(k)} = \rho_k \mathbf{y}_k \ (k=1,\dots,N) \text{ with } \prod_{k=1}^N \rho_k = 1 \text{ for some } \rho_1,\dots,\rho_N \in \mathbb{C}.
\end{aligned} \tag{2.2}
$$

It is well-known [9] that the tensor coupling

$$
\Big\langle \mathbf{x}^{(1)} \otimes \cdots \otimes \mathbf{x}^{(N)}, \phi^{(1)} \otimes \cdots \otimes \phi^{(N)} \Big\rangle := \prod_{k=1}^N \langle \mathbf{x}^{(k)}, \phi^{(k)}\rangle
$$

admits a bounded N-linear extension to the projective tensor product of the spaces $\mathbf{X}^{(k)}$ with the injective tensor product of the dual spaces, and the latter is a closed subspace in $\mathcal{B}(\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}) := \big\{$bounded N-linear functionals $\mathbf{X}^{(1)} \times \cdots \mathbf{X}^{(N)} \to \mathbb{C}\big\}$ with its natural sup-norm. We shall speak of *weak convergence* in $\mathbf{X}^{(1)} \otimes \cdots \otimes \mathbf{X}^{(N)}$ in the sense of the tensor coupling. In particular $\mathbf{x}_j^{(1)} \otimes \cdots \otimes \mathbf{x}_j^{(N)} \longrightarrow^{\mathrm{w}} \mathbf{x}^{(1)} \otimes \cdots \otimes \mathbf{x}^{(N)}$ if $\prod_k \phi^{(k)}(\mathbf{x}_j^{(k)}) \to \prod_k \phi^{(k)}(\mathbf{x}^{(k)})$ for fixed $\phi^{(k)} \in [\mathbf{X}^{(k)}]^*$ $(k=1,\dots,N)$.

LEMMA 2.3. *Let* $\big[\mathbf{e}_j^{(k)} : \ j \in \mathcal{J}\big]$ *be nets of unit vectors in the respective spaces* $\mathbf{X}^{(k)}$ *such that*

$$
\mathbf{e}_j^{(1)} \otimes \cdots \otimes \mathbf{e}_j^{(N)} \longrightarrow^{\mathrm{w}} \mathbf{e}^{(1)} \otimes \cdots \otimes \mathbf{e}^{(N)}
$$

where $\mathbf{e}^{(k)} \in \mathbf{X}^{(k)}$ $(k=1,\dots,N)$ *are also vectors with norm* 1. *Then there are nets of constants* $\big[\kappa_j^{(k)} : j \in \mathcal{J}\big]$ $\big(k=1,\dots,N\big)$ *in* $\mathbb{T} := \{\kappa \in \mathbb{C} : |\kappa| = 1\}$ *such that*

$$
\prod_{\ell=1}^N \kappa_j^{(\ell)} = 1, \quad \big\|\kappa_j^{(k)} \mathbf{e}_j^{(k)} - \mathbf{e}^{(k)}\big\| \to 0 \qquad (k=1,\dots,N).
$$

Proof. Let $\mathbb{T}_0^N := \big\{(\kappa_1,\dots,\kappa_N) \in \mathbb{T}^N : \ \prod_k \kappa_k = 1\big\}$, and for any index $j \in \mathcal{J}$ define ε_j resp. $\big(\kappa_j^{(1)},\dots,\kappa_j^{(N)}\big)$ as the minimal value resp. a minimum location of the continuous function $\Delta_j(\kappa_1,\dots,\kappa_N) := \sum_{k=1}^N \big\|\kappa_k \mathbf{e}_j^{(k)} - \mathbf{e}^{(k)}\big\|$ on the compact domain $\mathbb{T}_0^N$. We have to show $\varepsilon_j \to 0$. As a consequence of the compactness of $\mathbb{T}$ along with the weak compactness of the closed unit ball in each $\mathbf{X}^{(k)}$, we can choose a subnet $\big[j_\nu:\ \nu\in\mathcal{N}\big]$ of $\mathcal{J}$ along with vectors $\mathbf{y}^{(k)}$ of norm ≤ 1 such that

$$\varepsilon_{j_\nu}\to\limsup\nolimits_j \varepsilon_j,\ \ \big\langle \kappa_j^{(k)}\mathbf{e}_{j_\nu}^{(k)}-\mathbf{y}^{(k)},\phi^{(k)}\big\rangle\to 0 \text{ for all } \phi^{(k)}\in[\mathbf{X}^{(k)}]^* \tag{2.4}$$

for $k=1,\dots,N$. By passing to j_ν-limits we have

$$\mathbf{y}^{(1)}\otimes\cdots\otimes\mathbf{y}^{(N)}=\mathbf{e}^{(1)}\otimes\cdots\otimes\mathbf{e}^{(N)}.$$

Since $\prod_k \big\|\mathbf{y}^{(k)}\big\|=\big\|\mathbf{y}^{(1)}\otimes\cdots\otimes\mathbf{y}^{(N)}\big\|=\big\|\mathbf{e}^{(1)}\otimes\cdots\otimes\mathbf{e}^{(N)}\big\|$ and $\big\|\mathbf{y}^{(1)}\big\|,\dots,\big\|\mathbf{y}^{(1)}\big\|\leq 1$, all the $\mathbf{y}^{(k)}$ must be unit vectors. Since also $\big\|\mathbf{e}^{(k)}\big\|=1$, in view of (2.2) here we even have $\mathbf{y}^{(k)}=\rho^{(k)}\mathbf{e}^{(k)}$ with suitable constants $\rho^{(k)}\in\mathbb{T}$. Thus, by our basic topological assumption on the equivalence of weak and norm convergence on the unit sphere of the spaces $\mathbf{X}^{(k)}$, from (2.4) it follows

$$\begin{aligned}&\kappa_{j_\nu}^{(k)}\mathbf{e}_{j_\nu}^{(k)}\to\mathbf{y}^{(k)}=\rho^{(k)}\mathbf{e}^{(k)}\quad\text{in norm},\\ &\Delta_{j_\nu}(\kappa_{j_\nu}^{(1)},\dots,\kappa_{j_\nu}^{(N)})\to\sum_{k=1}^N\big\|\mathbf{y}^{(k)}-\mathbf{e}^{(k)}\big\|=\sum_{k=1}^N\big|\rho^{(k)}-1\big|.\end{aligned}$$

By the definition of the terms $\kappa_j^{(k)}$ by means of Δ_j, for any fixed tuple $(\kappa_1,\dots,\kappa_N)\in\mathbb{T}_0^N$ we have $\sum_k\big\|\kappa_k\mathbf{e}_j^{(k)}-\mathbf{e}^{(k)}\big\|\geq\sum_k\big\|\kappa_j^{(k)}\mathbf{e}_j^{(k)}-\mathbf{e}^{(k)}\big\|$. Since, by taking a convergent subnet $\big[(\kappa_{j_{\nu_\tau}}^{(1)},\dots,\kappa_{j_{\nu_\tau}}^{(N)})\ :\ \tau\in\mathcal{T}\big]$ with limit point $(\xi_1,\dots,\xi_N)\in\mathbb{T}_0^N$, we have $\mathbf{e}_{j_{\nu_\tau}}^{(k)}=[\kappa_{j_{\nu_\tau}}^{(k)}]^{-1}\kappa_{j_{\nu_\tau}}^{(k)}\mathbf{e}_{j_{\nu_\tau}}^{(k)}\to\overline{\xi_k}\rho^{(k)}\mathbf{e}^{(k)}$. Hence, we conclude that

$$\sum_{k=1}^N\big|\kappa_k\overline{\xi_k}-1\big|=\sum_{k=1}^N\big\|\kappa_k\overline{\xi_k}\rho^{(k)}\mathbf{e}^{(k)}-\mathbf{e}^{(k)}\big\|\geq\sum_{k=1}^N\big\|\rho^{(k)}\mathbf{e}^{(k)}-\mathbf{e}^{(k)}\big\|$$

for any $(\kappa_1,\dots,\kappa_N)\in\mathbb{T}_0^N$. With the choice $\kappa_k:=\xi_k$ $(k=1,\dots,N)$ we get $\rho^{(k)}=1$ $(k=1,\dots,N)$ and $\limsup_j\varepsilon_j=\lim_\nu\Delta_{j_\nu}(\kappa_{j_\nu}^{(1)},\dots,\kappa_{j_\nu}^{(N)})=0$ which completes the proof. $\square$

PROPOSITION 2.5. *Suppose* $\mathbf{e}_t^{(k)}\in\mathbf{X}^{(k)}$ $(t\in\mathbb{R},\ k=1,\dots,N)$ *are unit vectors such that function* $t\mapsto\mathbf{e}_t^{(1)}\otimes\cdots\otimes\mathbf{e}_t^{(N)}$ *is weakly continuous. Then one can find a function* $t\mapsto(\kappa_t^{(k)},\dots,\kappa_t^{(k)})$ *from* $\mathbb{R}$ *to* $\mathbb{T}_0^N:=\big\{(\kappa_1,\dots\kappa_N)\in\mathbb{T}\ :\ \prod_k\kappa_k=1\big\}$ *such that the functions* $t\mapsto\kappa_t^{(k)}\mathbf{e}_t^{(k)}$ $(k=1,\dots,N)$ *are norm-continuous.*

Proof. An analogous argument with the σ-compctness of the real line as in ([10], proof of Lemma 2.3) shows: it suffices to see that for any $\tau\in\mathbb{R}$ there is an open interval I_τ around τ where the statement holds. Let us fix

τ arbitrary. By choosing a family $\phi^{(k)} \in [\mathbf{X}^{(k)}]^*$ $(k = 1, \dots, N)$ of functionals with $\langle \mathbf{e}_\tau^{(k)}, \phi^{(k)} \rangle = \|\phi^{(k)}\| = 1$ (guaranteed by the Hahn-Banach theorem), define the interval I_τ as the connected component of τ in the open set $\{t \in \mathbb{R} : \prod_{k=1}^N \langle \mathbf{e}_t^{(k)}, \phi^{(k)} \rangle \neq 0\}$. For the parameters $t \in I_\tau$, we define the functions $t \mapsto \kappa_t^{(k)}$ as

$$\kappa_t^{(k)} := \left|\langle \mathbf{e}_t^{(k)}, \phi^{(k)} \rangle\right| / \langle \mathbf{e}_t^{(k)}, \phi^{(k)} \rangle \quad (k < N), \qquad \kappa_t^{(N)} := \overline{\kappa_t^{(1)} \cdots \kappa_t^{(N-1)}}.$$

To establish the norm continuity of the functions $t \mapsto \kappa_t^{(k)} \phi_t^{(k)}$ (on I_τ) we consider a convergent sequence $t_n \to t_0$ in I_τ and show that

$$\left\| \kappa_{t_n}^{(k)} \mathbf{e}_{t_n}^{(k)} - \kappa_{t_0}^{(k)} \mathbf{e}_{t_0}^{(k)} \right\| \to 0 \ . \tag{2.6}$$

In view of Lemma 2.3, there are convergent sequences $[\mu_{t_n}^{(k)} : n = 1, 2, \dots]$ $(k = 1, \dots, N)$ in $\mathbb{T}$ such that $\|\mu_{t_n}^{(k)} \mathbf{e}_{t_n}^{(k)} - \mathbf{e}_{t_0}^{(k)}\| \to 0$ $(k = 1, \dots, N)$. Fix any index k. To prove (2.6) we have to see that

$$\kappa_{t_n}^{(k)} / \mu_{t_n}^{(k)} \to \kappa_{t_0}^{(k)} \ .$$

Observe that $\kappa_t^{(k)} \langle \mathbf{e}_t^{(k)}, \phi^{(k)} \rangle = \left|\langle \mathbf{e}_t^{(k)}, \phi^{(k)} \rangle\right|$ for any $t \in I_0$. Therefore

$$\begin{aligned} \frac{\kappa_{t_n}^{(k)}}{\mu_{t_n}^{(k)}} &= \frac{\kappa_{t_n}^{(k)} \langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle}{\mu_{t_n}^{(k)} \langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle} = \frac{\left|\langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle\right|}{\mu_{t_n}^{(k)} \langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle} = \\ &= \frac{\left|\mu_{t_n}^{(k)} \langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle\right|}{\mu_{t_n}^{(k)} \langle \mathbf{e}_{t_n}^{(k)}, \phi^{(k)} \rangle} \longrightarrow \frac{\left|\langle \mathbf{e}_{t_0}^{(k)}, \phi^{(k)} \rangle\right|}{\langle \mathbf{e}_{t_0}^{(k)}, \phi^{(k)} \rangle} = \kappa_{t_0}^{(k)}. \quad \square \end{aligned}$$

Corollary 2.7. *If $t \mapsto \mathbf{x}_t^{(k)}$ $(k = 1, \dots, N)$ are nowhere vanishing functions $\mathbb{R} \to \mathbf{X}^{(k)}$ with weakly continuous tensor product $t \mapsto \mathbf{x}_t^{(1)} \otimes \cdots \otimes \mathbf{x}_t^{(N)}$ whose norm $t \mapsto \prod_{k=1}^N \|\mathbf{x}_t^{(k)}\|$ is also continuous, then there are multipliers $\kappa_t^{(k)} \in \mathbb{T}$ $(t \in \mathbb{R},\ k = 1, \dots, N)$ making the vector valued functions $t \mapsto \kappa_t^{(k)} \|\mathbf{x}_t^{(k)}\|^{-1} \prod_{j=1}^N \|\mathbf{x}_t^{(j)}\|^{1/N} \mathbf{x}_t^{(k)}$ norm-continuous.*

Proof. A choice in accordance with Proposition 2.5 for the multipliers $\kappa_t^{(k)}$ to the functions $t \mapsto \mathbf{e}_t^{(k)} := \|\mathbf{x}_t^{(k)}\|^{-1} \mathbf{x}_t^{(k)}$ suits the statement. $\square$

For the next two lemmas let $\mathbf{Z}$ denote a complex topological vector space with separating topological dual (thus, for any vector $0 \neq \mathbf{z} \in \mathbf{Z}$ there is a continuous linear functional $\psi : \mathbf{Z} \to \mathbb{C}$ with $\psi(\mathbf{z}) = 1$).

Lemma 2.8. *Assume $\mathbf{z}_1(t), \mathbf{z}_2(t)$ are linearly independent vectors in $\mathbf{Z}$ for all $t \in \mathbb{R}$ and let $\rho_1, \rho_2, \mu : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ be functions making the maps $\zeta_1 \mathbf{z}_1, \zeta_2 \mathbf{z}_2, \mu(\mathbf{z}_1 + \mathbf{z}_2)$ continuous. Then the functions $\zeta_2 \mathbf{z}_1, \zeta_1 \mathbf{z}_2$ are also continuous.*

Proof. Let us fix any point $t_0 \in \mathbb{R}$. It suffices to see the continuity of ζ_1/ζ_2 at t_0. To this aim let us choose a couple ψ_1, ψ_2 of continuous linear functionals $\mathbf{Z} \to \mathbb{C}$ with $\det\left[\begin{smallmatrix}\psi_1\mathbf{z}_1(t_0) & \psi_1\mathbf{z}_2(t_0)\\ \psi_2\mathbf{z}_1(t_0) & \psi_2\mathbf{z}_2(t_0)\end{smallmatrix}\right] \neq 0$ and define $P\mathbf{z} := \left[\begin{smallmatrix}\psi_1\mathbf{z}\\ \psi_2\mathbf{z}\end{smallmatrix}\right]$ ($\mathbf{z} \in \mathbf{Z}$). This can be done by the linear independence of $\mathbf{z}_1(t_0), \mathbf{z}_2(t_0)$. Then, by the continuity of both $t \mapsto \zeta_k(t)\mathbf{z}_k(t)$ $(k = 1, 2)$, there is a neighborhood I of t_0 where $\det\bigl[P\zeta_1(t)\mathbf{z}_1(t),\ P\zeta_2(t)\mathbf{z}_2(t)\bigr] \neq 0$ $(t \in I)$ and the matrix valued function $t \mapsto L_t := \bigl[P\zeta_1(t)\mathbf{z}_1(t),\ P\zeta_2(t)\mathbf{z}_2(t)\bigr]^{-1}$ is continuous on I. The continuity of $t \mapsto \mu(t)\mathbf{z}_1(t) + \mathbf{z}_2(t)]$ entails the continuity of the function $t \mapsto L_t P\mu(t)[\mathbf{z}_1(t) + \mathbf{z}_2(t)] = \left[\begin{smallmatrix}\mu(t)/\zeta_1(t)\\ \mu(t)/\zeta_2(t)\end{smallmatrix}\right]$ on I whence the continuity of $\zeta_1/\zeta_2 = [\mu/\zeta_1]/[\mu/\zeta_2]$ at t_0 is immediate. $\square$

LEMMA 2.9. *Let $t \mapsto U_t$ be a map $\mathbb{R} \to \mathrm{GL}(\mathbf{Z})$ and let $\rho : \mathbb{R} \to \mathbb{C}\setminus\{0\}$ resp. $\lambda : \mathbb{R}^2 \to \mathbb{C} \setminus \{0\}$ be functions such that all the maps $t \mapsto \rho(t)U_t\mathbf{z}$ $(\mathbf{z} \in \mathbf{Z})$ are continuous and we have $U_{s+t} = \lambda(s,t)U_sU_t$ for all $s, t \in \mathbb{R}$. Then there exists a function $\widetilde{\rho} : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ making $\bigl[\widetilde{\rho}(t)U_t :\ t \in \mathbb{R}\bigr]$ a one-parameter group.*

Proof. By induction on n, we see that there are functions $\lambda_n : \mathbb{R}^n \to \mathbb{C}\setminus\{0\}$ with

$$U_{s_1+\cdots+s_n} = \lambda_n(s_1, \dots, s_n)U_{s_1} \cdots U_{s_n}$$

for all possible choices. We establish the *symmetry* of each λ_n by showing that the family $\{U_t :\ t \in \mathbb{R}\}$ is Abelian. Since $U_0 = U_{0+0} = \lambda(0,0)U_0^2$, necessarily $U_0 = \lambda(0,0)^{-1}\mathrm{Id}$. Hence, the identity $\lambda(t,-t)U_tU_{-t} = U_0$ implies that U_{-t} is always a multiple of U_t^{-1}. Since, for $n = 2, 3, \dots$ we have $U_{nt} = \lambda_n(t, \dots, t)U_t^n$, the set $\{U_q :\ q \in \mathbb{Q}\} = \bigcup_{m=1}^{\infty}\{U_{n/m!} :\ n \in \mathbb{Z}\}$ is Abelian as being the union of an increasing sequence of Abelian families. By the density of the rational numbers within $\mathbb{R}$, our strong continuity assumption entails that $\{\rho(t)U_t :\ t \in \mathbb{R}\}$ and hence, $\{U_t :\ t \in \mathbb{R}\}$ are Abelian.

From the relation $U_{nt} \in \mathbb{C}U_t^n$ we see also that any operator of the form ζU_t admits arbitrary m-th roots of the form $\eta U_{t/m}$. Hence, for any given $t \in \mathbb{R}$, we can construct a sequence of group homomorphisms $g_{t,m} : \{n/m! :\ n \in \mathbb{Z}\} \to \mathrm{GL}(\mathbf{Z})$ $(m = 1, 2, \dots)$ with $g_{t,m+1}$ extending $g_{t,m}$ such that $g_{t,m}(1) = U_t$ and $g_{t,m}(n/m!) \in \mathbb{C}U_{t/m!}^n = \mathbb{C}U_{nt/m!}$ for all m, n. Thus, for any $t \in \mathbb{R}$, there is a homomorphism $g_t : \mathbb{R} \to \mathrm{GL}(\mathbf{Z})$ along with a function $\widetilde{\rho}_t : \mathbb{Q} \to \mathbb{C}$ such that

$$g_t(n/m!) = g_{t,m}(n/m!) = \widetilde{\rho}_t(n/m!)U_{nt/m!} \quad (n = 0, \pm1, \pm2, \dots;\ m = 1, 2, \dots).$$

Let us fix any Hamel basis $\mathcal{H}$ in $\mathbb{R}$ and define the map $g : \mathbb{R} \to \mathrm{GL}(\mathbf{Z})$ resp. the function $\widetilde{\rho} : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ as

$$\begin{aligned} &g(q_1t_1 + \cdots + q_nt_n) := g_{t_1}(q_1) \cdots g_{t_n}(q_n),\\ &\widetilde{\rho}(t_1q_1 + \cdots + t_nq_n) := \lambda_n(q_1t_1, \dots, q_nt_n)^{-1}\widetilde{\rho}_{t_1}(q_1) \cdots \widetilde{\rho}_{t_n}(q_n) \end{aligned}$$

for $q_1,\dots,q_n \in \mathbb{Q}$ and $t_1,\dots,t_n \in \mathcal{H}$. We complete the proof by observing that g is a group homemorphism with $g(1) = U_1$ and satisfying the identities $g(q_1t_1+\cdots+q_nt_n) = \lambda_n(q_1t_1,\dots,q_nt_n)^{-1}\widetilde{\rho}(q_1t_1+\cdots+q_nt_n)U_{q_1t_1}\cdots U_{q_nt_n}$. $\square$

Proof of Proposition 1.3.

By assumption, for any tuple $\left[\mathbf{x}^{(j)}\right] \in \mathbf{X}^{(1)} \times \cdots \mathbf{X}^{(N)}$ the map $t \mapsto U_{1,t}\mathbf{x}^{(1)} \otimes \cdots \otimes U_{N,t}\mathbf{x}^{(N)}$ is weakly continuous. An application of Corollary 2.7 establishes the existence of functions $\rho_{k,[\mathbf{x}^{(j)}]} : \mathbb{R} \to \mathbb{C}\setminus\{0\}$ such that all the maps $t \mapsto \rho_{k,[\mathbf{x}^{(k)}]}(t)U_{k,t}x^{(k)}(t)$ are continuous. Let us fix any family $0 \neq \mathbf{x}_0^{(j)} \in \mathbf{X}^{(k)}$ $(j = 1,\dots,N)$ and $\rho_k(t) := \rho_{k,[\mathbf{x}_0^{(j)}]}(t)$. To complete the proof that each $\left[\rho_k(t)U_{k,t} :\ t \in \mathbb{R}\right]$ is a strongly continuous family, it suffices to see that, given any index ℓ with vector $\mathbf{x}^{(\ell)} \in \mathbf{X}^{(\ell)}$ which is linearly independent of $\mathbf{x}_0^{(\ell)}$, the function $t \mapsto U_{\ell,t}\mathbf{x}^{(\ell)}$ is continuous. This fact follows immediately from Lemma 2.8 applied with $\mathbf{Z} := \mathbf{X}^{(\ell)}$, $\mathbf{z}_1(t) := U_{\ell,t}\mathbf{x}_0^{(\ell)}(t)$, $\mathbf{z}_z(t) := U_{\ell,t}\mathbf{x}^{(\ell)}(t)$, $\zeta_1(t) := \rho_\ell(t)$, $\zeta_2(t) := \rho_{\ell,[\mathbf{y}^{(k)}]}(t)$, $\mu(t) := \rho_{\ell,[\mathbf{u}^{(k)}]}(t)$ where $\mathbf{y}^{(k)} := (1-\delta_{k\ell})\mathbf{x}_0^{(k)} + \delta_{k\ell}\mathbf{y}^{(\ell)}$ and $\mathbf{u}^{(k)} := (1-\delta_{k\ell})\mathbf{x}_0^{(k)} + \delta_{k\ell}\mathbf{y}^{(\ell)}$ in terms of the Kronecker-delta $\delta_{k\ell} := [1 \text{ for } k = \ell \text{ and } 0 \text{ else}]$.

It is a well-known consequence of (2.2) and the Schur lemma that the factorization of a non-vanishing tensor products of linear operators is unique up to constant factors with product one. Hence, the semigroup property $\mathbf{U}(s+t) = \mathbf{U}(s)\mathbf{U}(t)$ entails the existence of functions $\lambda_k : \mathbb{R}^2 \to \mathbb{C}\setminus\{0\}$ such that $U_{k,s+t} = \lambda_k(s,t)U_{k,t}U_{s,t}$ $s,t,\in\mathbb{R};\ k{=}1,\dots,N)$. Given any index $\ell\in\{1,\dots,N\}$, by applying Lemma 2.9 again with $\mathbf{Z} := \mathbf{X}^{(\ell)}$, the operators $U_t := U_{\ell,t}$ and the function $\rho := \rho_\ell$ constructed above, we obtain the existence of a function $\widetilde{\rho}_\ell : \mathbb{R}\to\mathbb{C}$ making the family $\left[\widetilde{\rho}_\ell U_{\ell,t} : t\in\mathbb{R}\right]$ a one-parameter group. $\square$

3. CARTAN FACTORS OF TYPES 2,3; PROOF OF THEOREM 1.6

Throughout this section let $\mathbf{H}$ be an arbitrarily fixed complex Hilbert space. We shall write $\mathbf{h}^* := [\mathbf{H} \ni \mathbf{x} \mapsto \langle\mathbf{x}|\mathbf{h}\rangle]$ for the dual functionals, $\mathrm{Ball}(\mathbf{H}) := \{\mathbf{h} \in \mathbf{H} :\ \|\mathbf{h}\| < 1\}$ for the unit ball, $\partial\mathrm{Ball}(\mathbf{H}) := \{\mathbf{h} \in \mathbf{H} : \|\mathbf{h}\| = 1\}$ for the unit sphere and $\mathbf{g}^* \wedge \mathbf{h}^* := \mathbf{g}^* \otimes \mathbf{h}^* - \mathbf{h}^* \otimes \mathbf{g}^*$ for the basic antisymmetric functionals, respectively, where $\mathbf{g}^* \otimes \mathbf{h}^* := [(\mathbf{x},\mathbf{y}) \mapsto \mathbf{g}^*(\mathbf{x})\mathbf{h}^*(\mathbf{y})]$. The spaces $\mathcal{F}_2, \mathcal{F}_3$ in Theorem 1.6 are isomorphic copies of the Banach spaces $\mathfrak{F}_2, \mathfrak{F}_3$ of all symmetric resp. antisymmetric continuous bilinear functionals $\mathbf{H} \times \mathbf{H} \to \mathbb{C}$. Recall [11] that in both cases $k = 2,3$, the surjective linear isometries of $\mathfrak{F}_k$ are of the form $[U \otimes U]\Phi = \left[(\mathbf{x},\mathbf{y}) \mapsto \Phi(U\mathbf{x}, U\mathbf{y})\right]$ $(\Phi \in \mathfrak{F}_k)$. Therefore the conclusion that, given any of the indices $k \in \{2,3\}$ with a one-parameter group $\big[\mathbf{U}(t) : t \in \mathbb{R}\big]$ of the form $\mathbf{U}(t) = U_t \otimes U_t$ such that all the functions $t \mapsto \Phi(U_t\mathbf{x}, U_t\mathbf{y})$ ($\Phi \in \mathfrak{F}_k;\ \mathbf{x}, \mathbf{x} \in \mathbf{H}$) are continuous, there are functions $\kappa, \widetilde{\kappa} : \mathbb{R} \to \mathbb{T}$ making $\big[\kappa(t)U_t : t \in \mathbb{R}\big]$ resp. $\big[\widetilde{\kappa}(t)U_t : t \in \mathbb{R}\big]$ a strongly continuous family resp. a one parameter group is immediate from Propositions 3.3, 3.5 and Corollary 3.4 below even with $\kappa, \widetilde{\kappa}$ admitting only the values ± 1. Since the results of ([10], Sections 4, 5) concern only a single underlying Hilbert space, hence, we can conclude Theorem 1.6 (without refering to ([10], Section 6).

Lemma 3.1 ([10], Lemma 2.3). *Suppose* $\mathbf{F} : \mathbb{R} \to \mathbb{P}(\mathbf{H}) := \{\mathbb{T}\mathbf{g} : \langle \mathbf{g}|\mathbf{g}\rangle = 1\}$ *is a continuous map with respect to the distance* $\operatorname{dist}(\mathbb{T}\mathbf{g}, \mathbb{T}\mathbf{h}) := \min_{|\kappa|=|\lambda|=1} \big|\kappa\mathbf{g} - \lambda\mathbf{h}\big|$. *Then* $\mathbf{F}(t) = \mathbb{T}\mathbf{h}_t$ ($t \in \mathbb{R}$) *for some continuous function* $t \mapsto \mathbf{h}_t \in \partial\mathrm{Ball}(\mathbf{H})$.

Lemma 3.2. *Let* $t \mapsto \mathbf{e}_t$ *be a function* $\mathbb{R} \to \partial\mathrm{Ball}(\mathbf{H})$. *If for any convergent net* $t_\nu \to t$ *in* $\mathbb{R}$ *there exists a constant* $\sigma_* \in \{-1, 1\}$ *along with a subnet* t_{ν_α} *such that* $\mathbf{e}_{t_{\nu_\alpha}} \to \sigma_*\mathbf{e}_t$ *then there also exists* $\sigma : \mathbb{R} \to \{-1, 1\}$ *such that* $t \mapsto \sigma(t)\mathbf{e}_t$ *is continuous.*

Proof. We can apply Lemma 3.1 as follows. Observe that $t \mapsto \mathbb{T}\mathbf{e}_t$ is necessarily continuous. Indeed, if $t_\nu \to t$ then $\liminf_\nu \operatorname{dist}(\mathbb{T}\mathbf{e}_{t_\nu}, \mathbb{T}\mathbf{e}_t) \le \liminf_\nu \min_{\sigma_*=\pm 1} \big|\mathbf{e}_{t_\nu} - \sigma_*\mathbf{e}_t\big| = 0$. Thus, $\mathbb{T}\mathbf{e}_t = \mathbb{T}\mathbf{h}_t$ ($t \in \mathbb{R}$) for some continuous function $t \mapsto \mathbf{h}_t \in \partial\mathrm{Ball}(\mathbf{H})$, that is $t \mapsto \kappa(t)\mathbf{e}_t$ is continuous for a suitable function $\kappa : \mathbb{R} \to \mathbb{T}$. For any point $t \in \mathbb{R}$, we can choose an open interval I_t around it such that $\big\langle \kappa(s)\mathbf{e}_s \big| \kappa(t)\mathbf{e}_t\big\rangle > 0$ for all $s \in I_t$. Each function $\sigma_t(s) := \operatorname{sign}\mathrm{Re}\big(\kappa(s)\overline{|\kappa(t)}\big)$ is continuous on the the interval I_t. By the σ-compactness of $\mathbb{R}$, we can find a sequence $\cdots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \cdots$ such that each interval $[\tau_{n-1}, \tau_n]$ is included in some of member of the family $\{I_t : t \in \mathbb{R}\}$, say $[\tau_{n-1}, \tau_n] \subset I_{t_n}$ ($n = 0, \pm 1, \pm 2, \ldots$). Then we obtain a function σ suiting the requirements of the lemma by letting $\sigma(t) := \sigma_{t_0}(t)$ for $t \in [\tau_{-1}, \tau_0]$, and then, recursively for $k = 1, 2, \ldots,$ $\sigma(t) := \sigma(\tau_{-k})\sigma_{t_{-k}}(\tau_{-k})\sigma_{t_{-k}}(t)$ for $t \in [\tau_{-k-1}, \tau_{-k}]$ and $\sigma(t) := \sigma(\tau_{k-1})\sigma_{t_k}(\tau_{k-1})\sigma_{t_k}(t)$ for $t \in [\tau_{k-1}, \tau_k]$, respectively. $\square$

Proposition 3.3. *If* $t \mapsto \phi(\mathbf{e}_t, \mathbf{e}_t)$ *with unit vectors* $\mathbf{e}_t \in \mathbf{H}$ *is continuous for all* $\Phi \in \mathfrak{F}_2$ *then there is a function* $\sigma : \mathbb{R} \to \{-1, 1\}$ *such that* $t \mapsto \sigma(t)\mathbf{e}_t$ *is continuous.*

Proof. Given any vector $\mathbf{h} \in \mathbf{H}$, with the functional $\Phi_\mathbf{h}(\mathbf{x}, \mathbf{y}) := \mathbf{h}^* \otimes \mathbf{h}^* (\in \mathfrak{F}_2)$ we see that the function $t \mapsto \big\langle \mathbf{e}_t \big| \mathbf{h}\big\rangle^2$ is continuous. Consider a convergent net $t_\nu \to t \in \mathbb{R}$ and let $t_{\nu_\alpha} (\to t)$ be a *universal subnet* [4] for it. Since the range $\{\mathbf{e}_t : t \in \mathbb{R}\}$ is contained in the weakly compact closed unit ball $\overline{\mathrm{Ball}}(\mathbf{H})$, for some vector $\mathbf{g} \in \overline{\mathrm{Ball}}(\mathbf{H})$ we have

$$\langle \mathbf{e}_{t_{\nu_\alpha}} | \mathbf{h} \rangle \to \langle \mathbf{g} | \mathbf{h} \rangle, \quad \langle \mathbf{g} | \mathbf{h} \rangle^2 = \langle \mathbf{e}_t | \mathbf{h} \rangle^2 \qquad (\mathbf{h} \in \mathbf{H}).$$

Since the mapping $\mathbf{h} \to \langle \mathbf{g} | \mathbf{h} \rangle$ is conjugate-linear, it follows that $\mathbf{g} \in \{\pm \mathbf{e}_t\}$. That is, for some $\sigma_* \in \{\pm 1\}$ we have $\sigma_* \mathbf{e}_t = \text{weak} \lim_{\nu_\alpha} \mathbf{e}_{t_{\nu_\alpha}}$. It is well-known that weak convergence entails norm convergence for nets of unit vectors in Hilbert spaces. Therefore we even have $\sigma_* \mathbf{e}_t = \text{norm} \lim_{\nu_\alpha} \mathbf{e}_{t_{\nu_\alpha}}$ whence Lemma 3.2 applies. □

COROLLARY 3.4. *Let $t \mapsto [\mathbf{e}_t, \mathbf{f}_t]$ be a map $\mathbb{R} \to \{\text{orthonormed 2-frames}\}$. If the map $t \mapsto \Phi(\xi \mathbf{e}_t + \eta \mathbf{f}_t, \xi \mathbf{e}_t + \eta \mathbf{f}_t)$ is continuous for all $\Phi \in \mathfrak{F}_2$ and $\xi, \eta \in \{0, \pm 1\}$ then there is a function $\sigma : \mathbb{R} \to \{-1, 1\}$ such that $t \mapsto \sigma(t)[\mathbf{e}_t, \mathbf{f}_t]$ is continuous.*

Proof. By Proposition 3.3, for some $\{-1,1\}$-valued functions $t \mapsto \kappa(t), t \mapsto \lambda(t), t \mapsto \mu(t)$, the functions $t \mapsto \lambda(t)\mathbf{e}_t, t \mapsto \kappa(t)\mathbf{f}_t$, $t \mapsto \mu(t)2^{-1/2}[\mathbf{e}_t + \mathbf{f}_t]$ are continuous. Therefore also $t \mapsto \langle \lambda(t) | \mathbf{e}_t \rangle = \lambda(t)\mu(t)$ and similarly $t \mapsto \kappa(t)\mu(u)$ are continuous and necessarily constant with value 1 or -1. Hence, the statement is immediate. □

PROPOSITION 3.5. *Suppose $t \mapsto [\mathbf{e}_t, \mathbf{f}_t, \mathbf{g}_t]$ is a function $\mathbb{R} \to \{\text{orthonormed 3-frames}\}$ such that the functions $t \mapsto \phi(\mathbf{e}_t, \mathbf{f}_t)$, $t \mapsto \phi(\mathbf{e}_t, \mathbf{g}_t)$, $t \mapsto \phi(\mathbf{f}_t, \mathbf{g}_t)$ are continuous for all $\Phi \in \mathfrak{F}_3$. Then there exists a function $\sigma : \mathbb{R} \to \{-1, 1\}$ such that $t \mapsto \sigma(t)[\mathbf{e}_t, \mathbf{f}_t, \mathbf{g}_t]$ is continuous.*

Proof. Consider a convergent net $t_\nu \to t \in \mathbb{R}$ and let $t_{\nu_\alpha} (\to t)$ be again a universal subnet. By the weak compactness of $\overline{\mathrm{Ball}}(\mathbf{H})$, for some vectors $\mathbf{e}_*, \mathbf{f}_* \in \overline{\mathrm{Ball}}(\mathbf{H})$ we have $\langle \mathbf{e}_{t_{\nu_\alpha}} | \mathbf{h} \rangle \to \langle \mathbf{e}_* | \mathbf{h} \rangle$, $\langle \mathbf{f}_{t_{\nu_\alpha}} | \mathbf{h} \rangle \to \langle \mathbf{f}_* | \mathbf{h} \rangle$, $\langle \mathbf{g}_{t_{\nu_\alpha}} | \mathbf{h} \rangle \to \langle \mathbf{g}_* | \mathbf{h} \rangle$ for any fixed vector $\mathbf{h} \in \mathbf{H}$. Given any couple of vectors $\mathbf{x}, \mathbf{y} \in \mathbf{H}$, with the functional $\phi_{\mathbf{x},\mathbf{y}} := \mathbf{x}^* \wedge \mathbf{y}^* (\in F_3)$ we see that the function $\tau \mapsto \langle \mathbf{e}_\tau | \mathbf{x} \rangle \langle \mathbf{f}_\tau | \mathbf{y} \rangle - \langle \mathbf{f}_\tau | \mathbf{x} \rangle \langle \mathbf{e}_\tau | \mathbf{y} \rangle$ is continuous. Hence, we conclude that

$$\langle \mathbf{e}_\tau | \mathbf{x} \rangle \langle \mathbf{f}_t | \mathbf{y} \rangle - \langle \mathbf{f}_t | \mathbf{x} \rangle \langle \mathbf{e}_t | \mathbf{y} \rangle = \langle \mathbf{e}_* | \mathbf{x} \rangle \langle \mathbf{f}_* | \mathbf{y} \rangle - \langle \mathbf{f}_* | \mathbf{x} \rangle \langle \mathbf{e}_* | \mathbf{y} \rangle \qquad (\mathbf{x}, \mathbf{y} \in \mathbf{H}).$$

That is we have $\mathbf{e}_t^* \wedge \mathbf{f}_t^* = \mathbf{e}_*^* \wedge \mathbf{f}_*^*$. It is well-known from classical linear algebra that then for some constants $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22} \in \mathbb{C}$ we have

$$\mathbf{e}_* = \lambda_{11} \mathbf{e}_t + \lambda_{12} \mathbf{f}_t, \quad \mathbf{f}_* = \lambda_{21} \mathbf{e}_t + \lambda_{22} \mathbf{f}_t, \quad \det[\lambda_{k,\ell}] = 1.$$

In particular $\mathbf{e}_* \in \mathbb{C}\mathbf{e}_t + \mathbb{C}\mathbf{f}_t$. Similarly (with $\mathbf{g}_{t_{\nu_\alpha}}$ in place of $\mathbf{f}_{t_{\nu_\alpha}}$), also $\mathbf{e}_* \in \mathbb{C}\mathbf{e}_t + \mathbb{C}\mathbf{g}_t$ and therefore $\mathbf{e}_* \in \mathbb{C}\mathbf{e}_t$ that is $\mathbf{e}_* = \lambda \mathbf{e}_t$ with $|\lambda| \le 1$. Analogously $\mathbf{f}_* = \mu \mathbf{f}_t$ and $\mathbf{g}_* = \kappa \mathbf{g}_t$ with $|\mu|, |\kappa| \le 1$ as well. Since $\mathbf{e}_t^* \wedge \mathbf{f}_t^* = \mathbf{e}_*^* \wedge \mathbf{f}_*^*$, it follows $\lambda\mu = 1$. Similarly $\lambda\kappa = \mu\kappa = 1$. Therefore $\kappa = \lambda = \mu \in \{-1, 1\}$. By writing σ_t for the common value of κ, λ, μ, we get $\lim_{\nu_\alpha} \mathbf{e}_{t_{\nu_\alpha}} = \sigma_t \mathbf{e}_t$. According to Lemma 1, for suitable functions $t \mapsto \lambda(t), t \mapsto \mu(t), t \mapsto \kappa(t)$ ranging in $\{-1, 1\}$

the functions $t \mapsto \lambda(t)\mathbf{e}_t, t \mapsto \mu(t)\mathbf{f}_t, t \mapsto \kappa(t)\mathbf{g}_t$ are continuous. In particular also both $t \mapsto [\lambda(t)\mathbf{e}_t]^* \wedge [\lambda(t)\mathbf{f}_t]^*$ and $t \mapsto \mathbf{e}_t^* \wedge \mathbf{f}_t^*$ are continuous entailing the continuity and hence, the constant being of the $\{-1,1\}$-valued function $t \mapsto \lambda(t)\mu(t)$. $t \mapsto [\lambda(t)\mathbf{e}_t]^* \wedge [\lambda(t)\mathbf{f}_t]^*$. Thus, $\mu(t) \equiv \lambda(t)$ or $\mu(t) \equiv -\lambda(t)$. Similarly $\kappa(t) \equiv \lambda(t)$ or $\kappa(t) \equiv -\lambda(t)$. In any case $t \mapsto \lambda(t)[\mathbf{e}_t, \mathbf{f}_t, \mathbf{g}_t]$ must be continuous. $\square$

APPENDIX: PROOF OF PROPOSITION 1.5

Let $\mathbf{U} := \left[U^t : \ t \in \mathbb{R}\right]$ be a strongly continuous one-parameter subgroup of $\mathrm{GL}(\mathbf{H})$ with

$$M^{-1}\|\mathbf{x}\| \leq \|U^t\mathbf{x}\| \leq M\|\mathbf{x}\| \qquad (\mathbf{x} \in \mathbf{H},\ t \in \mathbb{R}) \ .$$

Consider a Banach limit $L_{R\to\infty}$ on the space $\mathcal{C}_{\mathrm{b}}[0,\infty)$ of all bounded continuous functions $[0,\infty)\to\mathbb{C}$. Thus, $L_{R\to\infty}$ is a linear functional $\mathcal{C}_{\mathrm{b}}[0,\infty)\to\mathbb{C}$ such that

$$\liminf_{R\to\infty} f(R) \leq L_{R\to\infty} f(R) \leq \limsup_{R\to\infty} f(R) \quad \text{whenever} \quad \mathrm{range}(f) \subset \mathbb{R}. \tag{4.1}$$

Since all the functions $t \mapsto U^t\mathbf{x}$ are continuous and bounded, the operation

$$\langle\langle \mathbf{x}|\mathbf{y}\rangle\rangle := L_{R\to\infty}\left(\frac{1}{2R}\int_{-R}^{R} \langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt\right)$$

is well-defined for every couple of vectors $\mathbf{x},\mathbf{y} \in \mathbf{H}$. The sesquilinearity of the product $\langle .|.\rangle$ along with the inequalities (4.1) entail immediately that $\langle\langle .|.\rangle\rangle$ is also a scalar product on $\mathbf{H}$ with

$$M^{-2}\langle \mathbf{x}|\mathbf{x}\rangle \leq \langle\langle \mathbf{x}|\mathbf{x}\rangle\rangle \leq M^2\langle \mathbf{x}|\mathbf{x}\rangle \qquad (\mathbf{x} \in \mathbf{H}) \ .$$

We complete the proof by showing the $\mathbf{U}$-invariance of the new scalar product as follows. Given any parameter $s \in \mathbb{R}$, we have $\left|\int_I \langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt\right| \leq M^2 s\|\mathbf{x}\|^2$ whenever I is an interval of length $|s|$. Therefore

$$\begin{aligned}
\langle\langle U^s\mathbf{x}|U^s\mathbf{y}\rangle\rangle &= L_{R\to\infty}\Big(\tfrac{1}{2R}\textstyle\int_{-R}^{R} \langle U^{t+s}\mathbf{x}|U^{t+s}\mathbf{y}\rangle \ dt\Big) = \\
&= L_{R\to\infty}\Big(\tfrac{1}{2R}\textstyle\int_{-R+s}^{R+s} \langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt\Big) = \\
&= L_{R\to\infty}\Big(\tfrac{1}{2R}\Big[\textstyle\int_{-R}^{R} + \int_{-R+s}^{-R} + \int_{R}^{R+s}\Big]\langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt\Big) = \\
&= L_{R\to\infty}\Big(\tfrac{1}{2R}\textstyle\int_{-R+s}^{R+s} \langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt + O(R^{-1})\Big) = \\
&= L_{R\to\infty}\Big(\tfrac{1}{2R}\textstyle\int_{-R+s}^{R+s} \langle U^t\mathbf{x}|U^t\mathbf{y}\rangle \ dt\Big) = \langle\langle \mathbf{x}|\mathbf{y}\rangle\rangle \ .
\end{aligned}$$

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