We consider a second-order differential inclusion and we obtain sufficient conditions for \( h \)-local controllability along a reference trajectory.

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1. INTRODUCTION

In this paper we are concerned with the following second-order differential inclusion

\[
(1.1) \quad x'' \in Ax + F(t,x), \quad x(0) \in X_0, \quad x'(0) \in X_1,
\]

where \( F : [0,T] \times X \to \mathcal{P}(X) \) is a set valued map, \( A \) is the infinitesimal generator of a strongly continuous cosine family of operators \( \{C(t); \ t \in \mathbb{R}\} \) on a separable Banach space \( X \) and \( X_0, X_1 \subset X \) are closed sets. Let \( S_F \) be the set of all mild solutions of (1.1) and let \( R_F(T) \) be the reachable set of (1.1). For a mild solution \( z(.) \in S_F \) and for a locally Lipschitz function \( h : X \to X \) we say that the differential inclusion (1.1) is \( h \)-locally controllable around \( z(.) \) if \( h(z(T)) \in \text{int}(h(R_F(T))) \). In particular, if \( h \) is the identity mapping the above definitions reduces to the usual concept of local controllability of systems around a solution.

The aim of the present paper is to obtain a sufficient condition for \( h \)-local controllability of inclusion (1.1) when \( X \) is finite dimensional. This result is derived using a technique developed by Tuan for differential inclusions ([15]). More exactly, we show that inclusion (1.1) is \( h \)-locally controllable around the mild solution \( z(.) \) if a certain linearized inclusion is \( \lambda \)-locally controllable around the null solution for every \( \lambda \in \partial h(z(T)) \), where \( \partial h(.) \) denotes Clarke’s generalized Jacobian of the locally Lipschitz function \( h \). The key tools in the proof of our result are a continuous version of Filippov’s theorem for mild solutions of problem (1.1) obtained in [5] and a certain generalization of the classical open mapping principle in [16].
Our result may be interpreted as an extension of the controllability results in [10] to $h$-controllability.

We note that existence results and qualitative properties of the mild solutions of problem (1.1) may be found in [2–10] etc.

The paper is organized as follows: in Section 2 we present some preliminary results to be used in the sequel and in Section 3 we present our main results.

2. PRELIMINARIES

Let denote by $I$ the interval $[0, T]$ and let $X$ be a real separable Banach space with the norm $||.||$ and with the corresponding metric $d(.,.)$. Denote by $\mathcal{L}(I)$ the $\sigma$-algebra of all Lebesgue measurable subsets of $I$, by $\mathcal{P}(X)$ the family of all nonempty subsets of $X$ and by $\mathcal{B}(X)$ the family of all Borel subsets of $X$. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $||x(.)||_C = \sup_{t \in I} ||x(t)||$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $||x(.)||_1 = \int_I ||x(t)||dt$.

We recall that a family $\{C(t); t \in \mathbb{R}\}$ of bounded linear operators from $X$ into $X$ is a strongly continuous cosine family if the following conditions are satisfied

(i) $C(0) = Id$, where $Id$ is the identity operator in $X$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s) \ \forall t, s \in \mathbb{R},$

(iii) the map $t \to C(t)x$ is strongly continuous $\forall x \in X$.

The strongly continuous sine family $\{S(t); t \in \mathbb{R}\}$ associated to a strongly continuous cosine family $\{C(t); t \in \mathbb{R}\}$ is defined by

$$S(t)y := \int_0^t C(s)yds, \quad y \in X, t \in \mathbb{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t); t \in \mathbb{R}\}$ is defined by

$$Ay = (\frac{d^2}{dt^2})C(t)y|_{t=0}.$$

For more details on strongly continuous cosine and sine family of operators we refer to [12, 14].
In what follows $A$ is infinitesimal generator of a cosine family $\{C(t); \ t \in \mathbb{R}\}$ and $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values, which define the following Cauchy problem associated to a second-order differential inclusion

\begin{align}
(2.1) \quad x''(t) & = Ax(t) + F(t, x(t)), \quad x(0) = x_0, \quad x'(0) = x_1. \\
\end{align}

A continuous mapping $x(.) \in C(I, X)$ is called a \textit{mild solution} of problem (2.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that:

\begin{align}
(2.2) \quad f(t) & \in F(t, x(t)) \quad a.e. (I), \\
(2.3) \quad x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-u)f(u)du \quad \forall t \in I,
\end{align}

i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(.,x(\cdot))$ and $x(.)$ is the mild solution of the Cauchy problem

\begin{align}
(2.4) \quad x''(t) & = Ax(t) + f(t), \quad x(0) = x_0, \quad x'(0) = x_1.
\end{align}

We shall call $(x(\cdot), f(\cdot))$ a \textit{trajectory-selection pair} of (2.1) if $f(.)$ verifies (2.2) and $x(.)$ is a mild solution of (2.4).

**Hypothesis 2.1.** i) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $L(.) \in L^1(I, \mathbb{R}_+)$ such that, for any $t \in I$, $F(t, .)$ is $L(t)$-Lipschitz in the sense that

\[ d_H(F(t, x_1), F(t, x_2)) \leq L(t)||x_1 - x_2|| \quad \forall x_1, x_2 \in X. \]

**Hypothesis 2.2.** Let $S$ be a separable metric space, $X_0, X_1 \subset X$ are closed sets, $a_0(.) : S \to X_0$, $a_1(.) : S \to X_1$ and $c(.) : S \to (0, \infty)$ are given continuous mappings.

The continuous mappings $g(.) : S \to L^1(I, X)$, $y(.) : S \to C(I, X)$ are given such that

\[ (y(s))''(t) = Ay(s)(t) + g(s)(t), \quad y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1. \]

There exists a continuous function $p(.) : S \to L^1(I, \mathbb{R}_+)$ such that

\[ d(g(s)(t), F(t, y(s)(t)))) \leq p(s)(t) \quad a.e. (I), \quad \forall s \in S. \]

**Theorem 2.3 ([5]).** Assume that Hypotheses 2.1 and 2.2 are satisfied.

Then there exist $M > 0$ and the continuous functions $x(.) : S \to L^1(I, X)$, $h(.) : S \to C(I, X)$ such that for any $s \in S$ $(x(s)(.), h(s)(.))$ is a trajectory-selection of (1.1) satisfying for any $(t, s) \in I \times S$

\[ x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s), \]
(2.5) \[ ||x(s)(t) - y(s)(t)|| \leq M[c(s) + ||a_0(s) - y(s)(0)|| + \int_0^t p(s)(u)du]. \]

In what follows we assume that \( X = \mathbb{R}^n \).

A closed convex cone \( C \subset \mathbb{R}^n \) is said to be regular tangent cone to the set \( X \) at \( x \in X \) ([13]) if there exists continuous mappings \( q_\lambda : C \cap B \to \mathbb{R}^n \), \( \forall \lambda > 0 \) satisfying
\[
\lim_{\lambda \to 0^+} \max_{v \in C \cap B} ||q_\lambda(v)|| = 0, \\
x + \lambda v + q_\lambda(v) \in X \quad \forall \lambda > 0, v \in C \cap B.
\]

From the multitude of the intrinsic tangent cones in the literature (e.g. [1]) the contingent, the quasitangent and Clarke’s tangent cones, defined, respectively, by
\[
K_xX = \{v \in \mathbb{R}^n; \exists s_m \to 0^+, x_m \in X : \frac{x_m - x}{s_m} \to v\} \\
Q_xX = \{v \in \mathbb{R}^n; \forall s_m \to 0^+, \exists x_m \in X : \frac{x_m - x}{s_m} \to v\} \\
C_xX = \{v \in \mathbb{R}^n; \forall (x_m, s_m) \to (x, 0^+), x_m \in X, \exists y_m \in X : \frac{y_m - x_m}{s_m} \to v\}
\]
seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings. We recall that, in contrast with \( K_xX, Q_xX \), the cone \( C_xX \) is convex and one has \( C_xX \subset Q_xX \subset K_xX \).

The results in the next section will be expressed, in the case when the mapping \( g(.) : X \subset \mathbb{R}^n \to \mathbb{R}^m \) is locally Lipschitz at \( x \), in terms of the Clarke generalized Jacobian, defined by ([11])
\[
\partial g(x) = \text{co}\{\lim_{i \to \infty} g'(x_i); \quad x_i \to x, \quad x_i \in X \setminus \Omega_g\},
\]
where \( \Omega_g \) is the set of points at which \( g \) is not differentiable.

Corresponding to each type of tangent cone, say \( \tau_xX \) one may introduce (e.g. [1]) a set-valued directional derivative of a multifunction \( G(.) : X \subset \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n) \) (in particular of a single-valued mapping) at a point \( (x, y) \in \text{graph}(G) \) as follows
\[
\tau_yG(x; v) = \{w \in \mathbb{R}^n; (v, w) \in \tau_{(x, y)}\text{graph}(G)\}, \quad v \in \tau_xX.
\]

We recall that a set-valued map, \( A(.) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n) \) is said to be a convex (respectively, closed convex) process if \( \text{graph}(A(.)) \subset \mathbb{R}^n \times \mathbb{R}^n \) is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

**Hypothesis 2.4.** i) **Hypothesis 2.1 is satisfied and** \( X_0, X_1 \subset \mathbb{R}^n \) **are closed sets.**
ii) \((z(.), f(.)) \in C(I, \mathbb{R}^n) \times L^1(I, \mathbb{R}^n)\) is a trajectory-selection pair of (1.1) and a family \(P(t, .) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n), t \in I\) of convex processes satisfying the condition
\[
P(t, u) \subset Q_{f(t)}F(t, .)(z(t); u) \quad \forall u \in \text{dom}(P(t, .)), \text{ a.e. } t \in I
\]
is assumed to be given and defines the variational inclusion
\[
v'' \in Av + P(t, v).
\]

**Remark 2.5.** We note that for any set-valued map \(F(., .)\), one may find an infinite number of families of convex process \(P(t, .), t \in I\), satisfying condition (2.6); in fact any family of closed convex subcones of the quasitangent cones, \(\mathcal{P}(t) \subset Q_{z(t), f(t)}\text{graph}(F(t, .))\), defines the family of closed convex process
\[
P(t, u) = \{v \in \mathbb{R}^n; (u, v) \in \mathcal{P}(t)\}, \quad u, v \in \mathbb{R}^n, t \in I
\]
that satisfy condition (2.6). One is tempted, of course, to take as an "intrinsic" family of such closed convex process, for example Clarke’s convex-valued directional derivatives \(C_{f(t)}F(t, .)(z(t); .)\).

We recall (e.g. [1]) that, since \(F(t, .)\) is assumed to be Lipschitz a.e. on \(I\), the quasitangent directional derivative is given by
\[
Q_{f(t)}F(t, .)((z(t); u)) = \{w \in \mathbb{R}^n; \lim_{\theta \to 0^+} \frac{1}{\theta} d(f(t) + \theta w, F(t, z(t) + \theta u)) = 0\}.
\]

In what follows \(B\) or \(B_{\mathbb{R}^n}\) denotes the closed unit ball in \(\mathbb{R}^n\) and \(0_n\) denotes the null element in \(\mathbb{R}^n\).

Consider \(h : \mathbb{R}^n \to \mathbb{R}^m\) an arbitrary given function.

**Definition 2.6.** Inclusion (1.1) is said to be \(h\)-locally controllable around \(z(.)) if \(h(z(T)) \in \text{int}(h(R_F(T)))\).

Inclusion (1.1) is said to be locally controllable around the solution \(z(.)\) if \(z(T) \in \text{int}(R_F(T))\).

Finally, a key tool in the proof of our results is the following generalization of the classical open mapping principle due to Warga ([16]).

For \(k \in \mathbb{N}\) we define
\[
\Sigma_k := \{\gamma = (\gamma_1, ..., \gamma_k); \sum_{i=1}^k \gamma_i \leq 1, \quad \gamma_i \geq 0, \ i = 1, 2, ..., k\}.
\]

**Lemma 2.7 ([16]).** Let \(\delta \leq 1\), let \(g(.) : \mathbb{R}^n \to \mathbb{R}^m\) be a mapping that is \(C^1\) in a neighborhood of \(0_n\) containing \(\delta B_{\mathbb{R}^n}\). Assume that there exists \(\beta > 0\) such that for every \(\theta \in \delta \Sigma_n, \beta B_{\mathbb{R}^m} \subset g'(\theta) \Sigma_n\). Then, for any continuous mapping \(\psi : \delta \Sigma_n \to \mathbb{R}^m\) that satisfies \(\sup_{\theta \in \delta \Sigma_n} ||g(\theta) - \psi(\theta)|| \leq \frac{\delta \beta}{32}\) we have
\[
\psi(0_n) + \frac{\delta \beta}{16} B_{\mathbb{R}^m} \subset \psi(\delta \Sigma_n).
\]
3. THE MAIN RESULT

In what follows $C_0$ is a regular tangent cone to $X_0$ at $z(0)$, $C_1$ is a regular tangent cone to $X_1$ at $z'(0)$, denote by $S_P$ the set of all mild solutions of the differential inclusion

$$v'' \in Av + P(t,v), \quad v(0) \in C_0, \quad v'(0) \in C_1$$

and by $R_P(T) = \{x(T); x(.) \in S_P\}$ its reachable set at time $T$.

**Theorem 3.1.** Assume that Hypothesis 2.4 is satisfied and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function with Lipschitz constant $l > 0$.

Then, inclusion (1.1) is $h$-local controllable around the solution $z(.)$ if

$$0_m \in \text{int}(\lambda R_P(T)) \quad \forall \lambda \in \partial h(z(T)).$$

**Proof.** By (3.1), since $\lambda R_P(T)$ is a convex cone, it follows that $\lambda R_P(T) = \mathbb{R}^m \forall \lambda \in \partial f(z(T))$. Therefore, using the compactness of $\partial f(z(T))$ (e.g. [11]), we have that for every $\beta > 0$ there exist $k \in \mathbb{N}$ and $u_j \in R_P(T)$ $j = 1, 2, ..., k$ such that

$$\beta B_{\mathbb{R}^m} \subset \lambda (u(\Sigma_k)) \quad \forall \lambda \in \partial f(z(T)),$$

where

$$u(\Sigma_k) = \{u(\gamma) := \sum_{j=1}^{k} \gamma_j u_j, \quad \gamma = (\gamma_1, ..., \gamma_k) \in \Sigma_k\}.$$ 

Using an usual separation theorem we deduce the existence of $\beta_1, \rho_1 > 0$ such that for all $\lambda \in L(\mathbb{R}^n, \mathbb{R}^m)$ with $d(\lambda, \partial f(z(T))) \leq \rho_1$ we have

$$\beta_1 B_{\mathbb{R}^m} \subset \lambda (u(\Sigma_k)).$$

Since $u_j \in R_P(T)$, $j = 1, ..., k$, there exist $(w_j(.), g_j(.))$, $j = 1, ..., k$ trajectory-selection pairs of (2.7) such that $u_j = w_j(T)$, $j = 1, ..., k$. We note that $\beta > 0$ can be taken small enough such that $||w_j(0)|| \leq 1$, $j = 1, ..., k$.

Define

$$w(t,s) = \sum_{j=1}^{k} s_j w_j(t), \quad \bar{g}(t,s) = \sum_{j=1}^{k} s_j g_j(t), \quad \forall s = (s_1, ..., s_k) \in \mathbb{R}^k.$$ 

Obviously, $w(.,s) \in S_P$, $\forall s \in \Sigma_k$.

Taking into account the definition of $C_0$ and $C_1$, for every $\varepsilon > 0$ there exists a continuous mapping $o_\varepsilon : \Sigma_k \to \mathbb{R}^n$ such that

$$z(0) + \varepsilon w(0,s) + o_\varepsilon(s) \in X_0, \quad z'(0) + \varepsilon \frac{\partial w}{\partial t}(0,s) + o_\varepsilon(s) \in X_1$$

$$\lim_{\varepsilon \to 0^+} \max_{s \in \Sigma_k} \frac{||o_\varepsilon(s)||}{\varepsilon} = 0.$$
Define

\[ p_\varepsilon(s)(t) := \frac{1}{\varepsilon}d(\bar{g}(t, s), F(t, z(t) + \varepsilon w(t, s)) - f(t)), \]

\[ q(t) := \sum_{j=1}^{k} ||g_j(t)|| + L(t)||w_j(t)||, \quad t \in I. \]

Then, for every \( s \in \Sigma_k \) one has

\[ p_\varepsilon(s)(t) \leq ||\bar{g}(t, s)|| + \frac{1}{\varepsilon}d_H(0_n, F(t, z(t) + \varepsilon w(t, s)) - f(t)) \leq ||\bar{g}(t, s)|| + \frac{1}{\varepsilon}d_H(F(t, z(t)), F(t, z(t) + \varepsilon w(t, s))) \leq ||\bar{g}(t, s)|| + L(t)||w(t, s)|| \leq q(t). \]

Next, if \( s_1, s_2 \in \Sigma_k \) one has

\[ |p_\varepsilon(s_1)(t) - p_\varepsilon(s_2)(t)| \leq ||\bar{g}(t, s_1) - \bar{g}(t, s_2)|| + \frac{1}{\varepsilon}d_H(F(t, z(t) + \varepsilon w(t, s_1)), F(t, z(t) + \varepsilon w(t, s_2))) \leq ||s_1 - s_2|| \cdot \max_{j=1,k} [||g_j(t)|| + L(t)||w_j(t)||], \]

thus, \( p_\varepsilon(.) (t) \) is Lipschitz with a Lipschitz constant not depending on \( \varepsilon \).

On the other hand, from (2.8) it follows that

\[ \lim_{\varepsilon \to 0^+} p_\varepsilon(s)(t) = 0 \quad a.e. (I), \quad \forall s \in \Sigma_k \]

and hence,

\[ \lim_{\varepsilon \to 0^+} \max_{s \in \Sigma_k} p_\varepsilon(s)(t) = 0 \quad a.e. (I). \]

Therefore, from (3.6), (3.7) and Lebesgue dominated convergence theorem we obtain

\[ \lim_{\varepsilon \to 0^+} \int_0^T \max_{s \in \Sigma_k} p_\varepsilon(s)(t) dt = 0. \]

By (3.4), (3.5), (3.8) and the upper semicontinuity of the Clarke generalized Jacobian we can find \( \varepsilon_0, e_0 > 0 \) such that

\[ \max_{s \in \Sigma_k} \frac{||o_{\varepsilon_0}(s)||}{\varepsilon_0} + \int_0^T \max_{s \in \Sigma_k} p_{\varepsilon_0}(s)(t) dt \leq \frac{\beta_1}{28 T^2}, \]

\[ \varepsilon_0 w(T, s) \leq \frac{e_0}{2} \quad \forall s \in \Sigma_k. \]

If we define

\[ y(s)(t) := z(t) + \varepsilon_0 w(t, s), \quad g(s)(t) := f(t) + \varepsilon_0 \bar{g}(t, s) \quad s \in \mathbb{R}^k, \]

\[ a_0(s) := z(0) + \varepsilon_0 w(0, s) + o_{\varepsilon_0}(s), \quad a_1(s) := z'(0) + \varepsilon_0 \frac{\partial w}{\partial t}(0, s) + o_{\varepsilon_0}(s), \quad s \in \mathbb{R}^k, \]

then we apply Theorem 2.3 and we find that there exists the continuous function \( x(.) : \Sigma_k \to C(I, \mathbb{R}^n) \) such that for any \( s \in \Sigma_k \) the function \( x(s)(.) \) is solution of the differential inclusion \( x'' \in A x + F(t, x), x(s)(0) = a_0(s), (x(s))'(0) = a_1(s) \forall s \in \Sigma_k \) and one has
where \( \chi \) satisfies (3.3) and verifies (3.12). In particular, we obtain (3.3)

\[
\|h(x) - h_0(x)\| \leq lb,
\]

(3.13)

\[
\phi(s) := h_0(z(T) + \varepsilon_0 w(T, s)),
\]

where \( \chi(.) : \mathbb{R}^n \to [0, 1] \) is a \( C^\infty \) function with the support contained in \( B_{\mathbb{R}^n} \) that satisfies \( \int_{\mathbb{R}^n} \chi(y)dy = 1 \) and \( b = \min\{\frac{\varepsilon_0}{2}, \frac{\varepsilon_0 \beta_1}{2^6 l}\} \).

Therefore, \( h_0(.) \) is of class \( C^\infty \) and verifies (3.12)

\[
\|h(x) - h_0(x)\| \leq lb,
\]

(3.13)

\[
\phi'(s) \mu = h_0'(z(T) + \varepsilon_0 w(T, \mu)) \quad \forall \mu \in \Sigma_k.
\]

Using again the upper semicontinuity of Clarke’s generalized Jacobian we obtain

\[
d(h_0'(z(T) + \varepsilon_0 w(T, s)), \partial h(z(T))) \leq \sup\{d(h_0'(u), \partial h(z(T))) ; \|u - z(t)\| \leq \|u - (z(T) + \varepsilon_0 w(T, s))\| + \|\varepsilon_0 w(t, s)\| \leq \varepsilon_0, \quad h'(u) \text{ exists}\} < \rho_1.
\]

The last inequality with (3.3) gives

\[
\varepsilon_0 \beta_1 B_{\mathbb{R}^m} \subset \phi'(s) \Sigma_k \quad \forall s \in \Sigma_k.
\]

Finally, for \( s \in \Sigma_k \), we put \( \psi(s) = h(x(s)(T)) \).

Obviously, \( \psi(.) \) is continuous and from (3.11), (3.12), (3.13) one has

\[
\|\psi(s) - \phi(s)\| = \|h(x(s)(T)) - h_0(y(s)(T))\| \leq \|h(x(s)(T)) - h(y(s)(T))\| + \|h(y(s)(T)) - h_0(y(s)(T))\| \leq l\|x(s)(T) - y(s)(T)\| + lb \leq \frac{\varepsilon_0 \beta_1}{64} + \frac{\varepsilon_0 \beta_1}{64} = \frac{\varepsilon_0 \beta_1}{32}.
\]

We apply Lemma 2.7 and we find that

\[
h(x(0_k)(T)) + \frac{\varepsilon_0 \beta_1}{16} B_{\mathbb{R}^m} \subset \psi(\Sigma_k) \subset h(R_F(T)).
\]

On the other hand, \( \|h(z(T)) - h(x(0_k)(T))\| \leq \frac{\varepsilon_0 \beta_1}{64} \), so we have \( h(z(T)) \in \text{int}(R_F(T)) \) and the proof is complete. \( \square \)

**Remark 3.2.** If \( m = n \) and \( h(x) \equiv x \), Theorem 3.1 yields Theorem 3.4 in [10].
REFERENCES


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