ANALYSIS OF A CONTACT PROBLEM WITH ADHESION FOR ELECTRO-VISCOELASTIC MATERIALS WITH LONG MEMORY

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We consider a mathematical model which describes a quasistatic frictional contact between a piezoelectric body and a deformable foundation. A nonlinear electro-viscoelastic constitutive law with long memory is used and the contact is modeled with a normal compliance condition and the associated Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account. Under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution of the problem. The proof is based on arguments of variational inequalities, differential equations and Banach fixed point theorem.

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1. INTRODUCTION

Contact problems with or without friction between deformable bodies or between a rigid and a deformable body are very common in industry and in daily life. One of the first publications on this topic is [19] due to Signorini where the contact problem between a linear elastic body and a rigid foundation is formulated. The variational formulation associated with this contact model was established by Fechera [8] and the problem was solved by using the arguments of variational inequalities. It follows the work of Duvaut and Lions [7] which added friction to the contact problems. Recall that frictional contact problems are studied in [13, 14, 16, 19, 23] and many others.

Given their wide range of applications, considerable progress has been completed in the modeling and processing of deformable materials, in particular, materials are called piezoelectric which reflect the act of polarize under the mechanical action or deform when an electric field is applied on this type of material. In mathematical models, this property is included in the law of material behavior, for a detailed presentation of piezoelectricity, see [1, 2, 12].

So, in addition to the equation of motion describing the evolution of the displacement field and traction-displacement conditions, the models also include an equation describing the evolution of the electric displacement field and the conditions on the electric potential and electric charges applied on the material.

In addition to piezoelectric effect, in recent models, adhesion is considered; this last is interfacial phenomena accompanying the motion when a glue is added to prevent the surfaces from relative motion. Following Frémond [8, 9], many models of adhesive contact in quasistatic or dynamic processes are studied in [4, 19, 22, 23] and some models for viscoelastic materials with adhesion can be found in [4, 11, 17, 20].

This paper is a contribution to the study of the contact problem for piezoelectric materials. In this work, we consider a mathematical model for adhesive contact between a body assumed to be electro-viscoelastic with long memory and a deformable foundation. The novelty consists in the fact that here the quasistatic problem is frictional and is modeled by a standard normal compliance condition associated with a Coulomb's law of dry friction. Note that the frictionless contact problem is resolved [1]. Other work are carried out in viscoelasticity with short memory (see [22]) and in some models where the frictional condition is modeled with subdifferential boundary (see [25]).

As in [8, 9], we use a bonding field of adhesion represented by a variable β defined on the area of contact which satisfies the restrictions $0 \leq \beta \leq 1$, if $\beta = 1$ at all point in the area of contact; the adhesion is complete and all the bonds are active, if $\beta = 0$; all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$, the adhesion is partial and only a fraction β of the bonds is active.

The paper is organized as follows. In Section 2, we introduce some notation and preliminaries concerning the different function spaces used in continuum mechanics. In Sections 3 and 4, we describe the mechanical problem, list the assumptions on the data and set it into a variational formulation. Finally, in Section 5, under the assumption of smallness of the coefficient of friction, we state and prove an existence and uniqueness result of the weak solution to the mechanical problem, Theorem 5.1.

2. PRELIMINARIES AND NOTATIONS

In this section, we present the notation and different function spaces used in this work. For further details, we refer the reader to [3, 6, 19].

We denote by $\mathbf{S}_d$ the space of symmetric tensors of second order on $\mathbb{R}^d (d = 2, 3)$, '.' and $\|.\|$ the inner product and the Euclidean norm on $\mathbf{S}_d$ and $\mathbb{R}^d$ respectively, i.e.:

$$\begin{aligned} u.v &= u_i v_i, \quad \|v\| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d, \quad 1 \le i,j \le d, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in \mathbf{S}_d, \quad 1 \le i,j \le d. \end{aligned}$$

Let $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$ a bounded domain with a Lipshitz boundary $\partial\Omega = \Gamma$, ν denotes the unit outer normal defined almost everywhere on Γ. Throughout this work we use standard notations for the spaces L^P and Sobolev spaces associated to Ω and Γ. We introduce the spaces

$$\begin{aligned} &H = \left[L^2(\Omega)\right]^d, \\ &Q = \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\} \\ &H_1 = \left\{u = (u_i)\,|u_i \in H^1(\Omega),\ i = \overline{1,d}\right\} \\ &H_{div} = \{\sigma \in Q\,|\mathrm{div}\sigma \in H\} \end{aligned}$$

H, Q, H_1, H_{div} are real Hilbert spaces equipped with the inner products:

$$\begin{aligned} &(u,v)_H = \textstyle\int_\Omega u_i v_i \mathrm{d}x, \quad \langle\sigma,\tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}\mathrm{d}x, \\ &(u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u),\varepsilon(v))_Q\,, \\ &(\sigma,\tau)_{H_{div}} = \langle\sigma,\tau\rangle_Q + (\mathrm{div}\sigma,\mathrm{div}\tau)_H\,, \end{aligned}$$

where ε is the *deformation* operator and div is the *divergence* operator,

$$\begin{aligned} \varepsilon : H_1 \to Q,\ \varepsilon(u) = (\varepsilon_{ij}(u))\,,\ \varepsilon_{ij}(u) = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \\ \forall i,j = \overline{1,d}, u \in H_1 \end{aligned}$$

$$\mathrm{div} : H_{div} \to H_1, \quad \mathrm{div}\sigma = (\partial_j\sigma_{ij}) \quad \forall i = \overline{1,d}, \sigma \in H_{div}.$$

We denote, respectively, the norms associated with $\|.\|_H$, $\|.\|_Q$, $\|.\|_{H_1}$ and $\|.\|_{H_{div}}$.

If Γ is of class C^1 then, the mapping $v \to u\,|_\Gamma$ defined on $C^1\left(\bar{\Omega}\right)^d$ in $L^2(\Gamma)^d$ extends to a continuous mapping γ of H_1 into $L^2(\Gamma)^d$. The image space of H_1 by this mapping denoted $H_\Gamma = \left[H^{\frac{1}{2}}(\Gamma)\right]^d$ is the Hilbert space, then the mapping $v \to \gamma v$ is linear continuous and surjective from H_1 in H_Γ. Let $H_\Gamma^{'}$ the dual space of H_Γ, for any $\sigma \in H_{div}$, the vector $\sigma\nu$ can be defined as an element of $H_\Gamma^{'}$ satisfying Green's formula as follows:

$$\langle\sigma\nu,\gamma v\rangle_{H_\Gamma^{'}\times H_\Gamma} = \langle\sigma,\varepsilon(v)\rangle_Q + (\mathrm{div}\sigma,v)_H \quad \forall v \in H_1.$$

For every $v \in H_1$, we also use the notation v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ, given by

$$v_\nu = v.\nu = v_i\nu_i\ , \quad v_\tau = v - v_\nu\nu. \tag{2.1}$$

We define, similarly, by σ_v and σ_r the normal and the tangential traces of the stress tensor $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu, \tag{2.2}$$

and we have the relation

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu.v\mathrm{d}a \quad \forall v \in H_1, \tag{2.3}$$

where da is the surface measure element Γ and Green's formula is written as

$$\langle \sigma, \varepsilon(v)\rangle_Q + (\mathrm{div}\sigma, v)_H = \int_\Gamma \sigma\nu.v\mathrm{d}a \quad \forall v \in H_1. \tag{2.4}$$

Let $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ a partition of Γ, such that Γ_1, Γ_2 and Γ_3 are disjoint open sets and let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 \,|\gamma v = 0 \text{ } a.e. \text{ on } \Gamma_1\}. \tag{2.5}$$

If $meas(\Gamma_1) > 0$, then, the Korn's inequality holds, i.e., there exists a constant $c_\Omega > 0$ depending only on Ω and Γ_1 such that:

$$\|\varepsilon(v)\|_Q \geqslant c_\Omega \|v\|_{H_1} \quad \forall v \in V. \tag{2.6}$$

The space V is equipped with the inner product:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q \ , \tag{2.7}$$

and the associated norm $\|.\|_V$. It follows by Korn's inequality (2.6) that $\|.\|_V$ is equivalent to the canonical norm $\|.\|_{H_1}$ and V is a real Hilbert space. Moreover, given the trace theorem (2.6) and (2.7), there exists a positive constant c_0 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leqslant c_0 \|v\|_V \quad \forall v \in V, \tag{2.8}$$

where c_0 depends on Ω and Γ.

Now, if we consider another partition of assuming $\Gamma = \bar{\Gamma}_a \cup \bar{\Gamma}_b$ where Γ_a, Γ_b are disjoint open sets with $mes(\Gamma_a) > 0$, we obtain similar results. We define the following spaces

$$\begin{aligned} W &= \{\psi \in H_1 \,|\gamma\psi = 0 \ \ a.e. \text{ on } \Gamma_a\}, \\ W_a &= \left\{D = (D_i) \,\middle|\, D_i \in L^2(\Omega)\right\}, \end{aligned}$$

W and W_a are Hilbert spaces endowed, respectively:

$$\begin{aligned} (\psi, \phi)_W &= (\nabla\psi, \nabla\phi)_H \,, \\ (D, E)_{W_a} &= (D, E)_H + (\mathrm{div}D, \mathrm{div}E)_{L^2(\Omega)} \,, \end{aligned}$$

where ∇ is the *gradient* operator

$$\nabla\psi = (\partial_i\psi) = (\psi_{,i}) \ \ \forall\psi \in W. \tag{2.9}$$

We denote the norms associated with $\|.\|_W$ and $\|.\|_{W_a}$.

As $meas(\Gamma_a) > 0$ then the *Friedrichs-Poincaré* inequality holds, which shows that there exists a constant C_F depending only on Ω and Γ such that:

$$\|\nabla\psi\| \geq C_F \|\psi\|_{H^1(\Omega)} \quad \forall\psi \in W. \tag{2.10}$$

Moreover, if $D \in W_d$ is a regular field, we have a result similar to (2.4), that is to say, that we have Green's formula as follows:

$$(D, \nabla\psi)_H + (\mathrm{div}D, \psi)_{L^2(\Omega)} = \int_{\Gamma_b} D\nu\psi \mathrm{da} \quad \forall\psi \in W. \tag{2.11}$$

We will also need the space of tensors of order four Q_∞ defined by

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkh})\, ;\ \mathcal{E}_{ijkh} = \mathcal{E}_{jikh} = \mathcal{E}_{khij} \in L^\infty(\Omega)\}.$$

Q_∞ is a Banach space with the norm defined by

$$\|\mathcal{E}\|_{Q_\infty} = \max_{0\leq i,j,k,h\leq d} \|\mathcal{E}_{ijkh}\|_{L^\infty(\Omega)}.$$

For every real Banach space X and T a positive real we use the notation $C([0,T]\,; X)$ for the space of continuous functions defined on $[0,T]$ in X, which is a Banach space with the norm:

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X.$$

For $p \in [0, \infty]$, the space $L^p(0,T;X)$ denotes the space of (class of) measurable functions $t \to f(t)$, which is a Banach space for the norm:

$$\|u\|_{L^p(0,T;X)} \begin{cases} \left(\int\limits_0^T \|v(t)\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup\limits_{(0,T)} \mathrm{ess}\, \|v(t)\|_X & \text{if } p = \infty. \end{cases}$$

In particular, the space $L^2(0,T;X)$ is a Hilbert space with the inner product:

$$(u,v)_{L^2(0,T;X)} = \int\limits_0^T (u(t), v(t))_X.$$

In addition, $W^{k,\infty}(0,T;V)$ is a Banach space for the norm defined by:

$$\|u\|_{W^{k,\infty}(0,T;X)} = \sum_{j=0}^{k} \sup_{(0,T)} \mathrm{ess} \left\|D^j u\right\|_X.$$

In particular, for $k = 0$, we have

$$W^{k,\infty}(0,T;X) = L^\infty(0,T;X).$$

For $k = 1$, the space $W^{1,\infty}(0,T;X)$ is defined by

$$W^{1,\infty}(0,T;X) = \{u : [0,T] \to X \text{ such that: } u \in L^{\infty}(0,T;X) \text{ and } \dot{u} \in L^{\infty}(0,T;X)\},$$

equipped with the norm:

$$\|u\|_{W^{1,\infty}(0,T;X)} = \|u\|_{L^{\infty}(0,T;X)} + \|\dot{u}\|_{L^{\infty}(0,T;X)} \cdot$$

Finally, we introduce the space of bonding the field denoted B by

$$B = \{\beta : [0,T] \longrightarrow L^2(\Gamma_3) \text{ such that : } 0 \leq \beta(t) \leq 1 \ \forall t \in [0,T], \quad a.e.\ x \in \Gamma_3\}.$$

3. THE CONTACT PROBLEM STATEMENT

We consider the following physical setting. An electro-viscoelastic with long memory body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$. The boundary $\partial\Omega = \Gamma$ is assumed to be regular and partitioned into three parts $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, such that Γ_1, Γ_2, Γ_3 are open, mutually disjoint and $meas\,(\Gamma_1) > 0$ on one hand, and partitioned on two measurable parts $\Gamma = \overline{\Gamma}_a \cup \overline{\Gamma}_b$ such that Γ_a, Γ_b are open, disjoint, $meas\,(\Gamma_a) > 0$ and $\overline{\Gamma}_3 \subset \overline{\Gamma}_b$, on the other hand. Let $T > 0$, $[0,T]$ denotes the time interval under consideration. We assume that the body is fixed on Γ_1 and is submitted to volume forces of density φ_0 given on $\Omega \times (0,T)$ and to surface tractions of φ_2 given on $\Gamma_2 \times (0,T)$. On the other hand, the body is submitted to electrical constraints.We assume that the electric potential is zero on Γ_a and that the body is submitted to an electric charge density q_0 act on Ω and an electric charge of density q_2 imposed on Γ_b. Along Γ_3 the body is in adhesive contact with a deformable foundation. The contact is modelled with a normal compliance condition associated with Coulomb friction. This problem may be stated as follows:

Problem (P). *Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to S^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that, for all $t \in [0,T]$,*

$$\sigma(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\mathrm{d}s - \mathcal{E}^* E(\varphi(t)) \tag{3.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)) \tag{3.2}$$

$$\mathrm{div}\sigma(t) + \varphi_0(t) = 0 \qquad \text{in } \Omega, \tag{3.3}$$

$$\mathrm{div}D(t) + q_1(t) = 0 \qquad \text{in } \Omega, \tag{3.4}$$

$$u(t) = 0 \qquad \text{on } \Gamma_1, \tag{3.5}$$

$$\sigma\nu(t) = \varphi_2(t) \qquad \text{on } \Gamma_2, \tag{3.6}$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - q) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) \qquad \text{on } \Gamma_3, \tag{3.7}$$

$$\dot\beta(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \qquad \text{on } \Gamma_3, \tag{3.8}$$

$$\varphi(t) = 0 \qquad \text{on } \Gamma_a, \tag{3.9}$$

$$D\nu(t) = q_2(t) \qquad \text{on } \Gamma_b, \tag{3.10}$$

$$\beta(0) = \beta_0 \qquad \text{on } \Gamma_3, \tag{3.11}$$

$$\begin{cases} \left\|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\right\| \leqslant \mu p\left(u_\nu(t) - q\right) \\ \left\|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\right\| < \mu p\left(u_\nu(t) - q\right) \Longrightarrow u_\tau\left(t\right) = 0 \\ \left\|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\right\| = \mu p\left(u_\nu(t) - q\right) \Longrightarrow \exists \lambda \geqslant 0 \\ \textit{such that:} \ \ \sigma_\tau\left(t\right) + \gamma_\tau \beta\left(t\right)^2 R_\tau(u_\tau(t)) = -\lambda u_\tau\left(t\right) \end{cases} \qquad \text{on } \Gamma_3. \tag{3.12}$$

We now provide some comments on equations and conditions (3.1)–(3.12).

(3.1) and (3.2) represent the constituve law of an electro-viscoelastic with long memory, in which ε is the linearized deformation tensor, $\mathcal{B}$ is an operator of elasticity, $\mathcal{F}$ is the tensor of relaxation, $\mathcal{E} = (e_{ijk})$ is the piezoelectric operator reflecting the proportionality between the electric charge and deformation at constant field or zero, $\mathcal{E}^* = (e^*_{ijk})$ is its transpose, $E(\varphi) = -\nabla\varphi$ is the electric field and $\mathcal{C} = (\mathcal{C}_{ij})$ denotes the electric permittivity tensor.

(3.3) is the equation of motion describing the evolution of the displacement $u : \Omega \times [0,T] \to \mathbb{R}^d$. (3.4) is the equation describing the evolution of the electric displacement $D : \Omega \times [0,T] \to \mathbb{R}^d$. (3.5) and (3.6) are the displacement and traction boundary conditions.

The equation (3.7) reflects the fact that the body is in adhesive contact with a deformable foundation. The normal constraint satisfies the condition called normal compliance modeling the interpenetration of the contact surface in the foundation. The contact area is not known a priory, q is the initial jump between the body and the foundation measured in the direction of the normal ν and p is a given non negative function. This condition shows that the foundation has a reaction on the body which depends on the penetration $(u_\nu - q)$.

The differential equation (3.8) describes the evolution of the bonding field β. Here γ_ν, γ_τ and ϵ_a are positive coefficients of adhesion where $[.]_+ = \max\{0, .\}$. R_ν is a truncation operator defined by:

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < L \\ -s & \text{if} \quad -L \le s \le 0 \\ 0 & \text{if} \quad s > L \end{cases} \tag{3.13}$$

where $L > 0$ is the characteristic length of the bond [17].

R_τ is also a truncation operator defined by

$$R_\tau(s) = \begin{cases} s & \text{if} \quad \|s\| < L, \\ L\dfrac{s}{\|s\|} & \text{if} \quad \|s\| > L. \end{cases} \tag{3.14}$$

In (3.9) we assume that the potential vanishes on Γ_a, (3.10) expresses the fact that the electric charge density q_2 is imposed on Γ_b and (3.11) is an initial condition.

Eqs. (3.12) represent a version of Coulomb's law of dry friction on its static version, where μ is the friction coefficient and $\mu p\,(u_\nu(t) - q)$ represents the so-called friction bound.

In the study of Problems (P) we consider the following assumptions on the problem data:

The elasticity operator satisfies:

$$\begin{cases} \text{(a)} \ \ \mathcal{B} : \Omega \times S_d \longrightarrow S_d \\ \text{(b)} \ \ \mathcal{B} \in Q_\infty \text{ and there exists a constant } M_\mathcal{B} > 0 \text{ such that:} \\ \qquad \|\mathcal{B}(x,\xi_1) - \mathcal{B}(x,\xi_2)\| \le M_\mathcal{B}\, \|\xi_1 - \xi_2\| \ \ \forall \xi_1, \xi_2 \in S_d, \ \text{a.e. in } \Omega. \\ \text{(c)} \ \ \text{There exists a constant } m_\mathcal{B} > 0 \text{ such that:} \ \ \mathcal{B}\xi.\xi \geqslant m_\mathcal{B}\, |\xi|^2 \\ \qquad \forall \xi \in S_d \ \ \text{a.e. in } \Omega. \\ \text{(d)} \ \ \text{The function } x \longrightarrow \mathcal{B}(x,\xi) \text{ is measurable on } \ \Omega \ \text{ a.e. } \xi \in S_d. \end{cases} \tag{3.15}$$

The relaxation tensor $\mathcal{F}$ satisfies:

$$\mathcal{F} \in C([0,T]; Q_\infty) \tag{3.16}$$

The electric permittivity tensor $\mathcal{C}$ satisfies:

$$\begin{cases} \text{(a)} \ \ \mathcal{C} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \\ \text{(b)} \ \ \mathcal{C}(x,E) = (c_{ij}(x)E_j) \quad \forall E = (E_{ij}) \in \mathbb{R}^d, \quad \text{a.e. in } \Omega \\ \qquad c_{ij} = c_{ji} \in L^\infty(\Omega). \\ \text{(c)} \ \ \text{There exists a constant } m_\mathcal{C} > 0 \text{ such that:} \\ \qquad c_{ij}(x)E_iE_j \geqslant m_\mathcal{C}\, \|E\|^2 \quad \forall \xi \in S_d, \quad \text{a.e. in } \Omega. \end{cases} \tag{3.17}$$

The piezoelectric tensor $\mathcal{E}$ satisfies:

$$\begin{cases} \text{(a)} \ \ \mathcal{E} : \Omega \times S_d \longrightarrow \mathbb{R}^d \\ \text{(b)} \ \ \mathcal{E}(x,\xi) = (e_{ijkh}(x)\xi_{ij}) \quad \forall \xi = (\xi_{ij}) \in S^d, \quad \text{a.e. in } \Omega \\ \text{(c)} \qquad e_{ijk} = e_{ikj} \in L^\infty(\Omega) \end{cases} \tag{3.18}$$

The tensor $\mathcal{E}^* = \mathcal{E}^*(e^*_{ijk})$ transposed $\mathcal{E}$ satisfies the property

$$\mathcal{E}\sigma.v = \sigma.\mathcal{E}^* v \quad \forall \sigma \in S^d, v \in \mathbb{R}^d. \tag{3.19}$$

The contact function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:

$$(3.20)\quad \begin{cases} (a) \text{ There exists } L_p > 0 \text{ such that:} \\ \quad |p(x,u_1) - p(x,u_2)| \leqslant L_p |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ (b)\ x \longmapsto p_r(x,u) \text{ is Lebesgue measurable on } \Gamma_3, \quad \forall\, u \in \mathbb{R}; \\ (c)\ x \longmapsto p_r(x,u) = 0 \text{ for } u \leqslant 0, \quad \text{a.e. } x \in \Gamma_3. \end{cases}$$

In addition, we assume that

$$(3.21)\qquad q \in L^2(\Gamma_3), \quad q \geqslant 0, \quad \text{a.e. } x \in \Gamma_3.$$

$$(3.22)\qquad \gamma_\tau, \gamma_\nu \in L^\infty(\Gamma_3), \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\tau, \gamma_\nu, \epsilon_a \geqslant 0, \quad \text{a.e. } x \in \Gamma_3.$$

The forces have regularity

$$(3.23)\qquad \varphi_0 \in C([0,T]; H), \quad \varphi_2 \in C\left([0,T]; L^2(\Gamma_2)^d\right).$$

The electric charges q_0 and q_2 check:

$$(3.24)\qquad q_0 \in C([0,T]; H), \quad q_2 \in C\left([0,T]; L^2(\Gamma_b)^d\right).$$

To reflect the fact that the foundation is isolating, we assume that:

$$(3.25)\qquad q_0(t) = 0 \text{ on } \Gamma_3 \quad \forall t \in [0,T].$$

And finally, the initial data β_0 satisfies

$$(3.26)\qquad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on } \Gamma_3.$$

4. VARIATIONAL FORMULATION

It follows from *Riesz-Fréchet*'s representation theorem that there exists a function $f : [0,T] \longrightarrow V$ such that

$$(4.1)\qquad (f(t), v)_V = \int_\Omega \varphi_0(t) v \mathrm{d}x + \int_{\Gamma_2} \varphi_2 v \mathrm{d}a \quad \forall v \in V.$$

By the same argument, it follows that there exists a function $q : [0,T] \longrightarrow W$ such that

$$(4.2)\qquad (q(t), \psi)_V = \int_\Omega q_0(t) \psi \mathrm{d}x + \int_{\Gamma_2} q_2 \psi \mathrm{d}a \quad \forall \psi \in W.$$

(3.23) and (3.24) imply that:

$$(4.3)\qquad f \in C([0,T]; H) \text{ and } q \in C([0,T]; W).$$

We define the adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$, the normal compliance functional $j_{cn} : V \times V \longrightarrow \mathbb{R}$ and the friction functional $j_{fr} : V \times V \longrightarrow \mathbb{R}$ as follows:

$$j_{ad}(\beta,u,v)=\int_{\Gamma_3}-(\gamma_\nu\beta^2R_\nu(u_\nu)v_\nu+\gamma_\tau\beta^2R_\tau(u_\tau)v_\tau)\mathrm{d}a, \tag{4.4}$$

$$j_{cn}(u,v)=\int_{\Gamma_3}p(u_\nu-q)v_\nu\mathrm{d}a, \tag{4.5}$$

$$j_{fr}(u,v)=\int_{\Gamma_3}\mu p\left(u_\nu(t)-q\right).\left|v_\tau\right|\mathrm{d}a. \tag{4.6}$$

By using a standard procedure based on Green's formula with (2.4) and (2.11) we prove that if u,σ,φ and D are regular functions and satisfying the equations and conditions (3.1)–(3.12), then

$$\begin{aligned}&(\sigma(t),\varepsilon(u(t)))_Q+j_{ad}(\beta(t),u(t),v)+j_{cn}(u(t),v-u(t))\\&+j_{fr}(u(t),v)-j_{fr}(u(t),u(t))\geq(f(t),v-u(t))_V\\&\qquad\forall v\in V,t\in[0,T],\end{aligned} \tag{4.7}$$

$$(D(t),\nabla\psi)_H+(q(t),\psi)_W=0\quad\forall\psi\in W. \tag{4.8}$$

Take $\sigma(t)$ in (4.7) by its expression given by (3.1) and $D(t)$ by its expression given by (3.2) and adding the equations (3.8) and (3.11) with $E(\varphi)=-\nabla\varphi$, we associate to Problem (P) the following variational formulation:

Problem (P_V). *Find a displacement field* $u:[0,T]\longrightarrow V$, *an electric potential* $\varphi:[0,T]\longrightarrow W$ *and a bonding field* $\beta:[0,T]\longrightarrow L^\infty(\Gamma_3)$, such as $u(t)\in V$, $\varphi(t)\in W$ and

$$\begin{aligned}&(\mathcal{B}\varepsilon(u(t)),\varepsilon(v-u(t)))_Q+\left(\int_0^t\mathcal{F}(t-s)\varepsilon(u(s))\mathrm{d}s,\varepsilon(v-u(t))\right)_Q\\&\quad+(\mathcal{E}^*\nabla\varphi(t),\varepsilon(v-u(t)))_Q+j_{ad}(\beta(t),u(t),v-u(t))\\&\quad+j_{cn}(u(t),v-u(t))+j_{fr}(u(t),v)-j_{fr}(u(t),u(t))\\&\qquad\geq(f(t),v-u(t))_V\quad\forall v\in V,t\in[0,T],\end{aligned} \tag{4.9}$$

$$(\mathcal{C}\nabla\varphi(t),\nabla\psi)_H-(\mathcal{E}\varepsilon(u(t),\nabla\psi)_H=(q(t),\psi)_W\quad\forall\psi\in W,t\in[0,T], \tag{4.10}$$

$$\begin{aligned}&\dot\beta(t)=-\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2+\gamma_\tau\left\|R_\tau(u_\tau(t))\right\|^2\right)-\epsilon_a\right]_+\\&\qquad\text{a.e. } t\in(0,T),\ \text{a.e. } x\in\Gamma_3,\\&\beta(0)=\beta_0.\end{aligned} \tag{4.11}$$

5. EXISTENCE AND UNIQUENESS

The main result we prove in this section is the following.

THEOREM 5.1. *Assume that (2.15)–(2.25) hold, then there exists a constant* $L_0 > 0$ *such that if* $L_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < L_0$, *the problem* (P_V) *has a unique solution* (u, φ, β) *with regularity:*

(5.1) $$u \in C([0,T]; V),$$

(5.2) $$\varphi \in C([0,T]; W),$$

(5.3) $$\beta \in W^{1,\infty}\left(0,T; L^2(\Gamma_2)\right) \cap B.$$

The proof of this theorem will be carried out in several steps. We assume in what follows that (3.1) – (3.26) hold and also, everywhere in this section, c will represent a strictly positive constant independent on time and whose value may change from place to place. First, for $\beta \in B$ we state the following auxiliary problem:

Problem (P_V^β). *Find a displacement field* $u_\beta : [0,T] \longrightarrow V$ *and an electric potential* $\varphi_\beta : [0,T] \longrightarrow W$ such that:

(5.4)
$$\begin{aligned} &(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + \left(\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\mathrm{d}s, \varepsilon(v - u(t))\right)_Q \\ &\quad + (\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t)))_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) \\ &\quad + j_{cn}(u_\beta(t), v - u_\beta(t)) + j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \\ &\quad \geq (f(t), v - u_\beta(t))_V \qquad \forall v \in V, t \in [0,T] \end{aligned}$$

(5.5)
$$\begin{aligned} (\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi)_H = (q(t), \psi)_W \\ \forall \psi \in W, t \in [0,T]. \end{aligned}$$

We have the following lemma:

LEMMA 1. *The problem* (P_V^β) *has a unique solution* $(u_\beta, \varphi_\beta) \in C([0,T]; V \times W)$.

Proof. To prove this lemma, consider the product Hilbert space $X = V \times W$ with the inner product defined by

(5.6) $$\langle x, y\rangle = \langle (u,\varphi), (v,\psi)\rangle = (u,v) + (\varphi,\psi), \qquad x, y \in X$$

and the associated norm $\|.\|_X$.

For all $\eta \in C([0,T]; Q)$ and $t \in [0,T]$, we introduce the operator $\Lambda_\beta : X \longrightarrow X$ and the element $f_\eta(t) \in X$ defined for all $x = (u, v)$ and $y = (\varphi, \psi)$ by

$$\begin{aligned}(5.7)\quad \langle \Lambda_\beta(t)x, y\rangle &= (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_Q + (\mathcal{C}\nabla\varphi, \nabla\psi)_H \\ &\quad - (\mathcal{E}\varepsilon(u), \nabla\psi)_H + j_{ad}(\beta(t), u, v),\end{aligned}$$

$$(5.8)\qquad \langle f_\eta(t), y\rangle = (f(t), v)_V + (q(t), \psi)_W - (\eta(t), \varepsilon(v))_Q$$

and let

$$(5.9)\qquad j(x, y) = j_{cn}(u, v) + j_{fr}(u, v). \quad \square$$

We introduce the following two problems:

Problem (P^1_η). *Find* $x_{\beta\eta} : [0, T] \longrightarrow X$ such that

$$\begin{aligned}&(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t)))_Q \\ &+ (\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)_H + (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q \\ (5.10)\qquad &+ j_{ad}(\beta(t), u_{\beta\eta}(t), v) + j_{cn}(u_{\beta\eta}(t), v - (u_{\beta\eta}(t)) \\ &+ j_{fr}(u_{\beta\eta}(t), v) - j_{fr}(u_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V \\ &\qquad \forall v \in V, t \in [0, T],\end{aligned}$$

$$(5.11)\qquad \begin{aligned}&(\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)_H = (q(t), \psi)_W \\ &\qquad \forall \psi \in W, t \in [0, T].\end{aligned}$$

Problem (P^2_η). *Find* $x_{\beta\eta} : [0, T] \longrightarrow X$ *such that*

$$(5.12)\qquad \begin{aligned}&\langle \Lambda_\beta(t)x_{\beta\eta}(t), y - x_{\beta\eta}(t)\rangle + j(y, x_{\beta\eta}(t)) - j(x_{\beta\eta}(t), x_{\beta\eta}(t)) \\ &\qquad \geq \langle f_\eta(t), y - x_{\beta\eta}(t)\rangle \ \ \forall y \in X,\ t \in [0, T].\end{aligned}$$

Remark 1. The two precedent problems are equivalent in the way that if $x_{\beta\eta} = (u_\beta, \varphi_{\beta\eta}) \in C([0, T]; X)$ is a solution of one of the problems it is also a solution of the other problem.

We now have the following lemma:

Lemma 2. *There exists a constant* $L_0 > 0$ *such that if* $L_p(1 + \|\mu\|_{L^\infty(\Gamma_3)}) < L_0$*, the problem* (P^2_η) *has a unique solution* $x_{\beta\eta} \in C([0, T]; X)$.

Proof. To prove Lemma 2, we proceed as follows:
The functional j_{ad} is linear over the third term and therefore,

$$(5.13)\qquad j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v).$$

Using the properties of truncation operators (see [22]), we deduce that there exists c such that

$$(5.14)\quad j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c \int |\beta_1 - \beta_2| \, \|u_1 - u_2\|_V \, \mathrm{d}s.$$

We take $\beta = \beta_1 = \beta_2$ in the last inequality, we obtain

$$(5.15)\qquad j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \leq 0.$$

Choose $u_1 = v$ and $u_2 = 0$ in (5.15) and use (5.13) and (5.14), we obtain

$$(5.16)\qquad j_{ad}(\beta, v, v) \geq 0.$$

The following assumptions (3.13)–(3.19) and (5.16), show that Λ_β is an operator Lipschitz continuous and strongly monotone, i.e.: there exists c such that

$$(5.17)\qquad \langle \Lambda_\beta(t) x_{\beta\eta}(t), x_{\beta\eta}(t) \rangle \geq c \, \|x_{\beta\eta}(t)\|_X^2 \qquad \forall x_{\beta\eta}(t) \in X.$$

We note that

$$j(y, x_{\beta\eta}(t)) - j(x_{\beta\eta}(t), x_{\beta\eta}(t)) = j(y, u_{\beta\eta}(t)) - j(u_{\beta\eta}(t), u_{\beta\eta}(t))$$

Next, let the set $L_+^2(\Gamma_3)$ defined by

$$L_+^2(\Gamma_3) = \left\{ \varphi \in L^2(\Gamma_3);\ \varphi \geqslant 0,\ a.e. \text{ on } \Gamma_3 \right\}.$$

For each $g = (g_1, g_2) \in L_+^2(\Gamma_3)^2$, we define the functional $h(g, .) : V \longrightarrow \mathbb{R}$ by

$$h(g, y) = \int_{\Gamma_3} g_1 w_\nu \mathrm{d}a + \int_{\Gamma_3} g_2 \|w_\tau\| \, \mathrm{d}a \quad \forall y = (w, \psi) \in X,$$

and introduce an intermediate problem as follows

Problem (P_1^g). *Find* $x_{\beta\eta} : [0, T] \longrightarrow X$ *such that*

$$(5.18)\qquad \begin{aligned} &\langle \Lambda_\beta(t) x_{\beta\eta}(t), y - x_{\beta\eta}(t) \rangle + h(g, y) - h(g, x_{\beta\eta}(t)) \geqslant (f, y - x_{\beta\eta}(t))_V \\ &\qquad\qquad \forall y \in X. \end{aligned}$$

LEMMA 3. *The problem* (P_1^g) *has a unique solution.*

Proof. The functional $h(g, .)$ is convex and lower semicontinuous, Λ_β is Lipschitz continuous and strongly monotone, we deduce that the problem (P_1^g) has a unique solution (see [24]). □

Now, to prove Lemma 2, we define the following mapping:

$$\begin{aligned} \Psi : L_+^2(\Gamma_3)^2 &\longrightarrow L_+^2(\Gamma_3)^2 \\ g &\longmapsto \Psi(g) = (p(u_{\beta\eta g\nu} - q), \mu p(u_{\beta\eta g\nu} - q)), \end{aligned}$$

then we show the following lemma.

Lemma 4. *Ψ has a unique fixed point g^* and $x_{\beta\eta g^*}$ is a unique solution of the problem (P_η^2).*

Proof. For $i = 1, 2$, define the following problem.

Problem$(P^2_{\eta gi})$. Find $u_{gi} \in V$ such that:

$$\langle \Lambda_\beta(t)x_{\beta\eta gi}, y\rangle + h\left(g_i, y\right) - h\left(g_i, x_{\beta\eta gi}\right) \geqslant \left(f, y - x_{\beta\eta gi}\right)_V \quad \forall y \in V.$$

Denote $x_{\beta\eta gi}$ by x_{g_i} and take $y = x_{g_2}$ in the previous inequality written for $g = g_1$, then take $y = x_{g_1}$ in the same inequality written for $g = g_2$, by adding the resulting inequalities, we get

$$\langle \Lambda_\beta(t)(x_{g_1} - x_{g_2}), x_{g_1} - x_{g_2}\rangle \le h\left(g_1, x_{g_1}\right) - h\left(g_1, y_{g_2}\right) + h\left(g_2, x_{g_2}\right) - h\left(g_2, y_{g_1}\right).$$

Then, using (2.8) and (5.17) it follows that there exists c such that

$$\left\|u_{g_1} - u_{g_2}\right\|_V \leqslant c\left\|g_1 - g_2\right\|_{L^2(\Gamma_3)^2}. \tag{5.19}$$

On the other hand, by the hypothesis on the function p and (2.8) we have

$$\left\|\Psi\left(g_1\right) - \Psi\left(g_2\right)\right\|_{L^2(\Gamma_3)^2} \leqslant cL_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right)\left\|x_{g_1} - x_{g_2}\right\|_X.$$

Hence, using (5.19) we deduce the estimate

$$\left\|\Psi\left(g_1\right) - \Psi\left(g_2\right)\right\|_{L^2(\Gamma_3)^2} \leqslant c'cL_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right)\left\|g_1 - g_2\right\|_{L^2(\Gamma_3)^2}.$$

By asking $L_0 = 1/c'c$, we deduce that $L_p\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < L_0$, Ψ is a contraction, thus, admitting a unique fixed point g^*.

Keeping in mind that there is a unique element x_{g^*} satisfying the equality:

$$\langle \Lambda_\beta(t)x_{g^*}, y - x_{g^*}\rangle + h\left(\Psi(g^*), y\right) - h\left(\Psi(g^*), x_{g^*}\right) \geqslant \left(f, y - x_{g^*}\right)_V \ \forall y \in Xx_{g^*},$$

and as $h \circ \Psi = j$, we have that $x_{\beta\eta}(t) = x_{g^*}$ is a unique solution of Problem (P_η^2). $\square$

Now, define the operator $\digamma_\beta : C([0, T]; Q) \longrightarrow C([0, T]; Q)$ by

$$\digamma_\beta\eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))\mathrm{d}s \qquad \forall \eta \in C(0, T; Q) \ \ \forall t \in [0, T]. \tag{5.20}$$

Lemma 5. *The operator $\digamma_\beta$ has a unique fixed point u_β.*

Proof. Let η_1, $\eta_1 \in C([0, T]; Q)$. By a standard computation based on (3.16) and (5.10), we obtain

$$\left\|\digamma_\beta\eta_1(t) - \digamma_\beta\eta_2(t)\right\|_Q \le c\int_0^t \left\|\eta_1(t) - \eta_2(t)\right\|_Q \mathrm{d}s \ \ \forall t \in [0, T].$$

By iteration, we deduce for any positive integer n, the estimate

$$\left\|\digamma_\beta^n\eta_1 - \digamma_\beta^n\eta_2\right\|_{C([0,T];Q)} \le \frac{c^nT^n}{n!}\left\|\eta_1 - \eta_2\right\|_{C([0,T];Q)},$$

which implies that a power $\mathcal{F}_\beta^n$ of $\mathcal{F}_\beta$ is a contraction on the space $C([0,T]\,;Q)$. Thus, $\mathcal{F}_\beta^n$ admits a unique fixed element $\eta_\beta \in C([0,T]\,;Q)$ which is also a unique fixed point of $\mathcal{F}_\beta$. $\square$

Now, denote $u_\beta = u_{\beta\eta}$ and $\varphi_\beta = \varphi_{\beta\eta}$, the couple (u_β, φ_β) is the unique solution of the problem (P_V^β). Indeed, the existence and uniqueness follows from Lemmas 2 and 5.

Also, in order to prove Theorem 1, we need other intermediate results.

Consider u_β the solution obtained above and define the following Cauchy problem stated as follows:

Problem (P_{ad}). *Find a bonding field $\beta^* : [0,T] \longrightarrow L^\infty(\Gamma_3)$ such that:*

(5.21) $$\dot{\beta}^*(t) = -\left[\beta^*(t)\left((\gamma_\nu R_\nu u_{\beta^*\nu}(t))^2 + \gamma_\tau \left\|R_\tau(u_{\beta^*\tau}(t))\right\|^2\right) - \epsilon_a\right]_+, \quad \text{a.e. } t \in [0,T],$$

(5.22) $$\beta^*(0) = \beta_0.$$

We have the following lemma:

Lemma 6. *The problem (P_{ad}) has a unique solution $\beta^* \in W^{1,\infty}(0,T; L^\infty(\Gamma_3)) \cap B$.*

Proof. The solution β^* belongs to the subset Θ of the set $C([0,T]\,;L^2(\Gamma_2)$ defined by

(5.23) $$\Theta = \left\{\beta \in C([0,T]\,;L^2(\Gamma_2)) \cap B\ ;\ \beta(0) = \beta_0\right\}.$$

Indeed, consider the mapping: $\Phi : \Theta \longrightarrow \Theta$ defined by

(5.24) $$\Phi(t) = \beta_0 - \int_0^t \left[\beta(s)\left((\gamma_\nu R_\nu u_{\beta\nu}(s))^2 + \gamma_\tau \left\|R_\tau(u_{\beta\tau}(s))\right\|^2\right) - \epsilon_a\right]_+ \mathrm{d}s \quad \forall t \in [0,T],$$

where u_β is the first component of the solution of Lemma 1. To prove the result in Lemma 6 we show that for a positive integer n the operator Φ^n is a contraction in Θ for an integer n large enough.

Indeed, let β_i, $i = 1,2$ two elements of Θ, denote $u_{\beta_i} = u_i$ and $x_i = (u_i, \varphi_i)$. Let $t \in [0,T]$, from (5.14), we show that there exists $c > 0$

(5.25) $$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \le c\left\|\beta_1 - \beta_2\right\|_{L^2(\Gamma_3)}\left\|u_1 - u_2\right\|_V$$

Using the assumptions (3.15) and (3.20), we also show that there exists a constant c such that

$$\left\|x_1(t) - x_2(t)\right\|_V \le c\left\|\beta_1 - \beta_2\right\|_{L^2(\Gamma_3)}\left\|u_1 - u_2\right\|_V,$$

and consequently,

(5.26) $$\left\|u_1(t) - u_2(t)\right\|_V + \left\|\varphi_1(t) - \varphi_2(t)\right\|_W \le c\left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)},$$

hence,

$$\|u_1(t) - u_2(t)\|_V \leq c\,\|\beta 1(t) - \beta 2(t)\|_{L^2(\Gamma 3)}\,. \tag{5.27}$$

We have

$$\begin{aligned} \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq c\textstyle\int_0^t \left\|\beta_1(s) R_\nu(u_{1\nu})^2 - \beta_2(s) R_\nu(u_{2\nu})^2\right\|_{L^2(\Gamma 3)} \mathrm{d}s \\ + \textstyle\int_0^t \left\|\beta_1(s)\left\|R_\tau(u_{1\tau})\right\|^2 - \beta_2(s)\left\|R_\tau(u_{2\tau})\right\|^2\right\|_{L^2(\Gamma 3)} \mathrm{d}s. \end{aligned} \tag{5.28}$$

Using the proprieties of the operators R_ν and R_τ (see [24]), it follows that

$$\begin{aligned} &\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq \\ &c\left(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma 3)} \mathrm{d}s + int_0^t \|u_1(s) - u_2(s)\|_{L^2(\Gamma 3)^d} \mathrm{d}s\right). \end{aligned} \tag{5.29}$$

Finally, by the Gronwall argument and (2.8), we get

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq c\int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s, \tag{5.30}$$

from which we deduce:

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma 3)} \leq c\int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s$$

and taking into account (5.26) we obtain

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma 3)} \leq c\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma 3)} \mathrm{d}s. \tag{5.31}$$

By iteration, we deduce for any positive integer n, the estimate

$$\|\Phi^n\beta_1 - \Phi^n\beta_2\|_{C([0,T],L^2(\Gamma 3))} \leq \frac{c^n T^n}{n!}\|\beta_1 - \beta_2\|_{C([0,T],L^2(\Gamma 3))}\,. \tag{5.32}$$

Then, for n large enough, the operator Φ^n is a contraction on $C([0,T]\,;L^2(\Gamma 3))$, then Θ^n admits a unique fixed point $\beta^* \in \Theta$ which is also a unique fixed point of Θ. □

Now, we have all the ingredients to prove Theorem 1:

Existence. Let β^* the fixed point of the operator Φ and $x^* = (u^*, \varphi^*)$ the solution of problem $(P_V^{\beta^*})$, again using the assumption (3.15) and one p, we have

$$\begin{aligned} \|u^*(t_1) - u^*(t_2)\|_V \leq c\big(\,&\|\beta^*(t_1) - \beta^*(t_2)\|_{L^2(\Gamma 3)} + \|q(t_1) - q(t_2)\|_W \\ &+ \|f(t_1) - f(t_2)\|_V\big), \quad \forall t_1, t_2 \in [0,T] \end{aligned}$$

and

$$\|\varphi^*(t_1) - \varphi^*(t_2)\|_V \leq c\left(\|q^*(t_1) - q^*(t_2)\|_W + \|q(t_1) - q(t_2)\|_W + \right.$$
$$\left. \|u^*(t_1) - u^*(t_2)\|_V\right), \forall t_1, t_2 \in [0, T].$$

Uniqueness. It follows by Lemmas 2, 4, 5 and 6 that the triple $(u^*, \varphi^*, \beta^*)$ is a unique solution of the problem (P_V) and with the regularity express (5.1), (5.2) and (5.3). Finally, the uniqueness follows from the uniqueness of the fixed point of the operator Φ, which completes the proof of Theorem 1.

REFERENCES

[1] R. Arhab, *Contribution à l'étude du contact piézoélectriques avec adhésion.* Thèse de doctorat de l'université de Perpignan en mathématiques appliquées, 2008.

[2] F. Auricchio, P. Bisegna and C. Lovadina, *Finite element approximation of piezoelectric plates.* Internat. J. Numer. Methods Engrg. **50** (2001), 1469–1499.

[3] R.C. Batra and J.S. Yang, *Saint-Venant's principle in linear piezoelectricity.* J. Elasticity **38** (1995), 209–218.

[4] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité.* Ann. Inst. Fourier (Grenoble) **18** (1968), 115–175.

[5] O. Chau, J.R. Fernaundez, M. Shillor and M. Sofonea, *Variational and numerical analysis of quasistatic viscoelastic contact problem with adhesion.* J. Comput. Appl. Math. **159** (2003), 431–465

[6] O. Chau, M. Shillor and M. Sofonea, *Dynamic frictionless contact with adhesion.* Z. Angew. Math. Phys. **55** (2004), 32–47.

[7] G. Duvaut and J-L. Lions, *Les inéquations en mécanique et en physique.* Dunod, Paris, (1972).

[8] G. Fichera, *Problemi elastostatici con vincoli unilaterali. II. Problema di Signorini con ambique condizioni al contorno.* Mem. Accad. Naz. Lincei, S. VIII, Vol VII, Sez. I, 5, (1964), 91–140.

[9] M. Frémond, *Adhérence des solides.* J. Méc. Théor. Appl. **6** (1980), 383–407.

[10] M. Frémond, *Equilibre des structures qui adhèrent à leur support.* C.R. Acad. Sci. Paris, Série II **295** (1982), 913–916.

[11] W. Han and M. Sofonea, *Analysis and numerical approximation of an elastic frictional contact problem with normal compliance.* Appl. Math. (Warsaw) **26** (1999), *4*, 415–435.

[12] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity.* AMS/IP Studies in Advanced Mathematics, vol. 30, American Mathematical Society, Rhode Island; International Press, Massachusetts, 2002.

[13] W. Heywang, K. Lubitz, W. Wersing, *Piezoelectricity. Evolution and Future of a Technology.* Springer, Berlin, 2008.

[14] A. Klarbring, A. Miklic and M. Shillor, *A global existence result for the quasistatic problem with normal compliance.* Internat. Ser. Numer. Math. **101** (1991), 85–111.

[15] D. Motreanu and M. Sofonea, *Quasivariational inequalities and applications in frictional contact problems with normal compliance.* Adv. Math. Sci. Appl. **10(1)** (2000), 103–118.

[16] M. Raous, L. Cangémi and M. Cocou, *A consistent model coupling adhesion, friction, and unilateral contact.* Comput. Methods Appl. Mech. Engrg. **177** (1999), 383–399.

[17] R. Rocca, *Existence of a solution for a quasistatic problem of unilateral contact with local friction.* Cr. Acad. Sci. Paris. Ser1, t.328 (1999), 1253–1258.

[18] M. Rochdi, M. Shillor and M. Sofonea, *Quasistatic viscoelastic contact problem with normal compliance and friction.* J. Elasticity **53** (1998), 105–126.

[19] A. Signorini, *Sopra alcune que stioni di elastostatica.* Atti Societa Italiana il Progresso Scienze, 1933.

[20] M. Shillor, M. Sofonea and J.J. Telega, *Models and Variational Analysis of Quasistatic Contact.* Lecture Notes in Phys. **655** (2004).

[21] M. Sofonea and El H. Essoufi, *A piezoelectric contact problem with slip-dependent coefficient of friction.* Math. Model. Anal. **9** (2004), 229–242.

[22] M. Sofonea and R. Arhab, *An Electro-viscoelastic contact problem with adhesion.* Dyn. Contin. Discrete Impuls. Syst., Ser. A: Math. Anal. **14** (2007), 577–591.

[23] M. Sofonea, R. Arhab and R. Tarraf, *Analysis of electrostatic frictionless contact problems with adhesion.* J. Appl. Math. (2006), 1–25.

[24] M. Sofonea, W. Han and M. Shillor, *Analysis and approximations of contact problems with adhesion or damage.* Pure Appl. Math. (Boca Raton) **276** (2006).

[25] S. Migórski, A. Ochal and M. Sofonea, *Analysis of a quasistatic contact problem for piezoelectric materials.* J. Math. Anal. Appl. **382** (2011), 701–713.

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