A GENERALIZATION OF THE BIG PICARD THEOREM

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The big Picard theorem was generalized for analytical maps of the punctured disc $\{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$ into Riemann surfaces, e.g. by Makoto Ohtsuka ([3] Theorem 1), Heinz Huber ([1] 6 Satz 2), H.L. Royden ([5] Theorem), H. Renggli ([4] Proposition 1), and others. In this paper we generalize these results for maps between Riemann surfaces by replacing the origin of the punctured disc with an isolated point of the Kerékjártó-Stoilow ideal boundary possessing some properties described by the hyperbolic metric of the surface.

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Definition 1. In this paper we call a Riemann surface R hyperbolic if the unit disc is its universal covering surface and denote by R^* its Kerékjártó-Stoilow compactification. A Picard point of R is an isolated point of the Kerékjártó-Stoilow ideal boundary of R which possesses a fundamental system of neighborhoods $(U_n)_{n\in\mathbb{N}}$ such that U_n is a domain of R^*, its boundary is an analytic Jordan curve of R, $\bar{U}_{n+1} \subset U_n$, and $\bar{U}_n \setminus U_{n+1}$ is a compact set of R for every $n \in \mathbb{N}$ such that the hyperbolic diameter of $\bar{U}_n \setminus U_{n+1}$ converges to 0 for n converging to infinite.

Lemma 2. *For every $\varepsilon > 0$ there is a compact hyperbolic Riemann surface with hyperbolic diameter less than ε.*

Let ϵ be a strictly positive number and γ_0 the arc of the circle

$$x^2 + y^2 = 1 + \epsilon^2 , \qquad y \geq 1 .$$

For every natural number k, $k \leq 3$, put

$$\gamma_{\pm k} := \gamma_0 \pm 2k\epsilon$$

and denote by γ_4 the arc of the circle

$$x^2 + y^2 = 1 + 49\epsilon^2 , \qquad y \geq 1 .$$

Then the arcs $(\gamma_k)_{-3\leq k\leq 4}$ enclose a domain D of the upper half plane. If we endow this half plane with the hyperbolic metric then the diameter of D is less than $14\epsilon + \log(1 + 49\epsilon^2)$. By taking the usual identifications of the boundary curves of D (e.g. [2] 1.8.3) we get a compact hyperbolic Riemann surface (of genus 2) with the corresponding hyperbolic diameter.

PROPOSITION 3. *There are Picard points of "infinite genus".*

By Lemma 2 there is a sequence $(R_n)_{n\mathbb{N}}$ of hyperbolic compact Riemann surfaces such that the hyperbolic diameter of R_n converges to 0 for n converging to infinite. If we paste R_n with R_{n+1} for every $n \in \mathbb{N}$ along a cut then the obtained chain is a Riemann surface with a unique Kerékjártó-Stoilow ideal boundary point. By making the length of the cuts sufficiently small we obtain a Picard point.

THEOREM 4. *Let R, S be hyperbolic Riemann surfaces and $\varphi : R \to S$ an analytic map. If p is a Picard point of R then $\lim_{z\to p} \varphi(z)$ exists in S^* and it is either a point of S or a Picard point of S.*

Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing sequence of neighborhoods of p with the properties formulated in the definition of the Picard point. Then $(\varphi(U_n \cap R))_{n\in\mathbb{N}}$ is a decreasing sequence of domains of S, $\varphi(\bar{U}_n \setminus U_{n+1})$ is a compact set of S the boundary of which consists of two Jordan curves (since φ is open if it is not constant) for all $n \in \mathbb{N}$, and (by the principle of hyperbolic metric) its hyperbolic diameter converges to 0 for n converging to infinite. Thus $\varphi(\bar{U}_n \setminus U_{n+1})$ converges either to a point of S or to a point of the Kerékjártó-Stoilow ideal boundary (since the hyperbolic distance of a point of the surface to the ideal boundary is infinite). It follows that φ converges to this point when the argument converges to p and in the last case that this point is also a Picard point.

Remark. If we put

$$R := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$$

then, 0 is a Picard point of R and so the above theorem generalizes the results mentioned in the abstract.

COROLLARY 5. *If the hyperbolic Riemann surface R has a Picard point p then for every meromorphic function f on R omitting three values $\lim_{z\to p} f(z)$ exists.*

COROLLARY 6. *Let R be a hyperbolic Riemann surface such that its Kerékjártó-Stoilow ideal boundary consists of a finite number n of Picard points. Then every non-constant analytic map of R into a hyperbolic Riemann surface S may be extended by continuity to a surjective map $R^* \to S^*$ and the Kerékjártó-Stoilow ideal boundary of S consists of $m \leq n$ Picard points. In particular if $n \leq 2$ then every meromorphic function on R omitting three values is constant.*

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