A NOTE ON AMALGAMATED MONOTONE, ANTI-MONOTONE, AND ORDERED-FREE PRODUCTS OF OPERATOR-VALUED QUANTUM PROBABILITY SPACES

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We directly prove that the amalgamated monotone or anti-monotone products (in N. Muraki's originary sense [17], for the scalar-valued case) (see also M. Popa's paper [22], for an operator-valued case) of some bimodule maps (in particular, conditional expectations), and ordered-free product (in T. Hasebe's originary sense [8] for the scalar-valued case) of pairs of some bimodule maps defined on $*$- or C^*-algebras preserve the (complete) positivity. As a by-product, in the same context, we get an extension and, in particular, a new proof of R. Speicher's theorem [24] concerning the (complete) positivity of D. Voiculescu's amalgamated free product of positive conditional expectations [28]. Our approach is made in terms of Schwarz maps. The proofs extend a scalar case technique due to M. Bożejko, M. Leinert, and R. Speicher from [4] concerning the conditionally free product of states.

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1. INTRODUCTION

The monotone product and the anti-monotone product (which is a monotone product with respect to the opposite order) of linear functionals on algebras (indexed by a totally ordered set) are defined on the associated universal free product algebra without unit, and on involutive algebras they preserve the positivity (see, e.g., [16], [6], [17]).

The monotone product was introduced by N. Muraki in C^*-algebraic setting to abstract the structure hidden in his monotone Fock space [17] or M. De Giosa and Y.G. Lu's chronological Fock space, and the corresponding arcsine Brownian motion [14], [15], [5], [12], [13]. The mentioned Fock space is

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a special case of some structures as twisted Fock space, deformed Fock space, or interacting Fock space (see, e.g., [17] and the references therein).

These products and the involved stochastic independences are fundamental in the so-called monotone, respectively anti-monotone quantum probability theory and related topics (see, e.g., [1], [2], [7], [8], [9], [14], [15], [16], [22] and the references therein).

These are two (dual) theories of the five noncommutative probability theories (the other being R.L. Hudson's Boson or Fermion probability theory, D.V. Voiculescu's free probability theory, and R. Speicher and W. von Waldenfels' Boolean probability theory) emerged from an associative (but noncommutative) product which fulfills a quasi-universal rule for mixed moments (according to Muraki's work [17] on the quasi-universal products of algebraic probability spaces) or even a natural product in A. Ben Ghorbal and M. Schürmann's spirit (due to Muraki's classification of his natural products [18] and U. Franz's axiomatic study in [6]).

Franz [7] revealed that the monotone (and, by duality, anti-monotone) product of unital functionals on algebras may be derived from a unital conditionally-free (c-free, for short) product of adequate functionals in M. Bożejko and R. Speicher's sense (see, e.g., [4]) defined on the associated universal free product algebra with unit. Then, M. Popa [22] expressed this connection in the more general frame involving conditional expectations defined on algebras over a common subalgebra, and showed the monotone product of positive conditional expectations defined on $*$-algebras over a common C^*-algebra is also positive.

The ordered-free product of pairs of linear functionals on algebras recently introduced by T. Hasebe may be defined on the corresponding universal free product algebra as a generalization of (Voiculescu's free product and as well of) both the monotone and the anti-monotone product, is associative, consists of parts of Bożejko and Speicher's c-free products, and is also a part of Hasebe's indented product; which is another kind of associative (and noncommutative) product, but defined for triples of linear functionals on algebras (see [8]).

In this Note, we consider monotone and anti-monotone products of conditional expectations and ordered-free products of pairs of conditional expectations (but also, more generally, of some bimodule maps), as parts of adequate amalgamated c-free product maps (in F. Boca's sense [3]) defined on ($*$-, C^*-) algebras over (respectively, endowed with a compatible bimodule structure with respect to) a common C^*-algebra and directly prove they preserve the (complete) positivity. As a by-product, in the same frame, we get an extension and, in particular, a new proof of Speicher's theorem about the (complete) positivity of Voiculescu's amalgamated free product [28] of positive conditional expectations (i.e., Theorem 3.5.6 in [24]). Our statements are formulated in terms of Schwarz maps (see, e.g., Chapter II, 9.2–9.3 in [26]). Thus, our result on the amalgamated monotone product implies the corresponding Popa's result in [22]. The proofs extend M. Bożejko, M. Leinert, and R. Speicher's method from the scalar-valued case [4] concerning the c-free product of states on unital involutive algebras (see also [10], [11]).

2. ($*$-, C^*-) ALGEBRA WITH ADJOINED ALGEBRA, COMPLETE POSITIVITY, SCHWARZ MAP, AMALGAMATED UNIVERSAL FREE PRODUCT ($*$-, C^*-) ALGEBRA, AMALGAMATED C-FREE PRODUCT MAP

Let B be an (associative) algebra. Let A be an algebra (over the same field) endowed with a compatible B-B-bimodule structure, such that: $b(a_1a_2) = (ba_1)a_2$, $(a_1b)a_2 = a_1(ba_2)$, $(a_1a_2)b = a_1(a_2b)$; for all $a_1, a_2 \in A$ and $b \in B$. Then the direct sum $\tilde{A} := B \oplus A$ is an (associative) algebra over B (i.e., A includes B as a subalgebra) with the multiplication

$$(b_1 \oplus a_1)(b_2 \oplus a_2) := b_1b_2 \oplus (b_1a_2 + a_1b_2 + a_1a_2), \quad b_i \in B,\ a_i \in A.$$

If B has a unit 1_B, and $1_Ba = a1_B = a$, for all $a \in A$, then $\tilde{A}$ has the unit $1_B \oplus 0$.

We may call $\tilde{A}$ the *algebra with adjoined algebra* B, corresponding to A. Remark that $A = 0 \oplus A$ is a two-sided ideal (of codimension equal to the dimension of B) in $\tilde{A}$; and every homomorphism π of A in an algebra D over B, which is a B-B-bimodule map (in the natural sense), extends to a similar homomorphism $\tilde{\pi}$ of $\tilde{A}$ in D such that the restriction $\tilde{\pi}|B$ coincides with the embedding of B into D. The algebra $\tilde{A}$ is uniquely determined up to an isomorphism by this universality property: $\tilde{A}$ is generated by B and A, and if ε is an injective homomorphism of A into an algebra D over B, such that ε is a B-B-bimodule map, and $\varepsilon(A) \cap j(B) = \{0\}$, denoting by j the embedding of B into D, then ε extends to an injective homomorphism of $\tilde{A}$ into D which is also a B-B-bimodule map.

If B and A are (complex) $*$-algebras (i.e., complex algebras endowed with conjugate linear involutions $*$, which are anti-isomorphisms), and A is endowed with a corresponding compatible B-B-bimodule structure (i.e., besides of the above properties, $(ab)^* = b^*a^*$, $(ba)^* = a^*b^*$; for all $a \in A$ and $b \in B$), then $\tilde{A} := B \oplus A$ becomes a $*$-algebra (with a corresponding universality property) by $(b \oplus a)^* = b^* \oplus a^*$; $b \in B$, $a \in A$; and every (injective) $*$-homomorphism of A in(to) an $*$-algebra D over B, which is a B-B-bimodule map, can be extended to a(n injective) $*$-homomorphism of $\tilde{A}$ in(to) D which is a B-B-bimodule map.

PROPOSITION 2.1. *Let Bbe a C^*-algebra, and A be a C^*-algebra endowed with a corresponding compatible B-B-bimodule structure.*

Then the $*$-algebra $\tilde{A} := B \oplus A$ endowed with the norm

$$\|b \oplus a\| := \sup_{\|x\| \leq 1,\, x \in A} \|bx + ax\|, \quad b \in B,\ a \in A,$$

becomes a C^*-algebra over B, and $\tilde{A}$ contains A as a closed two-sided ideal.

Proof. The norm on $\tilde{A}$ is a Banach algebra norm, because it is the norm induced from the Banach algebra of all bounded linear operators on the underlying Banach space of A:

$$\|b \oplus a\| = \|bId_A + L_a\|$$

denoting by L_a the left multiplication (by a) operator, and by Id_A the identity operator.

The embedding of A into $\tilde{A}$ is isometric because, for $a \neq 0$,

$$\|a\| = \|a(a^*/\|a\|)\| \leq \|0 \oplus a\| = \|L_a\| \leq \|a\|.$$

Since A is a C^*-algebra, one infers as in the scalar case ($B = \mathbf{C}$, the field of complex numbers)

$$\|b \oplus a\|^2 = \sup_{\|x\| \leq 1,\, x \in A} \|bx + ax\|^2 = \sup_{\|x\| \leq 1,\, x \in A} \|(bx + ax)^*(bx + ax)\| \leq$$

$$\leq \sup_{\|x\| \leq 1,\, x \in A} \|b^*bx + (a^*b + b^*a + a^*a)x\| = \|(b \oplus a)^*(b \oplus a)\| \leq \|(b \oplus a)^*\| \, \|b \oplus a\|.$$

Therefore, $\|b \oplus a\| \leq \|(b \oplus a)^*\|$, and thus $\|b \oplus a\| = \|(b \oplus a)^*\|$.

Hence $\|(b \oplus a)^*(b \oplus a)\| \leq \|b \oplus a\|^2$, and the C^*-algebra norm condition is verified.

The embedding of B into $\tilde{A}$ is of norm one and also isometric. $\square$

Let A be a (complex) $*$-algebra. We consider the cone A_+ of positive elements in A consisting of finite sums $\sum a_i^* a_i$, with $a_i \in A$. Thus, A_+ determines a preorder structure on the real linear subspace of self-adjoints elements in A.

For any positive integer n, let $M_n(A)$ be the $*$-algebra of $n \times n$ matrices $[a_{ij}]$ with entries from A. Every positive element M^*M in $M_n(A)$ can be expressed as $\sum_{k=1}^{n} [a_i^{(k)*} a_j^{(k)}]_{i,j=1,\ldots,n}$ with some $a_i^{(k)} \in A$. When A is a C^*-algebra, $A_+ = \{a^*a;\, a \in A\}$ determines an order structure on the real linear subspace of self-adjoint elements in A, and $M_n(A)$ becomes a C^*-algebra.

Let B be another $*$-algebra and $\varphi : A \to B$ be a linear map. We say φ is Hermitian if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$; φ is positive if $\varphi(A_+) \subset B_+$; and φ is a Schwarz map if $\varphi(a^*a) \geq \varphi(a)^*\varphi(a)$, for all $a \in A$.

For any positive integer n, let $\varphi_n : M_n(A) \to M_n(B)$ be the inflation map given by $\varphi_n([a_{ij}]) = [\varphi(a_{ij})]$, for $[a_{ij}] \in M_n(A)$. Then φ is called n-positive if the map φ_n induced by φ is positive. The map φ is completely positive if it is n-positive, for all positive integer n.

If B is a subalgebra of A, a linear map $\varphi : A \to B$ is a conditional expectation of A onto B, if φ is a B-B-bimodule map (i.e., $\varphi(ab) = \varphi(a)b$, $\varphi(ba) = b\varphi(a)$, for $a \in A$ and $b \in B$), which is a projection on B (i.e., $\varphi|B = id_B$); and we view (A, φ) or (A, φ, ψ), if ψ is another conditional expectation of A onto B, as quantum probability spaces over B, according to [27], [28].

More generally, if B and A are algebras, A being endowed with a compatible B-B-bimodule structure, and $\varphi, \psi : A \to B$ are B-B-bimodule maps, we regard (A, φ), or (A, φ, ψ) as quantum B-probability spaces.

If B and A are algebras, A being also endowed with a compatible B-B-bimodule structure, every B-B-bimodule map ϕ of A in B naturally extends to a conditional expectation $\tilde{\phi}$ of $\tilde{A} := B \oplus A$, the algebra with algebra B adjoined to A, onto B:

$$\tilde{\phi}(b \oplus a) := b + \phi(a), \quad \text{for all } b \in B,\ a \in A.$$

When B and A as above are $*$-algebras, $\tilde{A} := B \oplus A$ becomes a $*$-algebra and if ϕ is a Hermitian B-B-bimodule Schwarz map, then the conditional expectation $\tilde{\phi}$ is a Hermitian Schwarz map, too.

We say that a $*$-algebra A satisfies the Combes axiom if, for every $a \in A$, there exists a scalar $\lambda(a) > 0$ with $x^*a^*ax \leq \lambda(a)x^*x$, for all $x \in A$.

The following criterion of positivity for a matrix over a C^*-algebra is due to W.L. Paschke and E. Størmer (see Proposition 6.1 in [19] and Theorem 2.2 in [25]).

Proposition 2.2. *Let B be a C^*-algebra and $[b_{i,j}] \in M_n(B)$. Then $[b_{i,j}] \in M_n(B)_+$ if and only if $\sum_{i,j=1}^{n} b_i^* b_{i,j} b_j \in B_+$ for all $b_1, \ldots, b_n \in B$.* □

In consequence, by Proposition 2.2, we can observe the next fact (see also, in the unital case, Proposition 3.5.4 and Remark 3.5.5 in [24]; or Chapter II, 9.2–9.3 in [26], in the C^*-algebraic case).

Proposition 2.3. *Let B be a C^*-algebra, A be a $*$-algebra such that B is a $*$-subalgebra of A, and let $\varphi : A \to B$ be a positive Hermitian conditional expectation.*

Then φ is a completely positive Schwarz map. □

The assertion below is adapted from one stated, e.g., in Chapter II, 9.2 in [26].

Proposition 2.4. *Let A be a $*$-algebra, B be a C^*-algebra, and $\varphi : A \to B$ be a Schwarz map. If $\varphi(a^*a) = \varphi(a)^*\varphi(a)$, for some $a \in A$, then $\varphi(x^*a) = \varphi(x)^*\varphi(a)$, and $\varphi(a^*x) = \varphi(a)^*\varphi(x)$, for all $x \in A$.* □

Throughout B will denote a (non-necessary unital) fixed (complex) ($*$-, C^*-) algebra.

If A is a (complex) ($*$-, C^*-) algebra endowed with a ($*$-)homomorphism $\varepsilon : B \to A$, we say that A is a ($*$-, C^*-) B-algebra, and call ε the structural morphism of the B-algebra A. (When A and B are unital, consider, as usually, unital structural morphism, whenever it is not null.) When the structural morphism is injective, we say that the B-algebra A is an algebra over B. The same algebra can have different B-algebra structures. A B-algebra is an algebra naturally endowed with a compatible B-B-bimodule structure. Every algebra endowed with a compatible B-B-bimodule structure may be regarded as a B-algebra with respect to the null structural morphism.

Consider the category of the ($*$-, C^*-) B-algebras, with morphisms being the ($*$-)algebraic homomorphisms $\pi : A_1 \to A_2$ making the following diagrams commutative

$$\begin{array}{ccc} A_1 & \xrightarrow{\pi} & A_2 \\ \varepsilon_1 \nwarrow & & \nearrow \varepsilon_2 \\ & B & \end{array}$$

The amalgamated universal free product of a family of ($*$-, C^*-) B-algebras $(A_i)_{i\in I}$ of structural morphisms $\varepsilon_i : B \to A_i$, $i \in I$, or the universal free product of a family of ($*$-, C^*-)algebras $(A_i)_{i\in I}$, endowed with ($*$-) homomorphisms $\varepsilon_i : B \to A_i$, $i \in I$, with B amalgamated, denoted $A = \star_{i\in I}(A_i, \varepsilon_i, B)$, is the coproduct in this category.

This ($*$-, C^*-)algebra A is endowed with a structural morphism $\varepsilon : B \to A$, and canonical ($*$-)algebraic homomorphisms $j_i : A_i \to A$ such that $j_i \circ \varepsilon_i = \varepsilon$, for all $i \in I$, A is generated by $\bigcup_{i\in I} j_i(A_i)$, and satisfies the following universality property: whenever D is a ($*$-, C^*-) B-algebra endowed with a structural morphism $\eta : B \to D$, and $\lambda_i : A_i \to D$, $i \in I$, are ($*$-)homomorphisms satisfying $\lambda_i \circ \varepsilon_i = \eta$, for all $i \in I$, there exists a ($*$-)homomorphism $\lambda : A \to D$ such that $\lambda \circ j_i = \lambda_i$, for all $i \in I$.

When the structural morphisms are embeddings, i.e., A_i are algebras over B, this universal object A is the universal free product of $(A_i)_{i\in I}$ *with amalgamation over* B [20], [24], [3], [28], and B identifies to a subalgebra of A; A is commonly denoted by $*_B A_i$, although this notation hides the dependence of A on the embeddings ε_i, $i \in I$.

Moreover, as B-B-bimodule, a realization of the universal free product $*_B A_i$ associated to a family of ($*$-, C^*-) algebras $(A_i)_{i\in I}$ over B, such that $A_i = B \oplus A_i^o$ are direct sums of B-B-bimodules A_i^o being endowed with a compatible scalar multiplication, is (see, e.g., [3])

$$A = B \oplus_{n\geq 1} \oplus_{i_1\neq\cdots\neq i_n} A^o_{i_1} \otimes_B \ldots \otimes_B A^o_{i_n} =: B \oplus A^o.$$

When the ($*$-, C^*-)algebras A_i, $i \in I$ are endowed with compatible B-B-bimodule structures, a realization (as B-B-bimodule) of the universal free product $A^o = \star_{i\in I}(A_i,\, \varepsilon_i = 0,\, B)$, *with B amalgamated,* is

$$A^o = \oplus_{n\geq 1} \oplus_{i_1\neq\cdots\neq i_n} A_{i_1} \otimes_B \ldots \otimes_B A_{i_n};$$

and B does not identify to a ($*$-, C^*-) subalgebra of A^o, if B is not $\{0\}$.

By natural operations, the above B-B-bimodules A and A^o are organized as ($*$-)algebras.

In particular, if A_i and B are C^*-algebras, A and A^o satisfy the Combes axiom.

After separation and completion of the corresponding universal free product $*$-algebra A, respectively A^o, in its enveloping C^*-seminorm

$$\|a\| = \sup\Big\{\, \|\pi(a)\|\,;\pi\; * \text{-representation of } A \text{ (resp. } A^o) \\ \text{as bounded linear operators on a Hilbert space}\Big\},$$

one can realize the universal (or full) amalgamated free product $*_B A_i$, or the involved $\star_{i\in I}(A_i,\, \varepsilon_i = 0,\, B)$, in the category of C^*-algebras over B, respectively, of C^*-algebras endowed with (corresponding) compatible B-B-bimodule structures.

When A_i, $i \in I$, are ($*$-, C^*-)algebras over B, and, thus, they naturally are B-algebras, we may also consider A_i as B-algebras with respect to the null structural morphisms. If A is their universal free product *with amalgamation over B* (hence with the identification of B as a subalgebra of A), and A^o denotes their universal free product *with B amalgamated* (but, generally, with the non- identification of B to a subalgebra of A^o), there exists a canonical epimorphism of A^o onto A arising from the embeddings of A_i into A, $i \in I$, via the universality property.

In the following, let A_i, $i \in I$, be ($*$-)algebras over B, and φ_i, ψ_i, $i \in I$, be (Hermitian) conditional expectations of A_i onto B. (The set I having, of course, at least two elements.) Then $A_i = B \oplus A^\circ_i$ are direct sums of B-B-bimodules, with $A^o_i = \ker\psi_i$, the kernel of ψ_i. When ψ_i is a homomorphism (which is the identity on B), then A^o_i is an algebra.

The amalgamated c-free product $\varphi = *_B^{\{\psi_i\}}\varphi_i$, in Boca's sense, of $(\varphi_i)_{i\in I}$ with respect to $\psi_i, i \in I$, is the unique linear map [well-]defined on the universal free product $A = *_B A_i$ (see, e.g.,[3]) such that:

1) $\varphi|A_i = \varphi_i$, for each $i \in I$;

2) $\varphi(a_1 \ldots a_n) = \varphi_{i_1}(a_1)\ldots\varphi_{i_n}(a_n)$, for all $n \geq 2$, $i_1 \neq \cdots \neq i_n$, and $a_k \in A_{i_k}$, with $\psi_{i_k}(a_k) = 0$, if $k = 1, \ldots, n$; relatively to the natural embeddings of A_i into A arising from the free product construction.

Therefore, $\varphi = *_B^{\{\psi_i\}}\varphi_i$ is a (Hermitian) conditional expectation of A onto B, and $*_B^{\{\psi_i\}}\psi_i$ is Voiculescu's amalgamated free product [28] $\psi = *_B\psi_i$.

When $I = \{1, 2\}$, we denote $*_B^{\{\psi_i\}}\varphi_i$ by $\varphi_1 \ {}_{\psi_1}\!* {}_{\psi_2}\ \varphi_2$, adopting Franz's notation from [7]; and $*_B\psi_i$ by $\psi_1 *_B \psi_2$; moreover, if A_i are adequate (complex) $*$-algebras, we denote by $A_1 \star_0 A_2$, and $A_1 \star_1 A_2$,the non-unital, and respectively, the unital free product $*$-algebra.

3. AMALGAMATED MONOTONE, ANTI-MONOTONE, AND ORDERED-FREE PRODUCTS

The following definition comes from [17], [9], [22], [23].

Definition 3.1. Let B be an algebra, and I be a totally ordered set. Let A_i be algebras endowed with compatible B-B-bimodule structures and $\varphi_i : A_i \to B$ be B-B-bimodule maps; $i \in I$.

The amalgamated monotone product $\varphi = \rhd_B\varphi_i$ of $(\varphi_i)_{i\in I}$ is the unique linear map well-defined on the universal free product A of $(A_i)_{i\in I}$ with B amalgamated, such that:

1) $\varphi \mid A_i = \varphi_i$, for each $i \in I$;

2) for $n \geq 2, i_1 \neq \cdots \neq i_n$, and $a_k \in A_{i_k}$, with $k = 1, \ldots, n$:

i) $\varphi(a_1 \ldots a_n) = \varphi_{i_1}(a_1)\varphi(a_2 \ldots a_n)$, if $i_1 > i_2$;

ii) $\varphi(a_1 \ldots a_n) = \varphi(a_1 \ldots a_{n-1})\varphi_{i_n}(a_n)$, if $i_{n-1} < i_n$;

iii) $\varphi(a_1 \ldots a_n) = \varphi(a_1 \ldots a_{k-1}\varphi_{i_k}(a_k)a_{k+1} \ldots a_n)$, if $i_{k-1} < i_k > i_{k+1}$ for $2 \leq k \leq n-1$; with respect to the natural embeddings of A_i into A arising from the free product construction.

Thus, $\varphi = \rhd_B\varphi_i$ is a B-B-bimodule map. □

When $I = \{1, 2\}$, we denote $\rhd_B\varphi_i$ by $\varphi_1 \rhd_B \varphi_2$. By reversing the order structure, one can define the amalgamated anti-monotone product, denoted $\lhd_B\varphi_i$, respectively, for two maps, $\varphi_1 \lhd_B \varphi_2$, and derive from the two propositions below the assertions involving it.

The next statement describes the amalgamated monotone product as a part of an amalgamated c-free product (compare with [9] and [23]).

Proposition 3.2. *Let B be an algebra, A_i be two algebras endowed with compatible B-B-bimodule structures and $\varphi_i : A_i \to B$ be B-B-bimodule maps. Consider the algebras $\tilde{A}_i := B \oplus A_i$, with adjoined algebra B, define the conditional expectations $\tilde{\varphi}_i$ of $\tilde{A}_i$ onto B, and the homomorphisms δ_i of $\tilde{A}_i$ onto B by $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$, respectively $\delta_i(b \oplus a) := b$; if $b \oplus a \in B \oplus A_i = \tilde{A}_i$.*

Let A be the universal free product of A_1 and A_2 with B amalgamated, $\widetilde{A} := \widetilde{A_1} *_B \widetilde{A_2}$ be the universal free product of $\tilde{A}_1$ and $\tilde{A}_2$ with amalgamation over B, and $\delta := \delta_1 *_B \delta_2$.

Then $A = \ker\delta$, $\widetilde{A} = B \oplus A$ *is the algebra with adjoined algebra* B, *corresponding to* A, *and* $\varphi_1 \rhd_B \varphi_2 = \widetilde{\varphi_1}\ {}_{\delta_1}\!*_{\widetilde{\varphi_2}}\ \widetilde{\varphi_2} \mid A$. □

The proposition below presents the link between the amalgamated monotone product and the amalgamated c-free product map considered in [7] and [22] as the monotone product of unital functionals, respectively, of conditional expectations (compare with Proposition 3.1 in [7] or Theorem 6.6 in [22]).

PROPOSITION 3.3. *Let* B*be an algebra. Let* A_i *be two algebras over* B, *such that* $A_1^{\cdot} = B \oplus A_1^\circ$, A_1° *being an algebra endowed with compatible* B-B-*bimodule structure, and* φ_i *be conditional expectations of* A_i *onto* B. *Let* δ_1 *be the unique homomorphism of* $A_1^{\cdot}$ *onto* B *such that* $A_1^\circ = \ker\delta_1$, *and consider the* B-B-*bimodule map* $\varphi_1^\circ := \varphi_1 \mid A_1^\circ$.

Let $A := A_1 *_B A_2$ *be the universal free product of* $A_1^{\cdot}$ *and* $A_2^{\cdot}$, *with amalgamation over* B, *and* A° *be the universal free product of* A_1° *and* $A_2^{\cdot}$ *with* B *amalgamated. Let* $\varphi := \varphi_1\ {}_{\delta_1}\!*_{\varphi_2}\ \varphi_2 : A \to B$ *be the amalgamated* c-*free product conditional expectation, and* $\varphi^\circ := \varphi_1^\circ \rhd_B \varphi_2 : A^\circ \to B$ *be the amalgamated monotone product* B-B-*bimodule map.*

Then $\varphi^\circ = \varphi \circ j$, j *being the canonical homomorphism of* A° *in* A, *arising from the embeddings* $A_1^\circ \hookrightarrow A_1^{\cdot} \hookrightarrow A$ *and* $A_2^{\cdot} \hookrightarrow A$, *via the universality property.* □

If B and A_i are $*$-algebras, and φ_i and δ_1 are Hermitian, then the above maps φ, φ° and j preserve the natural involutions.

Adopting the terminology from [7], [8], and [22], we consider, in the sequel, $\varphi_1\ {}_{\delta_1}\!*_{\varphi_2}\ \varphi_2$ as the amalgamated monotone product conditional expectation of φ_i, if φ_i are conditional expectations, as above; thus, in the scalar-valued case $B = \mathbf{C}$, we consider this map as the unital monotone product. By duality, we may also consider unital versions of the anti-monotone products.

Let B be a $*$-algebra, as above, let A_i be $*$ -algebras over B, let ψ_i be (Hermitian) conditional expectations of A_i onto B, and $A_i^o = \ker\psi_i$, $i \in I$. Now I is an arbitrary set having at least two elements.

The lemma below is a generalization of Lemma 2.1 (established for states) in [4]. Its proof is similar to the scalar case, via the natural decompositions $A_i \ni a = \psi_i(a) + a^\circ \in B \oplus A_i^\circ$.

Denote by $W = \{a_1 \dots a_n;\ n \geq 1,\ a_k \in A_{i_k}^o,\ i_1 \neq \cdots \neq i_n\}$ the set of reduced words in $A = *_B A_i$.

For $w = a_1 \dots a_n \in W$, call n the length of w and a_1 the first letter of w. If $x = \sum_k w^{(k)} \in A^o$, call the length of x the maximal length in this representation of x.

LEMMA 3.4. *Let* $\varphi_i, \psi_i : A_i \to B$ *be Hermitian conditional expectations*; $i \in I$.

Let $\varphi = *_B^{\{\psi_i\}}\varphi_i$ *be the amalgamated* c*-free product of*$(\varphi_i)_{i\in I}$ *with respect to* ψ_i, $i \in I$, *defined on* $A = *_B A_i$.

Consider two words $x_1 = y_1 \dots y_n$, *and* $x_2 = z_1 \dots z_m$ *in* W.

(1) *If* y_1 *and* z_1 *do not belong to the same* A_i^o, *then*

$$\varphi(x_1^* x_2) = \varphi(x_1)^* \varphi(x_2).$$

(2) *Let* $a \in A_i$, *for some* $i \in I$. *If* $y_1, z_1 \notin A_i^o$, *then*

$$\varphi(x_1^* a x_2) = \varphi(x_1^* \psi_i(a) x_2) - \varphi(x_1)^* \psi_i(a) \varphi(x_2) + \varphi(x_1)^* \varphi_i(a) \varphi(x_2). \quad \square$$

The folowing assertion implies Proposition 6.2 in [22].

Throughout when we consider positive conditional expectations, we assume Hermitian maps.

Proposition 3.5. *Let* B *be a* C^**-algebra. Let* A_i *be two* $*$*-algebras over* B, *and* $A := A_1 *_B A_2$.

Let φ_1, δ_1 *be a positive conditional expectation, and, respectively, a* $*$*-homomorphism of* A_1 *onto* B *such that* $\delta_1|B = id_B$. *Let* φ_2 *be a positive conditional expectation of* A_2 *onto* B.

Then the $*$*-algebraic amalgamated monotone product* $\varphi = \varphi_1 \rhd_B \varphi_2 := \varphi_1 \ {}_{\delta_1} * {}_{\varphi_2} \ \varphi_2$ *is a Schwarz map.*

Thus, φ *is a positive conditional expectation of* A *onto* B. *As quantum probability spaces over* B, $(A_1, \varphi_1) \rhd_B (A_2, \varphi_2) = (A, \varphi_1 \rhd_B \varphi_2)$.

Proof. Let $A_1^o = \ker \delta_1$, and $A_2^o = \ker \varphi_2$. In view of Lemma 3.4, it suffices to prove the asserted property for every word $x(i)$ in A^o represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i^o; if $i \in \{1,2\}$.

Suppose that $x(i)$ has p terms of length one; else, the argument is similar.

Thus, let $x(i) = \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} y^{(k)}$, with all $a^{(k)} \in A_i^o$; and $y^{(k)} \in W$, but the first letter of $y^{(k)}$ does not belong to A_i^o.

Since $\varphi_i = \varphi|A_i$ are B-B-bimodule maps, we deduce, by Lemma 3.4,

$$\varphi(x(i)^* x(i)) - \varphi(x(i))^* \varphi(x(i)) = b(i) + \varphi_i(x_\circ(i)^* x_\circ(i)) - \varphi_i(x_\circ(i))^* \varphi_i(x_\circ(i)),$$

with

$$x_\circ(i) := \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} \varphi(y^{(k)}) \in A_i,$$

and

$$b(1) := \varphi(y^* y) - \varphi(y)^* \varphi(y),$$

where $y := \sum\limits_{k=p+1}^{N} \delta_1(a^{(k)})y^{(k)} \in A^\circ$ has the length less than the length of $x(1)$; respectively,

$$b(2) := \sum_{k,l=p+1}^{N} \left[\varphi(y^{(k)*}\varphi_2(a^{(k)*}a^{(l)})y^{(l)}) - \varphi(y^{(k)})^*\varphi_2(a^{(k)*}a^{(l)})\varphi(y^{(l)})\right].$$

One may represent $\varphi_2(a^{(k)*}a^{(l)}) = \sum\limits_r b_r^{(k)*}b_r^{(l)}$, with some $b_r^{(k)} \in B$, via the complete-positivity of φ_2 (according to Proposition 2.3), and thus one may express

$$b(2) = \sum_r \left[\varphi(x^{(r)*}x^{(r)}) - \varphi(x^{(r)})^*\varphi(x^{(r)})\right],$$

where $x^{(r)} = \sum\limits_{k=p+1}^{N} b_r^{(k)}y^{(k)} \in A^o$ has the length less than the length of $x(2)$.

Therefore, one concludes by induction on the length of the $x(i)'$ s, because every φ_i is a Schwarz map. $\square$

Corollary 3.6. *Let A_i be two unital (complex) $*$-algebras such that $A_1 = \mathbf{C} \oplus A_1^\circ$ (direct sum of linear spaces), A_1° being even an algebra, and φ_i be (unital positive functionals, i.e.) states of A_i.*

Let $\varphi = \varphi_1 \triangleright \varphi_2$ be their unital monotone product, defined on the $*$-algebra $A := A_1 *_1 A_2$.

Then $\varphi(a^*a) \geq |\varphi(a)|^2$, for all $a \in A$.

Thus, φ is a state, too. $\square$

From Boca's main result in [3, Theorem 3.1], M. Popa in [23] derived a theorem concerning the complete positivity of the amalgamated conditionally monotone product involving unital maps between C^*-algebras. The method presented in our Note is more elementary and does not use the above cited Boca's result. (In fact, our method also permits a simpler new proof of Boca's main result in [3], and a corresponding extension-for slightly more general maps- of this theorem stated by Popa.) Let expose our statement for the amalgamated monotone product, via Proposition 3.2.

Corollary 3.7. *Let B be a C^*-algebra. Let A_i be two $*$-algebras endowed with compatible B-B-bimodule structures, and $\varphi_i : A_i \to B$ be Hermitian B-B-bimodule Schwarz maps. Let A be the universal free product of $(A_i)_i$ with B amalgamated.*

Then the $*$-algebraic amalgamated monotone product of B-B-bimodule maps $\varphi = \varphi_1 \triangleright_B \varphi_2$ is a Schwarz map of A in B.

Thus, φ is a (completely) positive B-B-bimodule map. As quantum B-probability spaces, $(A_1, \varphi_1) \triangleright_B (A_2, \varphi_2) = (A, \varphi_1 \triangleright_B \varphi_2)$. $\square$

The statement involving the monotone product in Muraki's originary sense [17] is the following

Corollary 3.8. *Let A_i be two (complex) $*$-algebras, and φ_i be linear functionals on A_i, such that their unitizations are states. Let $\varphi = \varphi_1 \rhd \varphi_2$ be their monotone product, defined on the $*$-algebra $A := A_1 *_0 A_2$.*

Then $\varphi(a^*a) \geq |\varphi(a)|^2$, for all $a \in A$. Thus, φ is positive. □

By duality, we may establish the assertions corresponding to the (amalgamated) anti-monotone product, of course.

We should mention here that M. Popa has in [22] and [21] stated two theorems about amalgamated conditionally free products which are more general than the results exposed in this Note. However, his proofs are incomplete, as we may

Remark 3.9. Indeed, for $\varphi = \varphi_1 \ {}_{\psi_1} * {}_{\psi_2} \ \varphi_2$ defined on $A := A_1 *_B A_2$, and $a = a_1 a_2 \in A$, with arbitrary $a_k \in \ker \psi_{i_k}$, $i_1 \neq i_2$, his computation (see [22, page 323] and [21, page 310]) gives $\varphi(a_2^* a_1^* a_1 a_2) = \varphi(a_2)^* \varphi(a_1^* a_1) \varphi(a_2)$.

This is true only if $\psi_{i_1}(a_1^* a_1) = 0$; and this condition is fulfilled if ψ_{i_1} is a homomorphism.

Otherwise, the equation above is equivalent to $\varphi_{i_2}((ba_2)^* ba_2) = \varphi_{i_2}(ba_2)^* \varphi_{i_2}(ba_2)$, denoting $b^*b := \psi_{i_1}(a_1^* a_1) \in B_+$, with $b \neq 0$.

Therefore, in view of (Proposition 2.3 and) Proposition 2.4, one could deduce $\varphi_{i_2}(xba_2) = \varphi_{i_2}(x) b \varphi_{i_2}(a_2)$, for all $x \in A_{i_2}$, and all $a_2 \in \ker \psi_{i_2}$; and then φ_{i_2} should satisfy the condition $\varphi_{i_2}(xby) = \varphi_{i_2}(x) b \varphi_{i_2}(y)$, for all $x, y \in A_{i_2}$, since φ_{i_2} is a conditional expectation.

In particular, when $B = \mathbf{C}$ (the field of complex numbers, as before), it would result that φ_{i_2} must be a $*$-homomorphism.

In conclusion, the proofs in [22, Theorem 6.5] and [21, Theorem 2.3] work, in the most general case, when ψ_i are $*$-homomorphisms; i.e., for the amalgamated Boolean product $\varphi_1 \ {}_{\delta_1} * {}_{\delta_2} \ \varphi_2$, in Franz's notation; and for a unital C^*-algebra B, and unital maps, because it is used Speicher's Theorem 3.5.6 in [24]. □

The next statement extends Proposition 3.5.

Theorem 3.10. *Let B be a C^*-algebra. Let A_i be two $*$-algebras over B; and $A := A_1 *_B A_2$.*

Let φ_1, ψ_1 be positive conditional expectations of A_1 onto B, and let φ_2 be a positive conditional expectation of A_2 onto B.

Then the $*$-algebraic amalgamated c*-free product $\varphi := \varphi_1 \ {}_{\psi_1} * {}_{\varphi_2} \ \varphi_2$ is a Schwarz map.*

Thus, φ is a positive conditional expectation of A onto B.

Proof. Denote now $A_1^o = \ker \psi_1$, and $A_2^o = \ker \varphi_2$.

By Lemma 3.4, as above, it is enough to verify the necessary property for every $x(i)$ in A^o represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i^o; if $i \in \{1,2\}$.

In the same way, assume that $x(i)$ has p terms of length one, i.e., $x(i) = \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} y^{(k)}$, with all $a^{(k)} \in A_i^o$; and $y^{(k)} \in W$, but the first letter of $y^{(k)}$ does not belong to A_i^o; otherwise, the argument is similar. Then we infer, via the same Lemma 3.4, with $x_\circ(i) := \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)}\ \varphi(y^{(k)}) \in A_i$ again, that

$$\varphi(x(i)^* x(i)) - \varphi(x(i))^* \varphi(x(i)) = b(i) + \varphi_i(x_\circ(i)^* x_\circ(i)) - \varphi_i(x_\circ(i))^* \varphi_i(x_\circ(i)),$$

where this time

$$b(1) := \sum_{k,l=p+1}^{N} \left[\varphi(y^{(k)*} \psi_1(a^{(k)*} a^{(l)}) y^{(l)}) - \varphi(y^{(k)})^* \psi_1(a^{(k)*} a^{(l)}) \varphi(y^{(l)})\right],$$

and

$$b(2) := \sum_{k,l=p+1}^{N} \left[\varphi(y^{(k)*} \varphi_2(a^{(k)*} a^{(l)}) y^{(l)}) - \varphi(y^{(k)})^* \varphi_2(a^{(k)*} a^{(l)}) \varphi(y^{(l)})\right],$$

because $\varphi_i = \varphi | A_i$ are B-B-bimodule maps. The complete-positivity of both ψ_1 and φ_2, as before (by Proposition 2.3), ensures us we may represent $\psi_1(a^{(k)*} a^{(l)}) = \sum_r b_r^{(k)*} b_r^{(l)}$, with some $b_r^{(k)} \in B$, and, respectively, $\varphi_2(a^{(k)*} a^{(l)}) = \sum_s \overline{b}_s^{(k)*} \overline{b}_s^{(l)}$, with some $b_r^{(k)}, \overline{b}_s^{(k)} \in B$.

Consequently, we may express $b(1) = \sum_r \left[\varphi(x^{(r)}(1)^* x^{(r)}(1)) - \varphi(x^{(r)}(1))^* \varphi(x^{(r)}(1))\right]$, and $b(2) = \sum_s \left[\varphi(x^{(s)}(2)^* x^{(s)}(2)) - \varphi(x^{(s)}(2))^* \varphi(x^{(s)}(2))\right]$, where $x^{(r)}(1) = \sum_{k=p+1}^{N} b_r^{(k)} y^{(k)} \in A^o$, and $x^{(s)}(2) = \sum_{k=p+1}^{N} \overline{b}_s^{(k)} y^{(k)} \in A^o$ have the length less than the length of $x(1)$, and respectively, that of $x(2)$.

In conclusion, every φ_i being a Schwarz map (due to the same Proposition 2.3), the proof completes by induction on the length of the $x(i)'$s. □

Thus, we get a new proof concerning the (complete) positivity of Voiculescu's amalgamated free product, due to the well-known associativity property of this product. The next consequence implies Speicher's Theorem 3.5.6 in [24] (stated for a unital C^*-algebra B, and unital maps).

Corollary 3.11. *Let B be a C^*-algebra, and I be a set having at least two elements. Let A_i be $*$-algebras over B, endowed with positive conditional expectations of A_i onto B; $i \in I$. Let $A = *_B A_i$ be the $*$-algebraic amalgamated free product.*

Then the $*$-algebraic amalgamated free product $\varphi := *_B \varphi_i$ is a Schwarz map.

Thus, φ is a positive conditional expectation of A onto B. As quantum probability spaces over B, $*_B(A_i, \varphi_i) = (A, \varphi)$. □

The previous facts have versions in terms of more general B-B-bimodule maps.

Corollary 3.12. *Let B be a C^*-algebra, and I be a set having at least two elements. Let A_i be $*$-algebras endowed with compatible B-B-bimodule structures, and $\varphi_i : A_i \to B$ be Hermitian B-B-bimodule Schwarz maps. Consider the $*$-algebras $\tilde{A}_i := B \oplus A_i$, with adjoined algebra B, and the conditional expectations $\tilde{\varphi}_i$ of $\tilde{A}_i$ onto B, defined by $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$; if $b \oplus a \in B \oplus A_i = \tilde{A}_i$; $i \in I$.*

Let A be the universal free product of $(A_i)_{i \in I}$ with B amalgamated.

Then the $*$-algebraic amalgamated free product $\star_B \varphi_i := (\star_B \widetilde{\varphi_i}) \mid A$ is a B-B-bimodule Schwarz map. As quantum B-probability spaces, $*_B(A_i, \varphi_i) = (A, \star_B \varphi_i)$. □

Corollary 3.13. *Let Bbe a C^*-algebra. Let A_i be two $*$-algebras endowed with compatible B-B-bimodule structures, and $\varphi_i : A_i \to B$ be Hermitian B-B-bimodule Schwarz maps. Let ψ_1 be a Hermitian B-B-bimodule Schwarz map of A_1 in B.*

Consider the $*$-algebras $\tilde{A}_i := B \oplus A_i$, with adjoined algebra B, the conditional expectations $\tilde{\varphi}_i$ of $\tilde{A}_i$ onto B, and $\tilde{\psi}_1$ of $\tilde{A}_1$ onto B, defined by $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$, and $\tilde{\psi}_1(b \oplus a) := b + \psi_1(a)$; if $b \oplus a \in B \oplus A_i = \tilde{A}_i$.

Let A be the universal free product of $(A_i)_{i \in \{1,2\}}$ with B amalgamated.

Then the $*$-algebraic amalgamated c*-free product $\varphi_1 \ {}_{\psi_1}{*}_{\varphi_2} \varphi_2 := \widetilde{\varphi_1}_{\widetilde{\psi_1}} {*}_{\widetilde{\varphi_2}} \widetilde{\varphi_2} \mid A$ is a B-B-bimodule Schwarz map.* □

In order to recover the ordered-free product we need the following result, which can be proved analogously.

Theorem 3.14. *Let B be a C^*-algebra. Let A_i be two $*$-algebras over B; and $A := A_1 *_B A_2$.*

Let φ_1 be a positive conditional expectation of A_1 onto B, and let φ_2, ψ_2 be positive conditional expectations of A_2 onto B.

Then the $*$-algebraic amalgamated c*-free product $\varphi := \varphi_1 \ {}_{\varphi_1}{*}_{\psi_2} \varphi_2$ is a Schwarz map. Thus, φ is a positive conditional expectation of A onto B.*

Proof. We sketch the proof for the convenience of the reader. In the light of the preliminary Lemma 3.4, it suffices to check the asserted condition for every $x(i)$ in A^o represented as $\sum_k w^{(k)}$ with $w^{(k)} \in W$ having the first letter in a same A_i^o; if $i \in \{1,2\}$; letting now $A_1^o = \ker \varphi_1$, and $A_2^o = \ker \psi_2$.

Assuming that $x(i)$ has p terms of length one, i.e., $x(i) = \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} y^{(k)}$, with every $a^{(k)} \in A_i^o$; and $y^{(k)} \in W$, but the first letter of $y^{(k)}$ does not belong to A_i^o, we get again, in the same way as before, with

$$x_\circ(i) := \sum_{k=1}^{p} a^{(k)} + \sum_{k=p+1}^{N} a^{(k)} \varphi(y^{(k)}) \in A_i,$$

that

$$\varphi(x(i)^* x(i)) - \varphi(x(i))^* \varphi(x(i)) = b(i) + \varphi_i(x_\circ(i)^* x_\circ(i)) - \varphi_i(x_\circ(i))^* \varphi_i(x_\circ(i)),$$

where now

$$b(1) := \sum_{k,l=p+1}^{N} \left[\varphi(y^{(k)*} \varphi_1(a^{(k)*} a^{(l)}) y^{(l)}) - \varphi(y^{(k)})^* \varphi_1(a^{(k)*} a^{(l)}) \varphi(y^{(l)})\right],$$

and

$$b(2) := \sum_{k,l=p+1}^{N} \left[\varphi(y^{(k)*} \psi_2(a^{(k)*} a^{(l)}) y^{(l)}) - \varphi(y^{(k)})^* \psi_2(a^{(k)*} a^{(l)}) \varphi(y^{(l)})\right].$$

It remains only to use the complete-positivity of both φ_1 and ψ_2 (by Proposition 2.3), to represent $\varphi_1(a^{(k)*} a^{(l)}) = \sum_r b_r^{(k)*} b_r^{(l)}$, and respectively $\psi_2(a^{(k)*}\, a^{(l)}) = \sum_s \overline{b}_s^{(k)*} \overline{b}_s^{(l)}$; then, as in the previous proof, $b(1) = \sum_r \left[\varphi(x^{(r)}(1)^* x^{(r)}(1)) - \varphi(x^{(r)}(1))^* \varphi(x^{(r)}(1))\right]$, and $b(2) = \sum_s \left[\varphi(x^{(s)}(2)^* x^{(s)}(2)) - \varphi(x^{(s)}(2))^* \varphi(x^{(s)}\,(2))\right]$; denoting $x^{(r)}(1) = \sum_{k=p+1}^{N} b_r^{(k)} y^{(k)} \in A^o$, and $x^{(s)}(2) = \sum_{k=p+1}^{N} \overline{b}_s^{(k)} y^{(k)} \in A^o$, with some $b_r^{(k)}, \overline{b}_s^{(k)} \in B$. □

Corollary 3.15. *Let B be a C^*-algebra. Let A_i be two $*$-algebras over B; and $A := A_1 *_B A_2$.*

Let φ_1 be a positive conditional expectation of A_1 onto B. Let φ_2, δ_2 be a positive conditional expectation, and, respectively, a $*$-homomorphism of A_2 onto B such that $\delta_2 | B = id_B$.

Then the $*$-algebraic amalgamated anti-monotone product $\varphi = \varphi_1 \lhd_B \varphi_2 := \varphi_1 \;{}_{\varphi_1} *_{\delta_2}\; \varphi_2$ *is a Schwarz map. Thus,* φ *is a positive conditional expectation of* A *onto* B. *As quantum probability spaces over* B, $(A_1, \varphi_1) \lhd_B (A_2, \varphi_2) = (A, \varphi_1 \lhd_B \varphi_2)$. □

COROLLARY 3.16. *Let* B *be a* C^**-algebra. Let* A_i *be two* $*$*-algebras endowed with compatible* B-B*-bimodule structures, and* $\varphi_i : A_i \to B$ *be Hermitian* B-B*-bimodule Schwarz maps. Let* ψ_2 *be a Hermitian* B-B*-bimodule Schwarz map of* A_2 *in* B.

Consider the $*$*-algebras* $\tilde{A}_i := B \oplus A_i$, *with adjoined algebra* B, *the conditional expectations* $\tilde{\varphi}_i$ *of* $\tilde{A}_i$ *onto* B, *and* $\tilde{\psi}_2$ *of* $\tilde{A}_2$ *onto* B, *defined by* $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$, *and* $\tilde{\psi}_2(b \oplus a) := b + \psi_2(a)$; *if* $b \oplus a \in B \oplus A_i = \tilde{A}_i$; $i \in \{1, 2\}$.

Let A *be the universal free product of* $(A_i)_{i\in\{1,2\}}$ *with* B *amalgamated.*

Then the $*$*-algebraic amalgamated c-free product* $\varphi_1 \;{}_{\varphi_1} *_{\psi_2} \varphi_2 := \widetilde{\varphi_1} \;{}_{\widetilde{\varphi_1}} *_{\widetilde{\psi_2}} \widetilde{\varphi_2} \mid A$ *is a* B-B*-bimodule Schwarz map.* □

COROLLARY 3.17. *Let* A_i *be two unital (complex)* $*$*-algebras, such that* $A_2 = C \oplus A_2^\circ$ *(direct sum of linear spaces),* A_2° *being an algebra too, and* φ_i *be states of* A_i.

Let $\varphi = \varphi_1 \lhd \varphi_2$ *be their unital anti-monotone product, defined on the* $*$*-algebra* $A := A_1 *_1 A_2$. *Then* $\varphi(a^*a) \ge |\varphi(a)|^2$, *for all* $a \in A$. *Thus,* φ *is a state, too.* □

We get the following statement via the dual of Proposition 3.2.

COROLLARY 3.18. *Let* B *be a* C^**-algebra. Let* A_i *be two* $*$*-algebras endowed with compatible* B-B*-bimodule structures, and* $\varphi_i : A_i \to B$ *be Hermitian* B-B*-bimodule Schwarz maps. Consider the* $*$*-algebras* $\tilde{A}_i := B \oplus A_i$, *with adjoined algebra* B, *define the conditional expectations* $\tilde{\varphi}_i$ *of* $\tilde{A}_i$ *onto* B, *and the* $*$*-homomorphism* δ_2 *of* $\tilde{A}_2$ *onto* B *by* $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$, *respectively* $\delta_2(b \oplus a) := b$; *if* $b \oplus a \in B \oplus A_i = \tilde{A}_i$, $i \in \{1, 2\}$.

Let A *be the universal free product of* $(A_i)_{i\in\{1,2\}}$ *with* B *amalgamated.*

Then the $*$*-algebraic amalgamated anti-monotone product of* B-B*-bimodule maps*

$$\varphi = \varphi_1 \lhd_B \varphi_2 := \widetilde{\varphi_1} \;{}_{\widetilde{\varphi_1}} *_{\delta_2}\; \widetilde{\varphi_2} \mid A$$

is a Schwarz map of A *in* B. *Thus,* φ *is a positive* B-B*-bimodule map. As quantum* B*-probability spaces,* $(A_1, \varphi_1) \lhd_B (A_2, \varphi_2) = (A, \varphi_1 \lhd_B \varphi_2)$. □

The assertion involving the anti-monotone product in Muraki's originary sense [17] is

COROLLARY 3.19. *Let* A_i *be two (complex)* $*$*-algebras, and* φ_i *be linear functionals on* A_i, *such that their unitizations are states.*

Let $\varphi = \varphi_1 \lhd \varphi_2$ be the anti-monotone product, defined on the $*$-algebra $A := A_1 *_0 A_2$. Then $\varphi(a^*a) \geq |\varphi(a)|^2$, for all $a \in A$. Thus, φ is positive, too. $\square$

From Theorem 3.10 and Theorem 3.14 we derive the next

COROLLARY 3.20. *Let B be a C^*-algebra. Let A_i be two $*$-algebras over B, and φ_i, ψ_i be positive conditional expectations of A_i onto B.*

Then the $*$-algebraic amalgamated ordered-free product $(\varphi_1, \psi_1) \curlywedge_B (\varphi_2, \psi_2) := (\varphi_1 \ {}_{\psi_1}\!*_{\varphi_2}\ \varphi_2\ , \psi_1\ {}_{\psi_1}\!*_{\varphi_2}\ \psi_2)$ consists of Schwarz maps; i.e., of positive conditional expectations of $A := A_1 *_B A_2$ onto B.

As quantum probability spaces over B, $(A_1, \varphi_1, \psi_1) \curlywedge_B (A_2, \varphi_2, \psi_2) = (A, (\varphi_1, \psi_1) \curlywedge_B (\varphi_2, \psi_2))$. $\square$

COROLLARY 3.21. *Let B be a C^*-algebra. Let A_i be two $*$-algebras endowed with compatible B-B-bimodule structures, and φ_i, ψ_i be Hermitian B-B-bimodule Schwarz maps of A_i in B.*

Consider the $*$-algebras $\tilde{A}_i := B \oplus A_i$, with adjoined algebra B, and the conditional expectations $\tilde{\varphi}_i$, $\tilde{\psi}_i$ of $\tilde{A}_i$ onto B given by $\tilde{\varphi}_i(b \oplus a) := b + \varphi_i(a)$, and $\tilde{\psi}_i(b \oplus a) := b + \psi_i(a)$; if $b \oplus a \in B \oplus A_i = \tilde{A}_i$.

Let A be the universal free product of $(A_i)_{i \in \{1,2\}}$ with B amalgamated.

Then the $*$-algebraic amalgamated ordered-free product $(\varphi_1, \psi_1) \curlywedge_B (\varphi_2, \psi_2) := (\widetilde{\varphi_1}\ {}_{\widetilde{\psi_1}}\!*_{\widetilde{\varphi_2}}\ \widetilde{\varphi_2} \mid A,\ \widetilde{\psi_1}\ {}_{\widetilde{\psi_1}}\!*_{\widetilde{\varphi_2}}\ \widetilde{\psi_2} \mid A)$ consists of Hermitian B-B-bimodule Schwarz maps.

As quantum B-probability spaces,

$$(A_1, \varphi_1, \psi_1) \curlywedge_B (A_2, \varphi_2, \psi_2) = (A, (\varphi_1, \psi_1) \curlywedge_B (\varphi_2, \psi_2)). \quad \square$$

In particular, we obtain the fact below concerning the ordered-free product in Hasebe's sense [8].

COROLLARY 3.22. *Let A_i be two unital (complex) $*$-algebras, endowed with pairs of states φ_i, ψ_i. Then the ordered-free product $(\varphi_1, \psi_1) \curlywedge (\varphi_2, \psi_2) := (\varphi_1\ {}_{\psi_1}\!*_{\varphi_2}\ \varphi_2, \psi_1\ {}_{\psi_1}\!*_{\varphi_2}\ \psi_2)$ consists of states, too.* $\square$

Remark 3.23. One can show that Voiculescu's GNS construction [27] can also be performed for all unital positive conditional expectation defined on a unital $*$-algebra A satisfying the Combes axiom and valued onto a $*$-subalgebra of A containing the unit of A, such that this subalgebra has a structure of C^*-algebra.

Therefore, when A_i are C^*-algebras, the above amalgamated c-free, monotone or anti-monotone, and ordered-free product maps extend to corresponding Schwarz maps on the amalgamated universal (full) free product C^*-algebra $*_B A_i$, or the involved C^*-algebra $\star_{i \in I}(A_i,\ \varepsilon_i = 0,\ B)$; via such a GNS type construction (using, when it is necessary, Proposition 2.1). $\square$

In the same way as above, one can prove that the general amalgamated conditionally free product of some extended B-B-bimodule maps (in particular, conditional expectations) defined on $*$-algebras and valued in C^*-algebras preserves the (complete) positivity; thus, in the same context, one get, for example, that the amalgamated indented product (in Hasebe's originary sense [8], for the scalar-valued case) preserves the (complete) positivity; and these statements are also true in C^*-algebraic setting. Moreover, due to the associativity of the amalgamated indented product, these facts are valable for more than two pairs of maps.

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