ALGEBRAIC AND TOPOLOGICAL REFLEXIVITY OF SPACES OF LIPSCHITZ FUNCTIONS

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We establish algebraic and topological reflexivity for sets of isometries between scalar valued Lipschitz function spaces.

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1. INTRODUCTION

Given a Lipschitz function f between two metric spaces (X, d) and (Y, D), there exists a positive constant K such that

$$(*) \qquad D(f(x_0), f(x_1)) \leq K d(x_0, x_1), \text{ for every } x_0 \text{ and } x_1 \text{ in } X.$$

The infimum of all numbers K for which the inequalities in $(*)$ hold is called the Lipschitz constant of f, and is denoted by $L(f)$,

$$L(f) = \sup_{x_0 \neq x_1} \frac{D(f(x_0), f(x_1))}{d(x_0, x_1)}.$$

If $L(f) < 1$ then f is said to be a contraction. A bijective function f is a lipeomorphism if both f and f^{-1} satisfy a Lipschitz condition $(*)$. Given a compact metric space (X, d) we denote by $\mathrm{Lip}(X)$ the Banach space of all complex valued Lipschitz functions on X with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$. Throughout this paper 1_X denotes the function constantly equal to 1 on X.

In [13], the authors give a characterization of linear isometries between Banach spaces of scalar valued Lipschitz functions.

THEOREM 1.1 (cf. [13]). *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a linear isometry.*

(1) *If $T(1_X)$ is a contraction, then there exist Y_0, a closed subset of Y, a surjective Lipschitz map $\varphi : Y_0 \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$, and a function $\tau \in \mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$, for all $y \in Y_0$, such that*

$$T(f)(y) = \tau(y) f(\varphi(y)) \text{ for all } f \in \mathrm{Lip}(X) \text{ and } y \in Y_0.$$

(2) *If T is surjective and $T(1_X)$ is a nonvanishing contraction, then there exist $\tau \in \mathrm{Lip}(Y)$, with $|\tau(y)| = 1$ for all $y \in Y$, and φ a lipeomorphism from Y onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$, such that*

$$T(f)(y) = \tau(y)f(\varphi(y)) \text{ for all } f \in \mathrm{Lip}(X) \text{ and } y \in Y.$$

We use these representations to study algebraic and topological properties of classes of isometries between Lipschitz function spaces. Specifically we address the issue of algebraic reflexivity and topological reflexivity of subsets of the surjective isometries between $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$. The notion of algebraic reflexivity has been a topic of considerable interest and we refer the reader to the work done in the papers [3, 4] and references [5] through [8].

The restriction concerning $T(1_X)$ stated in the Theorem 1.1 is essential. If $T(1_X)$ is not a contraction, then isometries need not be weighted composition operators, see Weaver [14]. This distinguishes Lipschitz spaces from many classical function spaces where isometries are weighted composition operators and hence have the disjointness preserving property, see [1] and [9]. We give an example below showing that even the simplest disjointness preserving weighted composition operator fails to be an isometry. We first discuss some interesting examples that motivated the problems addressed in this paper.

Examples. Let $X = [0, \frac{1}{3}]$ and $Y = [0, 1]$. We define

$$\varphi(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{3}, \\ -\frac{1}{2}(x-1) & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}$$

and $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ given by $T(f)(y) = f(\varphi(y))$. The operator T is an non-surjective isometry. We observe that given $f \in \mathrm{Lip}(X)$, $T(f)$ restricted to the interval $[0, \frac{1}{3}]$ is identically equal to f and on $[\frac{1}{3}, 1]$, φ compresses the interval by a factor less than 1. These conditions are sufficient to ensure that T is an isometry, since τ is constant and equal to 1. This is not necessarily true for nonconstant Lipschitz functions τ. In fact, let $X = Y = [0, 1]$ and $S(f)(y) = \mathrm{e}^{\mathrm{i}y} f(y)$. We show that T_1 is not an isometry. We just consider $f(x) = \mathrm{i}x$. Clearly, $\|f\| = 1$, but $T_1(f)(y) = \mathrm{e}^{\mathrm{i}y}\,(\mathrm{i}y)$ has the Lipschitz constant greater than 1, $L(T_1(f)) > 1$. We notice that $T(1_X)$ is a contraction however the Lipschitz constant of $T_1(1_X)$ is equal to 1.

We also have that for $X = [0, \frac{1}{2}]$, $Y = [0, 1]$ and $\varphi(y) = \frac{y^2}{2}$, the operator $T_2(f)(y) = f(\varphi(y))$ is not an isometry, since for $f(x) = -(1-x)^2$, we have $\|f\| = 2$ and $\|T_2(f)\| < 2$. This example shows that the conditions in the Theorem 1.1 are not sufficient for a weighted composition operator to be an isometry.

2. ALGEBRAIC REFLEXIVITY OF LIPSCHITZ SPACES

In this section we consider classes of operators from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$, which are locally given by surjective isometries. We first set some preliminary notation to be used throughout this paper. We denote by $\mathcal{L}(\mathrm{Lip}(X),\mathrm{Lip}(Y))$ the set of all bounded linear operators between $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$ and by $\mathcal{G}(\mathrm{Lip}(X),\mathrm{Lip}(Y))$ the set of all surjective linear isometries between $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$. We also set

$$\mathcal{G}_*(\mathrm{Lip}(X),\mathrm{Lip}(Y)) =$$
$$= \{T \in \mathcal{G}(\mathrm{Lip}(X),\mathrm{Lip}(Y)) : T(1_X) \text{ is a nonvanishing contraction}\}.$$

If $X = Y$ we denote $\mathcal{G}_*(\mathrm{Lip}(X),\mathrm{Lip}(Y))$ simply by $\mathcal{G}_*(\mathrm{Lip}(X))$.

Definition. An operator $T \in \mathcal{L}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is locally in $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ if and only if for every $f \in \mathrm{Lip}(X)$ there exists $S_f \in \mathcal{G}_*(\mathrm{Lip}(X),\mathrm{Lip}(Y))$ such that $T(f) = S_f(f)$. The set $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is algebraically reflexive if and only if every operator locally in $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is also in $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$.

Proposition 2.1. *Let X and Y be compact metric spaces. If T is locally in $\mathcal{G}_*(\mathrm{Lip}(X),\mathrm{Lip}(Y))$, then there exist a closed subset Y_0 of Y, a Lipschitz function $\tau \in \mathrm{Lip}(Y_0)$ with $|\tau(y)| = 1$, for all $y \in Y_0$, and a lipeomorphism φ from Y_0 onto X such that*

$$T(f)(y) = \tau(y)\, f(\varphi(y)) \textit{ for every } f \in \mathrm{Lip}(X) \textit{ and } y \in Y_0.$$

Proof. Theorem 1.1 (1) asserts the existence of $\tau \in \mathrm{Lip}(Y)$ and the existence of an onto map $\varphi \in \mathrm{Lip}(Y_0, X)$ (with Y_0 a closed subset of Y) such that for every $f \in \mathrm{Lip}(X)$ and $y \in Y_0$

$$T(f)(y) = \tau(y)\, f(\varphi(y)). \tag{1}$$

For every $f \in \mathrm{Lip}(X)$, Theorem 1.1 (2) asserts the existence of τ_f and the existence of a lipeomorphism φ_f in $\mathrm{Lip}(Y, X)$, such that for every $y \in Y$

$$T(f)(y) = \tau_f(y)\, f(\varphi_f(y)). \tag{2}$$

Given $\xi \in Y_0$ with $\varphi(\xi) = x_0$ we define the following Lipschitz function on X

$$f(z) = \max\left\{0, 1 - \tfrac{1}{2}d(z, x_0)\right\}.$$

We observe that $\|f\| = \|f\|_\infty = 1$, since $L(f) \le \frac{1}{2}$. Furthermore, x_0 is the unique point in X at which f attains the value 1. Therefore, for every $y \in Y_0$, equations (1) and (2) imply

$$\tau(y)\, f(\varphi(y)) = \tau_f(y)\, f(\varphi_f(y)). \tag{3}$$

In particular, for $y = \xi$ we obtain $\tau(\xi) = \tau_f(\xi)\, f(\varphi_f(\xi))$ and hence $f(\varphi_f(\xi)) = 1$. This implies that $\varphi_f(\xi) = x_0$. If there exists $\xi_1 \in Y_0$, with $\varphi(\xi_1) = x_0$ then

$\varphi_f(\xi_1) = x_0$. We conclude that $\xi = \xi_1$ since φ_f is a injective. Therefore, φ is also injective. We have shown that $\varphi : Y_0 \to X$ is Lipschitz and bijective. The compactness of Y_0 implies that φ is a homeomorphism. We need to show that φ^{-1} is also Lipschitz. For a given $z \in Y_0$, we define the Lipschitz function on X, given by $f_z(x) = d(x, \varphi(z))$. Equation (3) implies that $d(\varphi(y), \varphi(z)) = d(\varphi_f(y), \varphi(z))$, and hence $\varphi_f(z) = \varphi(z)$. For $y \neq z$ we have

$$\frac{d(\varphi(y), \varphi(z))}{d(y,z)} = \frac{d(\varphi_f(y), \varphi(z))}{d(y,z)} \geq \frac{d(\varphi_f(y), \varphi_f(z))}{d(y,z)}.$$

We use the fact that φ_f is a lipeomorphism to assure the existence of a positive number K_f for which

$$\frac{d(\varphi_f(y), \varphi_f(z))}{d(y,z)} \geq K_f \text{ for every } y \in Y \setminus \{z\}.$$

We set $\widetilde{Y_0} = \{(y_1, y_2) \in Y_0 \times Y_0 : y_1 \neq y_2\}$, and $\beta\widetilde{Y_0}$ denotes the Stone-Čech compactification of $\widetilde{Y_0}$. Now we consider the function $F : \beta\widetilde{Y_0} \to \mathbb{R}$ given by

$$F(y_1, y_2) = \begin{cases} \frac{d(\varphi(y_1), \varphi(y_2))}{d(y_1, y_2)} & \text{if } (y_1, y_2) \in \widetilde{Y_0}, \\ \beta(F)(w) & \text{if } w \in \beta(\widetilde{Y_0}) \setminus \widetilde{Y_0}, \end{cases}$$

where $\beta(F)(w)$ represents the unique extension of F, restricted to $\widetilde{Y_0}$, to the point w. If 0 is in the range of F, then there exists $\xi \in \beta(\widetilde{Y_0}) \setminus \widetilde{Y_0}$ so that $F(\xi) = 0$. Therefore, there exists a net $\{(y_\alpha, z_\alpha)\}$ converging to ξ and so that $F(y_\alpha, z_\alpha)$ converges to zero. There exists a subnet of $\{z_\alpha\}$, also denoted by $\{z_\alpha\}$, which converges to a point in X, say z_0. Previous considerations imply $F(y, z_0) > N_{z_0} > 0$, for every $y \neq z_0$ and thus $F(y_\alpha, z_\alpha) > \frac{1}{2}N_{z_0}$, for sufficiently large α. This contradiction implies the existence of a positive number N such that for every y and z with $y \neq z$ we have

$$\frac{d(\varphi(y), \varphi(z))}{d(y,z)} > N.$$

Therefore, φ is a lipeomorphism between Y_0 and X. □

THEOREM 2.1. *Let X and Y be compact metric spaces.*

(1) *If there exists an injective real valued function $f \in \mathrm{Lip}(X)$, then $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is algebraically reflexive.*

(2) *If Y is an n-dimensional compact and connected manifold without boundary, then $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is algebraically reflexive.*

Proof. We consider an operator T, locally in $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$. We assume that there exists an injective real valued function f on X. Without loss of generality we suppose that f is positive. Proposition 2.1 asserts the existence of a closed subset Y_0 of Y, a Lipschitz function $\tau \in \mathrm{Lip}(Y_0)$ with

$|\tau(y)| = 1$, for all $y \in Y_0$, and a lipeomorphism φ from Y_0 onto X such that, for every $y \in Y_0$,

$$T(f)(y) = \tau(y)\, f(\varphi(y)).$$

On the other hand, there exists a surjective isometry S_f so that $S_f(1_X)$ is a non-vanishing contraction and

$$T(f)(y) = \tau(y)\, f(\varphi(y)) = S_f(f)(y).$$

Theorem 1.1 asserts the existence $\tau_f \in \mathrm{Lip}(Y)$ with $|\tau_f(y)| = 1$, for all $y \in Y$, and a lipeomorphism φ_f such that

$$S_f(f)(y) = \tau_f(y) f(\varphi_f(y)) \text{ for all } y \in Y.$$

Therefore, for $y \in Y_0$, $\tau_f(y)f(\varphi_f(y)) = \tau(y)f(\varphi(y))$ and $f(\varphi_f(y)) = f(\varphi(y))$. Furthermore, since f is injective, we have $\varphi(y) = \varphi_f(y)$. Since φ_f is a lipeomorphism and φ is onto, we have $Y_0 = Y$. We now show that T is surjective. Given $g \in \mathrm{Lip}(Y)$, we define $h(x) = \overline{\tau(\varphi^{-1}(x))} g(\varphi^{-1}(x))$. The function h is Lipschitz and $T(h)(y) = g(y)$. This completes the proof of (1). Now we assume that Y is a connected and compact n-manifold with empty boundary. Proposition 2.1 and Theorem 1.1 imply that Y is homeomorphic to Y_0, a closed subset of Y. This also implies that Y_0 is a compact and connected n-manifold without boundary. Hence Y_0 has empty interior in Y, equivalently Y_0 is both open and closed in Y. The connectedness assumption on Y_0 implies that $Y = Y_0$. □

Remark. Closed and bounded subsets of $\mathbb{R}$, satisfy the condition stated in Theorem 2.1 (1). Therefore, $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is algebraically reflexive for X a compact subset of $\mathbb{R}$ and Y an arbitrary compact metric space.

We also determine the algebraic reflexivity of a class of periodic isometries. We consider a compact metric space (X, d) and $n \in \mathbb{N}$. We define

$$\mathcal{P}_n(\mathrm{Lip}(X)) = \{T \in \mathcal{G}_*(\mathrm{Lip}(X)) : T^n = \mathrm{Id}_{\mathrm{Lip}(X)}\}.$$

We first derive a straightforward representation for isometries in $\mathcal{P}_n(\mathrm{Lip}(X))$, based in the Theorem 1.1,

Lemma 2.1. *If* $T \in \mathcal{P}_n(\mathrm{Lip}(X))$ *then there exist* φ *a lipeomorphism on* X, $\tau \in \mathrm{Lip}(X)$ *a Lipschitz map such that* $\varphi^n = \mathrm{Id}_X$, $\tau(x)\tau(\varphi(x)) \cdots \tau(\varphi^{n-1}(x)) = 1$, *and* $T(f)(x) = \tau(x)f(\varphi(x))$, *for all* $x \in X$.

Proof. If T is an isometry in $\mathcal{P}_n(\mathrm{Lip}(X))$, it can be represented as:

$$T(f)(x) = \tau(x)f(\varphi(x)) \text{ for all } f \in \mathrm{Lip}(X) \text{ and } x \in X, \tag{4}$$

with τ and φ as given in the Theorem 1.1 (2). We have that $T^n(f)(x) = \tau(x)\tau(\varphi(x)) \cdots \tau(\varphi^{n-1}(x))f(\varphi^n(x)) = f(x)$, for every $f \in \mathrm{Lip}(X)$. If there exists x such that $\varphi^n(x) \neq x$, then the Lipschitz function $f(z) = d(z, \varphi^n(x))$ would not satisfy Equation (4). This shows that $\varphi^n = \mathrm{Id}_X$. We also have that $T^n 1_X(x) = \tau(x)\tau(\varphi(x)) \cdots \tau(\varphi^{n-1}(x)) = 1$, for every $x \in X$. □

Definition. A linear operator $T \in \mathcal{L}(\mathrm{Lip}(X))$ is locally a surjective periodic isometry if and only if for every $f \in \mathrm{Lip}(X)$ there exists a surjective isometry T_f such that $T_f(1_X)$ is a nonvanishing contraction, $T_f^n = \mathrm{Id}$, for some $n \in \mathbb{N}$, and $T(f) = T_f(f)$.

Proposition 2.2. *Let X be a compact metric space and f be an injective real valued Lipschitz function defined on X. If T is locally a surjective periodic isometry then T is a surjective periodic isometry and $T(1_X)$ is a nonvanishing contraction.*

Proof. Without loss of generality we assume that f is strictly positive. Theorem 1.1 asserts that there exist X_0 a closed subset of X and Lipschitz maps τ and φ such that

$$T(f)(x) = \tau(x)f(\varphi(x)) \text{ for all } x \in X_0.$$

Since T is locally a surjective periodic isometry, there exists $T_f \in \mathcal{P}_n(\mathrm{Lip}(X))$ such that $T(f) = T_f(f)$. We consider the representation for T_f as stated in Theorem 1.1

$$T_f(f)(x) = \tau_f(x)f(\varphi_f(x)) \text{ for all } x \in X.$$

Therefore, $f(\varphi(x)) = f(\varphi_f(x))$ which implies that $\varphi(x) = \varphi_f(x)$ for all $x \in X_0$. Therefore, $X = X_0$ and φ is a periodic lipeomorphism, i.e., $\varphi^n = \mathrm{Id}_X$. Previous considerations also imply that $\tau(x) = \tau_f(x)$ for all $x \in X$. Consequently, $T(g)(x) = \tau_f(x)g(\varphi_f(x))$, for every $g \in \mathrm{Lip}(X)$ and $x \in X$. We have shown that T is a surjective periodic isometry. Since $T(1_X) = T_{1_X}(1_X)$ we have that $T(1_X)$ is a nonvanishing contraction. □

Remark. The previous results also imply that $\mathcal{P}_n(\mathrm{Lip}(X))$ is an algebraically reflexive subset of $\mathcal{G}_*(\mathrm{Lip}(X))$.

3. TOPOLOGICAL REFLEXIVITY OF SUBSETS OF THE ISOMETRY GROUP

We consider a pair of spaces of scalar valued Lipschitz functions defined on compact metric spaces and we study topological properties of isometries between these spaces. We start with a definition of topological surjective isometry. For related concepts see [2], [5], [7] or [11].

Definition. We say that an operator $T \in \mathcal{L}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is a topologically surjective isometry if and only if $T(1_X)$ is a contraction and for every $f \in \mathrm{Lip}(X)$ there exists a sequence $\{T_n^f\}_n$, in $\mathcal{G}_*(\mathrm{Lip}(X), \mathrm{Lip}(Y))$, such that $T(f) = \lim_n T_n^f(f)$.

Lemma 3.1. *If T is a topologically surjective isometry in $\mathcal{L}(\mathrm{Lip}(X),$ $\mathrm{Lip}(Y))$, then T is an isometry.*

Proof. Given $f \in \mathrm{Lip}(X)$, let $\{T_n^f\}$ be a sequence of surjective isometries such that

$$\lim_{n\to\infty} \|T_n(f) - T(f)\| = 0.$$

Therefore $|\|T_n(f)\| - \|T(f)\|| = |\|f\| - \|T(f)\|| \leq \|T_n(f) - T(f)\|$ and T is an isometry. □

Proposition 3.1. *If X is a compact metric space and Y is a compact and connected n-manifold without boundary, then $T \in \mathcal{L}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is surjective provided that T is a topologically surjective isometry.*

Proof. For every $f \in \mathrm{Lip}(X)$ there exists T_n^f a sequence of surjective isometries such that $T_n^f(1_X)$ is a nonvanishing contraction and $T(f) = \lim_n T_n^f(f)$. Theorem 1.1 (1) asserts that there exist Y_0 a closed subset of Y, $\varphi : Y_0 \to X$ a surjective Lipschitz function with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$, $\tau \in \mathrm{Lip}(Y)$, with $\|\tau\| = 1$, $|\tau(y)| = 1$ for all $y \in Y_0$, such that

$$T(f)(y) = \tau(y)\, f(\varphi(y)) \text{ for all } y \in Y_0.$$

Theorem 1.1 (2) also asserts the existence of a sequence of Lipchitz functions of norm 1, $\{\tau_n^f\}$ and a sequence of lipeomorphisms φ_n^f such that $L(\varphi_n^f) \leq \max\{1, \mathrm{diam}(X)\}$, $L((\varphi_n^f)^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$, and

$$T_n^f(g)(y) = \tau_n^f(y)\, g(\varphi_n^f(y)),$$

for all $g \in \mathrm{Lip}(Y)$ and $y \in Y$. We claim that φ is injective. We suppose there exist y_0 and y_1 in Y_0 such that $y_0 \neq y_1$ and $\varphi(y_0) = \varphi(y_1)$. We set $f(z) = d(z, \varphi(y_0)) \in \mathrm{Lip}(X)$. For this function we associate the sequence $\{T_n^f\}_n$ as described before. Hence

$$\lim_n \tau_n^f(y) f(\varphi_n^f(y)) = \tau(y) f(\varphi(y)) \text{ for all } y \in Y_0. \tag{5}$$

This implies that

$$\lim_n d(\varphi_n^f(y), \varphi(y_0)) = d(\varphi(y),\, \varphi(y_0)) \text{ for all } y \in Y_0.$$

In particular,

$$\lim_n d(\varphi_n^f(y_0), \varphi(y_0)) = \lim_n d(\varphi_n^f(y_1), \varphi(y_0)) = 0.$$

Therefore,

$$\lim_n \frac{d\big(\varphi_n^f(y_0), \varphi_n^f(y_1)\big)}{d(y_0, y_1)} = 0.$$

We choose n_0 so that

$$\frac{d\big(\varphi_{n_0}^f(y_0),\, \varphi_{n_0}^f(y_1)\big)}{d(y_0,\, y_1)} < \frac{1}{2\max\{\mathrm{diam}(Y), 1\}},$$

which implies that

$$\max\{\operatorname{diam}(Y),1\} \geq L((\varphi^f_{n_0})^{-1}) \geq \frac{d(y_0,y_1)}{d\big(\varphi^f_{n_0}(y_0),\varphi^f_{n_0}(y_1)\big)} > 2\max\{\operatorname{diam}(Y),1\}.$$

This leads to a contradiction. Therefore, φ is an injective Lipschitz map from Y_0 onto X. We have that Y_0 is homeomorphic to X, since X is homeomorphic to Y. Hence Y and Y_0 are homeomorphic. The assumption on Y implies that $Y = Y_0$. It now remains to show that φ^{-1} is also a Lipschitz function.

We assume that there exist sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n \neq z_n$ and $\lim_n \frac{d(\varphi(y_n),\varphi(z_n))}{d(y_n,z_n)} = 0$. We choose n_0 such that

$$\frac{d(\varphi(y_{n_0}),\varphi(z_{n_0}))}{d(y_{n_0},\, z_{n_0})} < \frac{1}{6\max\{1,\operatorname{diam}(Y)\}}.$$

We set $f(z) = \overline{\tau(z)}\, d(z,\, \varphi(z_{n_0}))$ in $\operatorname{Lip}(X)$. We consider the sequence of surjective isometries $\{T^f_n\}_n$ associated with f, as stated in Theorem 1.1 (2),

$$T^f_n(g)(y) = \tau^f_n(y)\, g(\varphi^f_n(y)) \text{ for every } n \in N \text{ and } g \in \operatorname{Lip}(X).$$

In particular, we have $\lim_n \|T^f_n(f) - T(f)\| = 0$. This implies

(i) $\lim_n \|T^f_n(f) - T(f)\|_\infty = 0$, equivalently $\|\tau^f_n(\cdot) f(\varphi^f_n(\cdot)) - \tau(\cdot) f(\varphi(\cdot))\|_\infty \to 0$, and

(ii) $\lim_n L(T^f_n(f) - T(f)) = 0$.

We set $H_n(w) = \tau^f_n(w)\, f(\varphi^f_n(w))$. The statement in (i) becomes

$$\lim_n \|H_n(w) - d(\varphi(w),\varphi(z_{n_0}))\|_\infty = 0.$$

Therefore, for every m,

$$\lim_n H_n(y_m) = d(\varphi(y_m),\varphi(z_{n_0})) \quad \text{and} \quad \lim_n H_n(z_m) = d(\varphi(z_m),\varphi(z_{n_0})).$$

In particular, for $m = n_0$ we have

$$\lim_n H_n(y_{n_0}) = d(\varphi(y_{n_0}),\varphi(z_{n_0})) \quad \text{and} \quad \lim_n H_n(z_{n_0}) = 0.$$

On the other hand, the statement in (ii) implies that

$$\lim_n \frac{|H_n(y_{n_0}) - d(\varphi(y_{n_0}),\varphi(z_{n_0})) - H_n(z_{n_0})|}{d(y_{n_0},\, z_{n_0})} = 0,$$

equivalently,

$$\lim_n \frac{|H_n(y_{n_0}) - H_n(z_{n_0})|}{d(y_{n_0},\, z_{n_0})} = \frac{d(\varphi(y_{n_0}),\, \varphi(z_{n_0}))}{d(y_{n_0},\, z_{n_0})}.$$

We choose n such that

$$\frac{|H_n(z_{n_0})|}{d(y_{n_0}, z_{n_0})} < \frac{1}{6\max\{1,\operatorname{diam}(Y)\}}$$

and

$$\frac{|H_n(y_{n_0})|}{d(y_{n_0},z_{n_0})} < \frac{d(\varphi(y_{n_0}),\varphi(z_{n_0}))}{d(y_{n_0},z_{n_0})} + \frac{1}{6\max\{1,\operatorname{diam}(Y)\}}.$$

Therefore,

$$\frac{d\big(\varphi_n^f(z_{n_0}),\varphi(z_{n_0})\big)}{d(y_{n_0},z_{n_0})} < \frac{1}{6\max\{1,\operatorname{diam}(Y)\}}$$

and

$$\frac{d\big(\varphi_n^f(y_{n_0}),\varphi(z_{n_0})\big)}{d(y_{n_0},z_{n_0})} < \frac{1}{3\max\{1,\operatorname{diam}(Y)\}},$$

which implies that

$$\frac{d\big(\varphi_n^f(y_{n_0}),\varphi_n^f(z_{n_0})\big)}{d(y_{n_0},z_{n_0})} < \frac{1}{2\max\{1,\operatorname{diam}(Y)\}}.$$

We now conclude that

$$\max\{1,\operatorname{diam}(Y)\} \geq L((\varphi_n^f)^{-1}) \geq \frac{d(y_{n_0},z_{n_0})}{d\big(\varphi_n^f(y_{n_0}),\varphi_n^f(z_{n_0})\big)} > 2\max\{1,\operatorname{diam}(Y)\}.$$

This contradiction implies that $\left\{\frac{d(\varphi(y),\varphi(z)))}{d(y,z)} : \text{with } y \neq x\right\}$ is bounded below by a positive number. Hence we define the operator $S : \operatorname{Lip}(Y) \to \operatorname{Lip}(X)$ as follows $S(g)(x) = \overline{\tau(\varphi^{-1}(x))}\, g(\varphi^{-1}(x))$. We have $T(S(g))(y) = g(y)$ and $S(T(f))(x) = f(x)$. The operator S is an isometry and T is onto. This completes the proof. □

Definition. We set $\mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y)) = \{T \in \mathcal{G}_*(\operatorname{Lip}(X),\operatorname{Lip}(Y)) : T(1_X) = 1_Y\}$. We say that $\mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y))$ is topologically reflexive whenever the following implication is true, cf. [6]:

If for every $f \in \operatorname{Lip}(X)$ there exists a sequence $\{T_n^f\}$ of isometries in $\mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y))$ such that $T(f) = \lim_n T_n^f(f)$, then $T \in \mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y))$.

Corollary 3.1. *If X is a compact metric space and Y is a compact and connected n-manifold without boundary, then $\mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y))$ is topologically reflexive.*

Proof. If T is an isometry and $f \in \operatorname{Lip}(X)$, then there exists a sequence T_n^f in $\mathcal{G}_1(\operatorname{Lip}(X),\operatorname{Lip}(Y))$ such that $T(f) = \lim_n T_n^f(f)$. Proposition 3.1 asserts that T is a surjective isometry which clearly satisfies $T(1_X) = 1_Y$ and completes the proof. □

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