ON THE STRUCTURE OF GAUSSIAN RANDOM VARIABLES

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We study when a given Gaussian random variable on a given probability space $(\Omega, \mathcal{F}, P)$ is equal almost surely to β_1 where β is a Brownian motion defined on the same (or possibly extended) probability space. As a consequence of this result, we prove that the distribution of a random variable in a finite sum of Wiener chaoses cannot be normal. This result also allows to understand better a characterization of the Gaussian variables obtained via Malliavin calculus.

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1. INTRODUCTION

We study when a Gaussian random variable defined on some probability space can be expressed almost surely as a Wiener integral with respect to a Brownian motion defined on the same space. The starting point of this work is provided by some recent results related to the distance between an arbitrary random variable X and the Gaussian law. This distance can be defined in various ways (the Kolmogorov distance, the total variations distance or others) and it can be expressed in terms of the Malliavin derivative DX of the random variable X when this derivative exists. These results lead to a characterization of Gaussian random variables through Malliavin calculus. Let us briefly recall the context. Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and let $(W_t)_{t\in[0,1]}$ be an $\mathcal{F}_t$ Brownian motion on this space, where $\mathcal{F}_t$ is its natural filtration. Equivalent conditions for the standard normality of a centered random variable X with variance 1 are the following: $\mathbf{E}\left(1 - \langle DX, D(-L)^{-1}\rangle_{L^2(\Omega)} | X\right) = 0$ or $\mathbf{E}\left(f_z'(X)(1 - \langle DX, D(-L)^{-1}\rangle_{L^2(\Omega)}\right) = 0$ for every z where D denotes the Malliavin derivative, L is the Ornstein-Uhlenbeck operator and the deterministic function f_z' is the solution of the Stein's equation (see [4]). This characterization is of course interesting and it can be useful in some cases. It is also easy to understand it for random variables that are Wiener integrals with respect to W. Indeed, assume that $X = W(h)$ where h is a deterministic function in $L^2([0,1])$ with $\|h\| = 1$. In this case $DX = h = D(-L)^{-1}X$ and then $\langle DX, D(-L)^{-1}\rangle_{L^2(\Omega)} = 1$ and the above equivalent conditions for the normality of X are easily verified. In some other cases, it is difficult, even impossible, to compute the quantity $\mathbf{E}\left(\langle DX, D(-L)^{-1}\rangle_{L^2(\Omega)}|X\right)$ or $\mathbf{E}\left(f_z'(X)(1-\langle DX, D(-L)^{-1}\rangle_{L^2(\Omega)}\right) = 0$. Let us consider for example the case of the random variable $Y = \int_0^1 \mathrm{sign}(W_s)\mathrm{d}W_s$. This is not a Wiener integral with respect to W. But it is well-known that it is standard Gaussian because the process $\beta_t = \int_0^t \mathrm{sign}(W_s)\mathrm{d}W_s$ is a Brownian motion as follows from the Lévy's characterization theorem. The chaos expansion of this random variable is known and it is recalled in Section 2. In fact Y can be expressed as an infinite sum of multiple Wiener-Itô stochastic integrals and it is impossible to check if the equivalent conditions for its normality are satisfied (it is even not differentiable in the Malliavin calculus sense). The phenomenon that happens here is that Y can be expressed as the value at time 1 of the Brownian motion β which is actually the Dambis-Dubbins-Schwarz (DDS in short) Brownian motion associated to the martingale $M^Y = (M_t^Y)_{t\in[0,1]}$, $M_t^Y = \mathbf{E}(Y|\mathcal{F}_t)$ (recall that $\mathcal{F}_t$ is the natural filtration under of W and β is defined on the same space Ω (or possibly on a extension of Ω) and is a $\mathcal{G}_s$-Brownian motion with respect to the filtration $\mathcal{G}_s = \mathcal{F}_{T(s)}$ where $T(s) = \inf(t \in [0,1]; \langle M^Y\rangle_t \geq s)$). This leads us to the following question: is any standard normal random variable X representable as the value at time 1 of the Brownian motion associated, via the Dambis-Dubbins-Schwarz theorem, to the martingale M^X, where for every t

$$M_t^X = \mathbf{E}(X|\mathcal{F}_t)? \tag{1}$$

By combining the techniques of Malliavin calculus and classical tools of the probability theory, we found the following answer: if the bracket of the $\mathcal{F}_t$ martingale M^X is bounded a.s. then this property is true, that is X can be represented as its DDS Brownian motion at time 1. If the bracket of M^X is not bounded, then this property is not true. An example when it fails is obtained by considering the standard normal random variable $W(h_1)\,\mathrm{sign}(W(h_2))$ where h_1, h_2 are two orthonormal elements of $L^2([0,1])$. Nevertheless, we will prove that we can construct a bigger probability space Ω_0 that includes Ω and a Brownian motion on Ω_0 such that X is equal almost surely with this Brownian motion at time 1. The construction is done by means of the Karhunen-Loève theorem. Some consequences of this result are discussed here; we believe that these consequences could be various. We prove that the standard normal random variables such that the bracket of the corresponding DDS martingale is bounded cannot live in a finite sum of Wiener chaoses: they can be or in the first chaos, or in an infinite sum of chaoses. We also make a connection with some results obtained recently via Stein's method and Malliavin calculus.

We structured our paper as follows. Section 2 starts with a short description of the elements of the Malliavin calculus and it also contains our main result on the structure of Gaussian random variables. In Section 3 we discuss some consequences of our characterization. In particular we prove that the random variables whose associated DDS martingale has bounded bracket cannot belong to a finite sum of Wiener chaoses and we relate our work to recent results on standard normal random variables obtained via Malliavin calculus.

2. ON THE STRUCTURE OF GAUSSIAN RANDOM VARIABLES

Let us consider a probability space $(\Omega, \mathcal{F}, P)$ and assume that $(W_t)_{t\in[0,1]}$ is a Brownian motion on this space with respect to its natural filtration $(\mathcal{F}_t)_{t\in[0,1]}$. Let I_n denote the multiple Wiener-Itô integral with respect to W. The elements of the stochastic calculus for multiple integrals and of Malliavin calculus can be found in [3] or [6]. We will just introduce very briefly some notation. Other formulas from Malliavin calculus will be recalled in the paper at the times when they are used. We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n\geq 0} I_n(f_n), \tag{2}$$

where $f_n \in L^2([0,1]^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

The isometry of multiple integrals can be written as: for m, n positive integers and $f \in L^2([0,1]^n)$, $g \in L^2([0,1]^m)$

$$\begin{aligned} &\mathbf{E}\left(I_n(f)I_m(g)\right) = n!\langle f, g\rangle_{L^2([0,1])^{\otimes n}} && \text{if } m = n, \\ &\mathbf{E}\left(I_n(f)I_m(g)\right) = 0 && \text{if } m \neq n. \end{aligned} \tag{3}$$

It also holds that

$$I_n(f) = I_n\big(\tilde{f}\big),$$

where $\tilde{f}$ denotes the symmetrization of f defined by

$$\tilde{f}(x_1, \ldots, x_x) = \frac{1}{n!} \sum_{\sigma\in\mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We will need the general formula for calculating products of Wiener chaos integrals of any orders m, n for any symmetric integrands $f \in L^2([0,1]^{\otimes m})$

and $g \in L^2([0,1]^{\otimes n})$; it is

$$I_m(f)I_n(g) = \sum_{r=0}^{p\wedge q} r! C_m^r C_n^r I_{m+m-2r}(f \otimes_r g), \tag{4}$$

where the contraction $f \otimes_r g$ is defined by

$$(f \otimes_\ell g)(s_1, \dots, s_{n-\ell}, t_1, \dots, t_{m-\ell}) \tag{5}$$
$$= \int_{[0,T]^{m+n-2\ell}} f(s_1, \dots, s_{n-\ell}, u_1, \dots, u_\ell) g(t_1, \dots, t_{m-\ell}, u_1, \dots, u_\ell) \mathrm{d}u_1 \dots \mathrm{d}u_\ell.$$

Note that the contraction $(f \otimes_\ell g)$ is an element of $L^2([0,1]^{m+n-2\ell})$ but it is not necessary symmetric. We will by $(f \tilde{\otimes}_\ell g)$ its symmetrization.

We denote by $\mathbb{D}^{1,2}$ the domain of the Malliavin derivative with respect to W which takes values in $L^2([0,1]\times\Omega)$. We just recall that D acts on functionals of the form $f(X)$, with $X \in \mathbb{D}^{1,2}$ and f differentiable in the following way: $D_\alpha f(X) = f'(X) D_\alpha X$ for every $\alpha \in (0,1]$ and on multiple integrals $I_n(f)$ with $f \in L^2([0,1]^n)$ as $D_\alpha I_n(f) = nI_{n-1} f(\cdot, \alpha)$.

The Malliavin derivative D admits a dual operator which is the divergence integral $\delta(u) \in L^2(\Omega)$ if $u \in \mathrm{Dom}(\delta)$ and we have the duality relationship

$$\mathbf{E}(F\delta(u)) = \mathbf{E}\langle DF, u\rangle, \quad F \in \mathbb{D}^{1,2},\ u \in \mathrm{Dom}(\delta). \tag{6}$$

For adapted integrands, the divergence integral coincides with the classical Itô integral.

Let us fix the probability space $(\Omega, \mathcal{F}, P)$ and let us assume that the Wiener process $(W_t)_{t\in[0,1]}$ lives on this space. Let X be a centered square integrable random variable on this space. Assume that X is measurable with respect to the sigma-algebra $\mathcal{F}_1$. After Proposition 1.1 the random variable X will be assumed to have standard normal law.

The following result is an immediate consequence of the Dambis-Dubbins-Schwarz theorem (DDS theorem for short, see [2], Section 3.4, or [8], Chapter V).

Proposition 1.1. *Let X be a random variable in $L^1(\Omega)$. Then there exists a Brownian motion $(\beta_s)_{s\geq 0}$ (possibly defined on an extension of the probability space) with respect to a filtration $(\mathcal{G}_s)_{s\geq 0}$ such that*

$$X = \beta_{\langle M^X\rangle_1},$$

where $M^X = (M_t^X)_{t\in[0,1]}$ is the martingale given by (1). *Moreover the random time $T = \langle M^X\rangle_1$ is a stopping time for the filtration $\mathcal{G}_s$ and it satisfies $T > 0$ a.s. and $\mathbf{E}T = \mathbf{E}X^2$.*

Proof. Let $T(s) = \inf\big(t \geq 0, \langle M^X\rangle_t \geq s\big)$. By applying Dambis-Dubbins-Schwarz theorem

$$\beta_s := M_{T(s)}$$

is a standard Brownian motion with respect to the filtration $\mathcal{G}_s := \mathcal{F}_{T(s)}$ and for every $t \in [0,1]$ we have $M_t^X = \beta_{\langle M^X\rangle_t}$ a.s. P. Taking $t = 1$ we get

$$X = \beta_{\langle M^X\rangle_1} \quad \text{a.s..}$$

The fact that T is a $(\mathcal{G}_s)_{s\geq 0}$ stopping time is well known. It is true because $(\langle M^X\rangle_1 \leq s) = (T(s) \geq 1) \in \mathcal{F}_{T(s)} = \mathcal{G}_s$. Also clearly $T > 0$ a.s. and $\mathbf{E}T = \mathbf{E}X^2$. $\square$

In the sequel we will call the Brownian β obtained via the DDS theorem as the DDS Brownian associated to X.

Recall the Ocone-Clark formula: if X is a random variable in $\mathbb{D}^{1,2}$ then

$$X = \mathbf{E}X + \int_0^1 \mathbf{E}\left(D_\alpha X|\mathcal{F}_\alpha\right) \mathrm{d}W_\alpha. \tag{7}$$

Remark 1.1. If the random variable X has zero mean and it belongs to the space $\mathbb{D}^{1,2}$ then by the Ocone-Clark formula (7) we have

$$M_t^X = \int_0^t \mathbf{E}\left(D_\alpha X|\mathcal{F}_\alpha\right) \mathrm{d}W_\alpha$$

and consequently,

$$X = \beta_{\int_0^1 (\mathbf{E}(D_\alpha X|\mathcal{F}_\alpha))^2 \mathrm{d}\alpha},$$

where β is the DDS Brownian motion associated to X.

Assume from now on that $X \sim N(0,1)$. As we have seen, X can be written as the value at a random time of a Brownian motion β (which is fact the Dambis-Dubbins-Schwarz Brownian associated to the martingale M^X). Note that β has the time interval $\mathbb{R}_+$ even if W is indexed over $[0,1]$. So, if we know that β_T has a standard normal law, what can we say about the random time T? Is it equal to 1 almost surely? This is for example the case of the variable $X = \int_0^1 \operatorname{sign}(W_s)\mathrm{d}W_s$ because here, for every $t \in [0,1]$, $M_t^X = \int_0^t \operatorname{sign}(W_s)\mathrm{d}W_s$ and $\langle M^X\rangle_t = \int_0^t (\operatorname{sign}(B_s)^2 \mathrm{d}s = t$. Another situation when this is true is related to Bessel processes. Let $(B^{(1)}, \ldots, B^{(d)})$ be a d-dimensional Brownian motion and consider the random variable

$$X = \int_0^1 \frac{B_s^{(1)}}{\sqrt{(B_s^{(1)})^2 + \ldots + (B_s^{(d)})^2}} \mathrm{d}B_s^{(1)} + \cdots + \int_0^1 \frac{B_s^{(d)}}{\sqrt{(B_s^{(1)})^2 + \ldots + (B_s^{(d)})^2}} \mathrm{d}B_s^{(d)}. \tag{8}$$

It also satisfies $\langle M^X\rangle_t = t$ for every $t \in [0,1]$ and in particular $\langle M^X\rangle_1 = 1$ a.s.. We will see below that the fact that any $N(0,1)$ random variable is equal a.s. to β_1 (its associated DDS Brownian evaluate d at time 1) is true only for random variables for which the bracket of their associated DDS martingale is almost surely bounded.

We will assume the following condition on the stopping time T.

$$\text{There exist a constant } M > 0 \text{ such that } T \leq M \text{ a.s..} \tag{9}$$

The problem we address in this section is then the following: let $(\beta_t)_{t\geq 0}$ be a $\mathcal{G}_t$-Brownian motion and let T be a almost surely positive stopping time for its filtration such that $\mathbf{E}(T) = 1$ and T satisfies (9). We will show that $T = 1$ a.s..

Let us start with the following result.

PROPOSITION 1.2. *Assume that T satisfies* (9). *Then for any $\lambda \in \mathbb{R}$ we have*

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\beta_T\big) = \mathrm{i}\lambda\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}T\big). \tag{10}$$

Proof. By using the duality relation (6) between D and δ and noting that $\beta_T = \delta(1_{[0,T]}(\cdot))$ (note that we are now using Malliavin calculus with respect to the Wiener process β) we obtain for every t

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_t}\beta_T\big) = \mathbf{E}\big(\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda\beta_t}\langle D\beta_t, 1_{[0,T]}\rangle\big) = \mathbf{E}\big(\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda\beta_t}(T\wedge t)\big)$$

and letting $t \to M$ we obtain

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_M}\beta_T\big) = \mathbf{E}\big(\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda\beta_M}T\big).$$

But, since T and β_T and $\mathcal{G}_T$ measurable, we have

$$\begin{aligned}\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_M}\beta_T\big) &= \mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\mathrm{e}^{\mathrm{i}\lambda(\beta_M-\beta_T)}\beta_T\big) = \mathbf{E}\big(\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\mathrm{e}^{\mathrm{i}\lambda(\beta_M-\beta_T)}\beta_T|\mathcal{G}_T\big)\big)\\ &= \mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\beta_T\big)\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda(\beta_M-\beta_T)}\big)\end{aligned}$$

where we used the strong Markov property of the Brownian motion. Similarly,

$$\mathbf{E}\big(\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda\beta_M}T\big) = \mathbf{E}\big(\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda\beta_T}T\big)\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda(\beta_M-\beta_T)}\big).$$

Consequently, (10) is proved. $\square$

As a consequence we can prove the following

PROPOSITION 1.3. *If T satisfies* (9) *then for every $\lambda \in \mathbb{R}$*

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}T\big) = \mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\big) = \mathrm{e}^{-\frac{\lambda^2}{2}}. \tag{11}$$

Proof. We know that β_T and β_1 have the same standard normal law. Therefore,

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\beta_T\big) = \mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_1}\beta_1\big) = \mathrm{i}\lambda\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_1}\big) = \mathrm{i}\lambda\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}\big) = \mathrm{i}\lambda\mathrm{e}^{-\frac{\lambda^2}{2}}$$

where for the second equality we applied the duality formula (6). Combining this with (10) we obtain (11). $\square$

Remark 1.2. As a conclusion of (10) and (11) we obtain that β_T is simultaneously a standard normal random variable and P and under the measure $T \cdot P$. We can wonder whether is possible for a random variable Z to be standard normal under a probability P and in the same time under $F \cdot P$ where F is a positive random variable. This is possible when $F = 1 + G$ where $\mathbf{E}(G|Z) = 0$.

COROLLARY 1.4. *Under* (9), *we have* $\mathbf{E}(T|\beta_T) = 1$ *almost surely.*

Proof. From (10) and (11), $\mathbf{E}\big(\mathrm{e}^{\mathrm{i}\lambda\beta_T}(T-1)\big) = 0$ and this gives $\mathbf{E}_Q\big(\mathrm{e}^{\mathrm{i}tB_T}\big) = 0$ where Q is the measure $Q(A) = \mathbf{E}((T-1)1_A)$ for every $A \in \sigma(B_T)$. The uniqueness of the Fourier transform shows that Q is zero so its Radon-Nykodim derivative with respect to P, which is $\mathbf{E}(T-1|B_T)$, is also almost surely zero. $\square$

PROPOSITION 1.5. *Under* (9), *we have that* $\mathbf{E}T^2 = 1$.

Proof. Let us apply Itô's formula to the $\mathcal{G}_t$ martingale $\beta_{T\wedge t}$. Letting $t \to \infty$ (recall that T is a.s. bounded) we get

$$\mathbf{E}\beta_T^4 = 6\mathbf{E}\int_0^T \beta_s^2 \mathrm{d}s.$$

Since β_T has $N(0,1)$ law, we have that $\mathbf{E}\beta_T^4 = 3$. Consequently,

$$\mathbf{E}\int_0^T \beta_s^2 \mathrm{d}s = \frac{1}{2}.$$

Now, by Corollary 1, $\mathbf{E}(T\beta_T^2) = \mathbf{E}\beta_T^2 = 1$. Applying again Itô formula to $\beta_{T\wedge t}$ with $f(t,x) = tx^2$ we get

$$\mathbf{E}T\beta_T^2 = \mathbf{E}\int_0^T \beta_s^2 \mathrm{d}s + \mathbf{E}\int_0^T s\mathrm{d}s.$$

Therefore, $\mathbf{E}\int_0^T s\mathrm{d}s = \frac{1}{2}$ and then $\mathbf{E}T^2 = 1$. $\square$

THEOREM 1.6. *Let* $(\beta_t)_{t\geq 0}$ *a* $\mathcal{G}_t$ *Wiener process and let* T *be a* $\mathcal{G}_t$ *bounded stopping time with* $\mathbf{E}T = 1$. *Suppose* β_T *has a* $N(0,1)$ *law. Then* $T = 1$ *a.s..*

Proof. It is a consequence of the above proposition, since $\mathbf{E}(T-1)^2 = \mathbf{E}T^2 - 2\mathbf{E}(T) + 1 = 0$. $\square$

Next, we will try to understand if this property is always true without the assumption that the bracket of the martingale M^X is finite almost surely. To this end, we will consider the following example. Let $(W_t)_{t\in[0,1]}$ a standard $\mathcal{F}_t$ Wiener process. Consider h_1, h_2 two functions in $L^2([0,1])$ such that $\langle h_1, h_2\rangle_{L^2([0,1])} = 0$ and $\|h_1\|_{L^2([0,1])} = \|h_2\|_{L^2([0,1])} = 1$. For example we can choose

$$h_1(x) = \sqrt{2}1_{[0,\frac{1}{2}]}(x) \quad \text{and} \quad h_2(x) = \sqrt{2}1_{[\frac{1}{2},1]}(x)$$

(so, in addition, h_1 and h_2 have disjoint support). Define the random variable

$$X = W(h_1)\,\mathrm{sign}(W(h_2). \tag{12}$$

It is well-known that X is standard normal. Note in particular that $X^2 = W(h_1)^2$. We will see that it cannot be written as the value at time 1 of its associated DDS martingale. To this end we will use the chaos expansion of X into multiple Wiener-Itô integrals.

Recall that if $h \in L^2([0,1])$ with $\|h\|_{L^2([0,1])} = 1$ then (see [1])

$$\mathrm{sign}(W(h)) = \sum_{k\geq 0} b_{2k+1} I_{2k+1}(h^{\otimes(2k+1)}) \text{ with } b_{2k+1} = \frac{2(-1)^k}{\sqrt{2\pi}(2k+1)k!2^k},\ k \geq 0.$$

We have

Proposition 1.7. *The standard normal random variable X given by* (12) *is not equal a.s. to β_1 where β is its associated DDS martingale.*

Proof. By the product formula (4) we can express X as (note that h_1 and h_2 are orthogonal and there are not contractions of order $l \geq 1$)

$$X = \sum_{k\geq 0} b_{2k+1} I_{2k+2}\big(h_1 \tilde{\otimes} h_2^{\otimes 2k+1}\big)$$

and

$$\mathbf{E}\big(X|\mathcal{F}_t\big) = \sum_{k\geq 0} b_{2k+1} I_{2k+2}\big((h_1 \tilde{\otimes} h_2^{\otimes 2k+1})1_{[0,t]}^{\otimes 2k+2}(\cdot)\big).$$

We have

$$(h_1 \tilde{\otimes} h_2^{\otimes 2k+1})(t_1, \dots, t_{2k+2}) = \frac{1}{2k+2} \sum_{i=1}^{2k+1} h_1(t_i) h_2^{\otimes 2k+1}(t_1, \dots, \hat{t}_i, \dots, t_{2k+2}), \tag{13}$$

where $\hat{t}_i$ means that the variable t_i is missing. Now, $M_t^X = \mathbf{E}\big(X|\mathcal{F}_t\big) = \int_0^t u_s \mathrm{d}W_s$ where, by (13)

$$\begin{aligned} u_s &= \sum_{k\geq 0} b_{2k+1}(2k+2) I_{2k+1}\big((h_1 \tilde{\otimes} h_2^{2k+1})(\cdot\,, s)1_{[0,s]}^{\otimes 2k+1}(\cdot)\big) \\ &= \sum_{k\geq 0} b_{2k+1} \Big[h_1(s) I_{2k+1}\big(h_2^{\otimes 2k+1} 1_{[0,s]}^{\otimes 2k+1}(\cdot)\big) \\ &\quad + (2k+1) h_2(s) I_1(h_1 1_{[0,s]}(\cdot)) I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big)\Big] \end{aligned}$$

for every $s \in [0,1]$. Note first that, due to the choice of the functions h_1 and h_2,

$$h_1(s)h_2(u)1_{[0,s]}(u) = 0 \quad \text{for every } s, u \in [0,1].$$

Thus the first summand of u_s vanishes and

$$u_s = \sum_{k \geq 0} b_{2k+1}(2k+1)h_2(s)I_1(h_1 1_{[0,s]}(\cdot))I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big).$$

Note also that $h_1(x)1_{[0,s]}(x) = h_1(x)$ for every s in the interval $[\frac{1}{2}, 1]$. Consequently,

$$u_s = W(h_1) \sum_{k \geq 0} b_{2k+1}(2k+1)h_2(s)I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big).$$

Let us compute the chaos decomposition of the random variable $\int_0^1 u_s^2 \mathrm{d}s$. Taking into account the fact that h_1 and h_2 have disjoint supports we can write

$$\int_0^1 u_s^2 \mathrm{d}s = \sum_{k,l \geq 0} b_{2k+1}b_{2l+1}(2k+1)(2l+1)W(h_1)^2 \cdot \int_0^1 \mathrm{d}s h_2(s)^2 I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big) I_{2l}\big(h_2^{\otimes 2l} 1_{[0,s]}^{\otimes 2l}(\cdot)\big).$$

Since

$$W(h_1)^2 = I_2\big(h_1^{\otimes 2}\big) + \int_0^1 h_1(u)^2 \mathrm{d}u = I_2\big(h_1^{\otimes 2}\big) + 1$$

and

$$\mathbf{E}\big(\operatorname{sign}(W(h_2)\big)^2 = \int_0^1 \mathrm{d}s h_2^2(s) \mathbf{E}\bigg(\sum_{k \geq 0} b_{2k+1}(2k+1)I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big)\bigg)^2 = 1$$

we get

$$\int_0^1 u_s^2 \mathrm{d}s = \big(1 + I_2\big(h_1^{\otimes 2}\big)\big) \times \bigg(1 + \sum_{k,l \geq 0} b_{2k+1}b_{2l+1}(2k+1)(2l+1) \int_0^1 \mathrm{d}s h_2(s)^2 \cdot \Big[I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big) I_{2l}\big(h_2^{\otimes 2l} 1_{[0,s]}^{\otimes 2l}(\cdot)\big) - \mathbf{E} I_{2k}\big(h_2^{\otimes 2k} 1_{[0,s]}^{\otimes 2k}(\cdot)\big) I_{2l}\big(h_2^{\otimes 2l} 1_{[0,s]}^{\otimes 2l}(\cdot)\big)\Big]\bigg)$$
$$=: \big(1 + I_2\big(h_1^{\otimes 2}\big)\big)(1 + A).$$

Therefore, we obtain that $\int_0^1 u_s^2 \mathrm{d}s = 1$ almost surely if and only if $\big(1 + I_2\big(h_1^{\otimes 2}\big)\big)(1+A) = 1$ almost surely which implies that $I_2(h_1^{\otimes 2})(1+A) + A = 0$ a.s. and this is impossible because $I_2(h_1^{\otimes 2})$ and A are independent. □

We obtain an interesting consequence of the above result.

Corollary 1.8. *Let X be given by* (12). *Then the bracket of the martingale M^X with $M_t^X = \mathbf{E}\big(X|\mathcal{F}_t\big)$ is not bounded.*

Proof. It is a consequence of Proposition 1.7 and of Theorem 1.6. □

Remark 1.3. Proposition 1.6 provides an interesting example of a Brownian motion β and of a stopping time T for its filtration such that β_T is standard normal and T is not almost surely equal to 1.

Let us make a short summary of the results in the first part of our paper: if X is a standard normal random variable and the bracket of M^X is bounded a.s. then X can be expressed almost surely as a Wiener integral with respect to a Brownian motion on the same (or possibly extended) probability space. The Brownian is obtained via DDS theorem. If the bracket of M^X is not bounded, then X is not necessarily equal with β_1, β being its associated DDS Brownian motion. This is the case of the variable (12).

Nevertheless, we will see that after a suitable extension of the probability space, any standard normal random variable can be written as the value at time 1 of a Brownian motion constructed on this extended probability space.

Proposition 1.9. *Let X_1 be a standard normal random variable on $(\Omega_1, \mathcal{F}_1, P_1)$ and for every $i \geq 2$ let $(\Omega_i, \mathcal{F}_i, P_i, X_i)$ be independent copies of $(\Omega_1, \mathcal{F}_1, P_1, X_1)$. Let $(\Omega_0, \mathcal{F}_0, P_0)$ be the product probability space. On Ω_0 define for every t*

$$W_t^0 = \sum_{k \geq 1} f_k(t) X_k,$$

where $f_k(t) = \sqrt{2} \frac{\sin(k-\frac{1}{2})\pi t}{(k-\frac{1}{2})\pi}$. Then W^0 is a Brownian motion on Ω_0 and $X = \int_0^1 f_1(s) \mathrm{d}W_s^0$ a.s..

Proof. The proof is a consequence of the Karhunen-Loève theorem for the Brownian motion. □

3. CONSEQUENCES

We think that the consequences of this result are multiple. We will prove here first that a random variable X which lives in a finite sum of Wiener chaoses cannot be Gaussian. Again we fix a Wiener process $(W_t)_{t \in [0,1]}$ on Ω. Let us start with the following lemma.

Lemma 3.1. *Fix $N \geq 1$. Let $g \in L^2([0,1]^{\otimes N+1})$ symmetric in its first N variables such that $\int_0^1 \mathrm{d}s g(\cdot, s) \tilde{\otimes} g(\cdot, s) = 0$ almost everywhere on $[0,1]^{\otimes 2N}$. Then for every $k = 1, \ldots, N-1$ we have*

$$\int_0^1 \mathrm{d}s g(\cdot, s) \tilde{\otimes}_k g(\cdot, s) = 0 \quad a.e. \text{ on } [0,1]^{2N-2k}.$$

Proof. Without loss of generality we can assume that g vanishes on the diagonals $(t_i = t_j)$ of $[0,1]^{\otimes(N+1)}$. This is possible from the construction of multiple stochastic integrals. From the hypothesis, the function

$$(t_1,\dots,t_{2N}) \to \frac{1}{(2N)!}\sum_{\sigma\in S_{2N}}\int_0^1 \mathrm{d}s g(t_{\sigma(1)},\dots,t_{\sigma(N)},x,s) \cdot g(t_{\sigma(N+1)},\dots,t_{\sigma(2N)},x,s)$$

vanishes almost everywhere on $[0,1]^{\otimes 2N}$. Put $t_{2N-1} = t_{2N} = x \in [0,1]$. Then for every x, the function

$$(t_1,\dots t_{2N-2}) \to \sum_{\sigma\in S_{2N-2}}\int_0^1 \mathrm{d}s g(t_{\sigma(1)},\dots,t_{\sigma(N-1)},s) g(t_{\sigma(N)},\dots,t_{\sigma(2N-2)},s)$$

is zero a.e. on $[0,1]^{\otimes(2N-2)}$ and integrating with respect to x we get $\int_0^1 \mathrm{d}s g(\cdot\,,s)\,\tilde{\otimes}_1 g(\cdot\,,s) = 0$ a.e. on $[0,1]^{\otimes(2N-2)}$. By repeating the procedure we obtain the conclusion. $\square$

Let us also recall the following result from [7].

Proposition 3.2. *Suppose that $F = I_N(f_N)$ with $f \in L^2([0,1]^N)$ symmetric and $N \geq 2$ fixed. Then the distribution of F cannot be normal.*

We are going to prove the same property for variables that can be expanded into a finite sum of multiple integrals.

Theorem 3.3. *Fix $N \geq 1$ and let X be a centered random variable such that $X = \sum_{n=1}^{N+1} I_n(f_n)$ where $f \in L^2([0,1]^n)$ are symmetric functions. Then the law of X cannot be normal.*

Proof. We will assume that $\mathbf{E}X^2 = 1$. Suppose that X is standard normal. We can write $X = \int_0^1 u_s \mathrm{d}W_s$ where $u_s = \sum_{n=1}^{N} I_n(g_n(\cdot\,,s))$. As a consequence of Proposition 1.9,

$$\int_0^1 u_s^2 \mathrm{d}s = 1 \quad \text{a.s.}$$

But from the product formula (4)

$$\begin{aligned}\int_0^1 u_s^2 \mathrm{d}s &= \int_0^1 \mathrm{d}s\Big(\sum_{n=1}^{N} I_n(g_n(\cdot\,,s))\Big)^2\\ &= \int_0^1 \mathrm{d}s \sum_{m,n=1}^{N}\sum_{k=1}^{m\wedge n} k! C_n^k C_m^k I_{m+n-2k}(g_n(\cdot\,,s)\otimes g_m(\cdot\,,s))\mathrm{d}s.\end{aligned}$$

The idea is to use the fact that the highest order chaos, which appears only once in the above expression, vanishes. Let us look at the chaos of order $2N$

in the above decomposition. As we said, it appears only when we multiply I_N by I_N and consists in the random variable $I_{2N}\big(\int_0^1 g_N(\cdot\,,s)\otimes g_N(\cdot\,,s)\mathrm{d}s\big)$. The isometry of multiple integrals (3) implies that

$$\int_0^1 g_N(\cdot\,,s)\tilde{\otimes} g_N(\cdot\,,s)\mathrm{d}s = 0 \quad \text{a.e. on } [0,1]^{2N}$$

and by Lemma 3.1, for every $k=1,\dots,N-1$,

$$\int_0^1 g_N(\cdot\,,s)\tilde{\otimes}_k g_N(\cdot\,,s)\mathrm{d}s = 0 \quad \text{a.e. on } [0,1]^{2N-2k}. \tag{14}$$

Consider now the the random variable $Y := I_{N+1}(f_{N+1})$. It can be written as $Y=\int_0^1 I_N(g_N(\cdot\,,s))\mathrm{d}W_s$ and by the DDS theorem, $Y=\beta^Y_{\int_0^1 \mathrm{d}s(I_N(g_N(\cdot\,,s)))^2}$. The multiplication formula together with (14) shows that $\int_0^1 \mathrm{d}s(I_N(g_N(\cdot\,,s)))^2$ is deterministic and as a consequence Y is Gaussian. This is in contradiction with Proposition 3.2. $\square$

Finally, let us make a connection with several recent results obtained via Stein's method and Malliavin calculus. Recall that the Ornstein-Uhlenbeck operator is defined as $LF=-\sum_{n\geq 0} nI_n(f_n)$ if F is given by (2). There exists a connection between δ, D and L in the sense that a random variable F belongs to the domain of L if and only if $F\in\mathbb{D}^{1,2}$ and $DF\in\mathrm{Dom}(\delta)$ and then $\delta DF=-LF$.

Let us denote by D the Malliavin derivative with respect to W and let, for any $X\in\mathbb{D}^{1,2}$

$$G_X=\langle DX, D(-L)^{-1}X\rangle.$$

The following theorem is a collection of results in several recent papers.

Theorem 3.4. *Let X be a random variable in the space $\mathbb{D}^{1,2}$. Then the following assertions are equivalent.*

1. *X is a standard normal random variable.*
2. *For every $t\in\mathbb{R}$, one has $\mathbf{E}\big(\mathrm{e}^{\mathrm{i}tX}(1-G_X)\big)=0$.*
3. $\mathbf{E}\big((1-G_X)/X\big)=0$.
4. *For every $z\in\mathbb{R}$, $\mathbf{E}\big(f_z'(1-G_X)\big)=0$, where f_z' is the solution of the Stein's equation* (see [4]).

Proof. We will show that 1. ⇒ 2. ⇒ 3. ⇒ 4. ⇒ 1. First suppose that $X\sim N(0,1)$. Then

$$\begin{aligned}\mathbf{E}\big(\mathrm{e}^{\mathrm{i}tX}(1-G_X)\big) &= \mathbf{E}(\mathrm{e}^{\mathrm{i}tX})-\frac{1}{\mathrm{i}t}\mathbf{E}\langle D\mathrm{e}^{\mathrm{i}tX}, D(-L)^{-1}X\rangle\\ &= \mathbf{E}(\mathrm{e}^{\mathrm{i}tX_n})-\frac{1}{\mathrm{i}t}\mathbf{E}\big(X\mathrm{e}^{\mathrm{i}tX}\big)=\varphi_X(t)-\frac{1}{t}\varphi'_X(t)=0.\end{aligned}$$

Let us now prove the implication 2. $\Rightarrow$ 3. It has also proven in [5], Corollary 3.4. Set $F = 1 - G_X$. The random variable $\mathbf{E}(F|X)$ is the Radon-Nykodim derivative with respect to P of the measure $Q(A) = \mathbf{E}(F1_A)$, $A \in \sigma(X)$. Relation 1. means that $\mathbf{E}\big(\mathrm{e}^{\mathrm{i}tX}\mathbf{E}(F/X)\big) = \mathbf{E}_Q(\mathrm{e}^{\mathrm{i}tX}) = 0$ and consequently $Q(A) = \mathbf{E}(F1_A) = 0$ for any $A \in \sigma(X_n)$. In other words, $\mathbf{E}(F|X) = 0$. The implication 3. $\Rightarrow$ 4. is trivial and the implication 4. $\Rightarrow$ 1. is a consequence of a result in [4]. $\square$

As we said, this property can be easily understood and checked if X is in the first Wiener chaos with respect to W. Indeed, if $X = W(f)$ with $\|f\|_{L^2([0,1])} = 1$ then $DX = D(-L)^{-1}X = f$ and clearly $G_X = 1$. There is no need to compute the conditional expectation given X, which is in practice a very difficult task. Let us consider now the case of the random variable $Y = \int_0^1 \operatorname{sign}(W_s)\mathrm{d}W_s$. The chaos expansion of this variable is known. But Y is not even differentiable in the Malliavin sense so it is not possible to check the conditions from Theorem 14. Another example is related to the value at time 1 of a Bessel process (8). Here again the chaos expansion of X can be obtained (see e.g. [1]) but is it impossible to compute the conditional expectation given X.

But on the other hand, for both variables treated above there is another explanation of their normality which comes from Lévy's characterization theorem. Another explanation can be obtained from the results in Section 2. Note that these two examples are random variables such that the bracket of M^X is bounded a.s..

Corollary 3.5. *Let X be an integrable random variable on $(\Omega, \mathcal{F}, P)$. Then X is a standard normal random variable if and only if there exists a Brownian motion $(\beta_t)_{t\geq 0}$ on an extension of Ω such that*

$$\langle D^\beta X, D^\beta(-L^\beta)^{-1}X\rangle = 1. \tag{15}$$

Proof. Assume that $X \sim N(0,1)$. Then by Proposition 1.9, $X = \beta_1$ where β is a Brownian motion on an extended probability space. Clearly (15) holds. Suppose that there exists β a Brownian motion on $(\Omega, \mathcal{F}, P)$ such that (15) holds. Then for any continuous and piecewise differentiable function f with $\mathbf{E}f'(Z) < \infty$ we have

$$\begin{aligned}\mathbf{E}\big(f'(Z) - f(X)X\big) &= \mathbf{E}\big(f'(X) - f'(X)\langle D^\beta X, D^\beta(-L^\beta)^{-1}X\rangle\big) \\ &= \mathbf{E}\big(f'(Z)(1 - \langle D^\beta X, D^\beta(-L^\beta)^{-1}X\rangle\big) = 0\end{aligned}$$

and this implies that $X \sim N(0,1)$ (see [4], Lemma 1.2). $\square$

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