ON THE CONVERGENCE OF A HYPERBOLOID APPROXIMATION PROCEDURE FOR A PERTURBED GENERALIZED EUCLIDEAN MULTIFACILITY LOCATION PROBLEM

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For a perturbed generalized multifacility location problem, we prove that a hyperboloid approximation procedure is convergent under certain conditions.

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1. INTRODUCTION

Let $a_1, a_2, \ldots, a_m$ be m points in $\mathbf{R}^d$, the d-dimensional Euclidean space. Let w_{ji}, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, and v_{jk}, $1 \le j < k \le n$ be given nonnegative numbers. For convenience, we assume $v_{jk} = v_{kj}$ whenever $j > k$ and $v_{jj} = 0$ for all j. Also, all vectors are assumed to be column vectors in this paper. Let B_{ji}, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, and D_{jk}, $1 \le j < k \le n$ be given symmetric positive definite square d-matrices. For convenience, we assume $D_{jk} = D_{kj}$ whenever $j > k$. Preda and Niculescu [5] defined the generalized Euclidean multifacility location (GEMFL) problem, which is to find a point $x = (x_1^T, x_2^T, \ldots, x_n^T)^T \in \mathbf{R}^{n\times d}$ that will minimize

$$f(x)=\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)}+\sum_{1\le j<k\le n} v_{jk}\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)},$$

where ^{T} means transpose. When $B_{ji} = I$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, and $D_{jk} = I$, $1 \le j < k \le n$, where I is the unit square d-matrix, is obtained the Euclidean multifacility location (EMFL) problem.

In this problem, $a_1, a_2, \ldots, a_m$ represent the location of m existing facilities; $x_1, x_2, \ldots, x_n$ represent the locations of n new facilities.

To avoid nondifferentiability, following the idea of Eyster, White and Wierwille [1], Preda and Niculescu [5] introduced a small positive number ε to the original problem, getting the following smooth perturbed objective function

$$f_\varepsilon(x) = \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon} + \sum_{1\le j<k\le n} v_{jk}\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}. \tag{1}$$

A minimum point of $f_\varepsilon(x)$ is called an ε-*optimal solution* to the GEMFL problem.

In this paper, we continue the work of Preda and Niculescu [5], further generalizing results obtained by Rosen and Xue [10] for the perturbed EMFL problem. On the line of papers Preda et al. [6], Preda and Batatorescu [7], Preda and Chitescu [8] and Preda [9] we can formulate some problems of this type.

The gradient of f_ε with respect to the jth new facility x_j is

$$\nabla_j f_\varepsilon(x) = \sum_{i=1}^{m} w_{ji}\frac{B_{ji}(x_j-a_i)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\ne j} v_{jk}\frac{D_{jk}(x_j-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\,. \tag{2}$$

As in Weiszfeld [11] and Miehle [3], we may get an improved location x_j^+ of the jth new facility with respect to the existing facilities and the other new facilities by solving the system of linear equations

$$\sum_{i=1}^{m} w_{ji}\frac{B_{ji}(x_j^+-a_i)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\ne j} v_{jk}\frac{D_{jk}(x_j^+-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}} = 0 \tag{3}$$

for x_j^+. This gives

$$x_j^+ = T_j(x_1, x_2, \ldots, x_n), \tag{4}$$

where $T_j : \mathbf{R}^{n\times d} \to \mathbf{R}^d$, $j = 1, 2, \ldots, n$, is defined by

$$T_j(x_1, x_2, \ldots, x_n) \tag{5}$$

$$= \left[\sum_{i=1}^{m} \frac{w_{ji}B_{ji}}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\ne j} \frac{v_{jk}D_{jk}}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\right]^{-1}$$

$$= \left[\sum_{i=1}^{m} w_{ji}\frac{B_{ji}a_i}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\ne j} v_{jk}\frac{D_{jk}x_k}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\right].$$

2. THE HYPERBOLOID APPROXIMATION PROCEDURE AND SOME PRELIMINARY RESULTS

Algorithm HAP

Step_0. Choose any initial point $x^0 \in \mathbf{R}^{n\times d}$. Set $s = 0$ and go to Step_1.

Step_1. For $j := 1, 2, \ldots, n$ do $x_j^{s+1} = T_j(x_1^s, x_2^s, \ldots, x_n^s)$.

Step_2. If $x^{s+1} = x^s$, stop; otherwise, replace s with $s+1$ and go to Step_1.

Let x^0 be given by

$$x_j^0 = \begin{cases} \left(\sum_{i=1}^m w_{ji}B_{ji}\right)^{-1}\left(\sum_{i=1}^m w_{ji}B_{ji}a_i\right) & \text{if } \sum_{i=1}^m w_{ji} > 0 \\ \dfrac{\sum_{i=1}^m B_{ji}a_i}{m} & \text{otherwise} \end{cases} \qquad j = 1, 2, \ldots, n.$$

A new facility x_j and an existing facility a_i are said to have an *exchange* whenever $w_{ji} > 0$. Two new facilities x_j and x_k are said to have an *exchange* whenever $v_{jk} > 0$. A new facility x_{j_0} is said to be *chained* if there exist $j_1, j_2, \ldots, j_l \in \{1, 2, \ldots, n\}$ and $i_0 \in \{1, 2, \ldots, m\}$ such that $v_{j_0j_1} \ldots v_{j_{l-1}j_l} w_{j_li_0} \neq 0$. A variable x_j which is not chained is called a *free* variable. Let F and C be the index sets for free variables and chained variables, respectively. We can rewrite $f_\varepsilon(x)$ as

$$\text{(6)} \qquad f_\varepsilon(x) = \sum_{j\in C}\sum_{i=1}^m w_{ji}\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon} + \sum_{\substack{j,k\in C \\ 1\le j<k\le n}} v_{jk}\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon} + \sum_{\substack{j,k\in F \\ 1\le j<k\le n}} v_{jk}\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}.$$

With no loss of generality [6], we assume in the rest of this paper that there is no free variable in the GEMFL problem or, in terms of Francis and Cabot [2] that the problem is *fully chained.*

With this assumption, $f_\varepsilon(x)$ is a strictly convex function for $\varepsilon > 0$ and has arbitrarily high derivatives.

PROPOSITION 1. $\lim_{\|x\|\to\infty} f_\varepsilon(x) = +\infty$. *Therefore, the GEMFL problem has a unique ε-optimal solution for each $\varepsilon > 0$.*

Proof. If $\|x\| \to \infty$, then $\|x_{j_0}\| \to \infty$ for some $j_0 \in \{1, 2, \ldots, n\}$. Since GEMFL is fully chained, there exist $j_1, j_2, \ldots, j_l \in \{1, 2, \ldots, n\}$, $i_0 \in$

$\{1, 2, \ldots, m\}$ such that $v_{j_0 j_1} \ldots v_{j_{l-1} j_l} w_{j_l i_0} \neq 0$. As a consequence, we have

$$\lim_{\|x\|\to\infty} \Bigg[\sum_{0\le k<l} v_{j_k j_{k+1}} \sqrt{\left(x_{j_k} - x_{j_{k+1}}\right)^T D_{j_k j_{k+1}} \left(x_{j_k} - x_{j_{k+1}}\right) + \varepsilon} + w_{j_l i_0} \sqrt{\left(x_{j_l} - a_{i_0}\right)^T B_{j_l i_0} \left(x_{j_l} - a_{i_0}\right) + \varepsilon} \Bigg] = +\infty. \tag{7}$$

Therefore, $\lim\limits_{\|x\|\to\infty} f_\varepsilon(x) = +\infty$. This, together with the continuity of $f_\varepsilon(x)$, guarantees the existence of a minimizer of $f_\varepsilon(x)$. Since $f_\varepsilon(x)$ is strictly convex, the minimizer is unique. □

Proposition 2. *If $B_{ji} = \alpha_{ji} B_j$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, and $D_{jk} = \beta_{jk} B_j$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, where $\alpha_{ji} > 0$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $\beta_{jk} > 0$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, and B_j is symmetric and positive definite, for any $j \in \{1, 2, \ldots, n\}$, then for any $j \in \{1, 2, \ldots, n\}$, $T_j(x_1, x_2, \ldots, x_n)$ is in the convex hull of the points $a_1, a_2, \ldots, a_m$ and $x_1, x_2, \ldots, x_n$.*

Proof. It follows from (5) that

$$T_j(x_1, x_2, \ldots, x_n)$$

$$= \left[\sum_{i=1}^m \frac{w_{ji}\alpha_{ji}B_j}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} \frac{v_{jk}\beta_{jk}B_j}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right]^{-1}$$

$$\left[\sum_{i=1}^m w_{ji} \frac{\alpha_{ji}B_j a_i}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk} \frac{\beta_{jk}B_j x_k}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right]$$

$$= \left[\sum_{i=1}^m \frac{w_{ji}\alpha_{ji}}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} \frac{v_{jk}\beta_{jk}}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right]^{-1}$$

$$B_j^{-1} \left[\sum_{i=1}^m w_{ji} \frac{\alpha_{ji}B_j a_i}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk} \frac{\beta_{jk}B_j x_k}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right]$$

$$= \left[\sum_{i=1}^m \frac{w_{ji}\alpha_{ji}}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} \frac{v_{jk}\beta_{jk}}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right]^{-1}$$

$$= \left[\sum_{i=1}^m w_{ji} \frac{\alpha_{ji} a_i}{\sqrt{\alpha_{ji}(x_j - a_i)^T B_j (x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk} \frac{\beta_{jk} x_k}{\sqrt{\beta_{jk}(x_j - x_k)^T B_j (x_j - x_k) + \varepsilon}} \right],$$

and this is a convex combination of the points $a_1, a_2, \ldots, a_m$ and $x_1, x_2, \ldots, x_n$. □

THEOREM 1. *Let* $x = (x_1, x_2, \ldots, x_n)$ *be any point in* $\mathbf{R}^{n\times d}$. *Let* $y = (y_1, \ldots, y_n)$ *be the point generated by*

$$y_j = T_j(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, n. \tag{8}$$

Then the following descent property for the **HAP** *holds as long as* $\varepsilon > 0$.

$$2f_\varepsilon(x) - 2f_\varepsilon(y) \geq \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{\mathcal{A}}{\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}} \tag{9}$$
$$+ \sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{\left[\sqrt{(y_j - y_k)^T D_{jk}(y_j - y_k) + \varepsilon} - \sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}\right]^2}{\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}},$$

where

$$\mathcal{A} = \left[\sqrt{(y_j - a_i)^T B_{ji}(y_j - a_i) + \varepsilon} - \sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}\right]^2 + (y_j - x_j)^T B_{ji}(y_j - x_j).$$

Proof. Since $y_j = T_j(x_1, x_2, \ldots, x_n)$, it follows from (3) that

$$\sum_{i=1}^{m} w_{ji}\frac{B_{ji}(y_j - a_i)}{\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk}\frac{D_{jk}(y_j - x_k)}{\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}} = 0. \tag{10}$$

Multiplying both sides of (10) by $(y_j - x_j)^T$ and rearranging terms, we get

$$\sum_{i=1}^{m} w_{ji}\frac{(y_j - a_i)^T B_{ji}(y_j - a_i)}{\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk}\frac{(y_j - x_k)^T D_{jk}(y_j - x_k)}{\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}} - (y_j - x_j)^T \nabla_j f_\varepsilon(x) \tag{11}$$
$$= \sum_{i=1}^{m} w_{ji}\frac{(x_j - a_i)^T B_{ji}(x_j - a_i)}{\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk}\frac{(x_j - x_k)^T D_{jk}(x_j - x_k)}{\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}}.$$

By definition,

$$y_j = x_j - \left[\sum_{i=1}^{m} w_{ji}\frac{B_{ji}}{\sqrt{(x_j - a_i)^T B_{ji}(x_j - a_i) + \varepsilon}} + \sum_{k\neq j} v_{jk}\frac{D_{jk}}{\sqrt{(x_j - x_k)^T D_{jk}(x_j - x_k) + \varepsilon}}\right]^{-1} \nabla_j f_\varepsilon(x). \tag{12}$$

Therefore,

$$(13)\quad -(y_j-x_j)^T\nabla_j f_\varepsilon(x) = (y_j-x_j)^T\Bigg[\sum_{i=1}^{m} w_{ji}\frac{B_{ji}}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\neq j} v_{jk}\frac{D_{jk}}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\Bigg](y_j-x_j).$$

Combining (11) and (13), we get

$$(14)\quad \sum_{i=1}^{m} w_{ji}\frac{(y_j-a_i)^T B_{ji}(y_j-a_i)+(y_j-x_j)^T B_{ji}(y_j-x_j)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{k\neq j} v_{jk}\frac{(y_j-x_k)^T D_{jk}(y_j-x_k)+(y_j-x_j)^T D_{jk}(y_j-x_j)-\frac{1}{2}(x_j-x_k)^T D_{jk}(x_j-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}$$
$$= \sum_{i=1}^{m} w_{ji}\frac{(x_j-a_i)^T B_{ji}(x_j-a_i)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \frac{1}{2}\sum_{k\neq j} v_{jk}\frac{(x_j-x_k)^T D_{jk}(x_j-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}.$$

Summing (14) over j (note that (14) is true for all j), since $D_{jk}=D_{kj}$, $\forall j\neq k$, we get

$$(15)\quad \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{(y_j-a_i)^T B_{ji}(y_j-a_i)+(y_j-x_j)^T B_{ji}(y_j-x_j)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{\mathcal{B}}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}} =$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{(x_j-a_i)^T B_{ji}(x_j-a_i)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} + \sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{(x_j-x_k)^T D_{jk}(x_j-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}},$$

where $\mathcal{B}=(y_j-x_k)^T D_{jk}(y_j-x_k)+(y_k-x_j)^T D_{jk}(y_k-x_j)+(y_j-x_j)^T D_{jk}(y_j-x_j)+(y_k-x_k)^T D_{jk}(y_k-x_k)-(x_j-x_k)^T D_{jk}(x_j-x_k)$.

Since D_{jk} is symmetrical and positive definite, we have $(y_j-x_k)^T D_{jk}(y_j-x_k)+(y_k-x_j)^T D_{jk}(y_k-x_j)+(y_j-x_j)^T D_{jk}(y_j-x_j)+(y_k-x_k)^T D_{jk}(y_k-x_k)-(x_j-x_k)^T D_{jk}(x_j-x_k)-(y_j-y_k)^T D_{jk}(y_j-y_k)=(y_j-x_k+y_k-x_j)^T D_{jk}(y_j-x_k+y_k-x_j)\geq 0$. Therefore, we get

$$(16)\quad (y_j-x_k)^T D_{jk}(y_j-x_k)+(y_k-x_j)^T D_{jk}(y_k-x_j) + (y_j-x_j)^T D_{jk}(y_j-x_j)+(y_k-x_k)^T D_{jk}(y_k-x_k) \geq (x_j-x_k)^T D_{jk}(x_j-x_k)+(y_j-y_k)^T D_{jk}(y_j-y_k).$$

Combining (15) and (16), we get

$$\begin{aligned}(17)\qquad &\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{(y_j-a_i)^T B_{ji}(y_j-a_i)+(y_j-x_j)^T B_{ji}(y_j-x_j)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}}\\ &+\sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{(y_j-y_k)^T D_{jk}(y_j-y_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\\ \le &\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{(x_j-a_i)^T B_{ji}(x_j-a_i)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}}+\sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{(x_j-x_k)^T D_{jk}(x_j-x_k)}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}.\end{aligned}$$

Adding $\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{\varepsilon}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}}+\sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{\varepsilon}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}$ to both sizes of (17), we get

$$\begin{aligned}(18)\qquad &\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{(y_j-a_i)^T B_{ji}(y_j-a_i)+\varepsilon+(y_j-x_j)^T B_{ji}(y_j-x_j)}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}}\\ &+\sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{(y_j-y_k)^T D_{jk}(y_j-y_k)+\varepsilon}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}\le f_\varepsilon(x).\end{aligned}$$

Following the ideas of Ostresh [4] and Weiszfeld [11], we have

$$\begin{aligned}(19)\qquad &(y_j-a_i)^T B_{ji}(y_j-a_i)+\varepsilon\\ &=\left[\sqrt{(y_j-a_i)^T B_{ji}(y_j-a_i)+\varepsilon}-\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}\right]^2\\ &+\left[\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}\right]^2+2\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}\\ &\cdot\left[\sqrt{(y_j-a_i)^T B_{ji}(y_j-a_i)+\varepsilon}-\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}\right].\end{aligned}$$

Also,

$$\begin{aligned}(20)\qquad &(y_j-y_k)^T D_{jk}(y_j-y_k)+\varepsilon\\ &=\left[\sqrt{(y_j-y_k)^T D_{jk}(y_j-y_k)+\varepsilon}-\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}\right]^2\\ &+\left[\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}\right]^2+2\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}\\ &\cdot\left[\sqrt{(y_j-y_k)^T D_{jk}(y_j-y_k)+\varepsilon}-\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}\right].\end{aligned}$$

Substituting (19) and (20) into (18), we get

$$\sum_{j=1}^{n}\sum_{i=1}^{m} w_{ji}\frac{\mathcal{C}}{\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}} \tag{21}$$

$$+\sum_{j=1}^{n}\sum_{k>j} v_{jk}\frac{\left[\sqrt{(y_j-y_k)^T D_{jk}(y_j-y_k)+\varepsilon}-\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}\right]^2}{\sqrt{(x_j-x_k)^T D_{jk}(x_j-x_k)+\varepsilon}}$$

$$+2f_\varepsilon(y)-f_\varepsilon(x)\le f_\varepsilon(x),$$

where

$$\mathcal{C}=\left[\sqrt{(y_j-a_i)^T B_{ji}(y_j-a_i)+\varepsilon}-\sqrt{(x_j-a_i)^T B_{ji}(x_j-a_i)+\varepsilon}\right]^2$$

$$+(y_j-x_j)^T B_{ji}(y_j-x_j).$$

This is equivalent to (9). □

3. THE CONVERGENCE THEOREM AND SOME COROLLARIES

We assume throughout this section that $y=(y_1,\ldots,y_n)=(T(x_1),\ldots,T(x_n))$.

LEMMA 1. *If there exist* $j_0\in\{1,2,\ldots,n\}$ *and* $i_0\in\{1,2,\ldots,m\}$ *such that*

$$w_{j_0i_0}>0,\quad \nabla_{j_0}f_\varepsilon(x)\neq 0, \tag{22}$$

then

$$f_\varepsilon(y)\le f_\varepsilon(x)-\frac{w_{j_0i_0}}{2\sqrt{(x_{j_0}-a_{i_0})^T B_{j_0i_0}(x_{j_0}-a_{i_0})+\varepsilon}}(y_{j_0}-x_{j_0})^T$$

$$\cdot B_{j_0i_0}(y_{j_0}-x_{j_0})<f_\varepsilon(x).$$

Proof. From Theorem 1 we have

$$f_\varepsilon(y)\le f_\varepsilon(x)-\frac{w_{j_0i_0}}{2\sqrt{(x_{j_0}-a_{i_0})^T B_{j_0i_0}(x_{j_0}-a_{i_0})+\varepsilon}}(y_{j_0}-x_{j_0})^T B_{j_0i_0}(y_{j_0}-x_{j_0}).$$

Combining (12) with (22) we get

$$\frac{w_{j_0i_0}}{2\sqrt{(x_{j_0}-a_{i_0})^T B_{j_0i_0}(x_{j_0}-a_{i_0})+\varepsilon}}(y_{j_0}-x_{j_0})^T B_{j_0i_0}(y_{j_0}-x_{j_0})>0. \tag{23}$$

This completes the proof. □

LEMMA 2. *If there exist $j_0, k_0 \in \{1, 2, \ldots, n\}$ such that $v_{j_0k_0} > 0$, $\nabla_{j_0} f_\varepsilon(x) \neq 0$ and $\nabla_{k_0} f_\varepsilon(x) = 0$, then*

$$f_\varepsilon(y) \leq f_\varepsilon(x) - \frac{v_{j_0k_0}}{2\sqrt{(x_{j_0} - x_{k_0})^T B_{j_0k_0}(x_{j_0} - x_{k_0}) + \varepsilon}}(y_{j_0} - x_{j_0})^T B_{j_0k_0}(y_{j_0} - x_{j_0})$$
$$< f_\varepsilon(x).$$

Proof. It follows from $\nabla_{k_0} f_\varepsilon(x) = 0$ and (12) that $y_{k_0} = x_{k_0}$. If we fix x_{k_0} at its current value, treat x_{k_0} as an extra existing facility instead of a new facility, and treat v_{jk_0} as the weight on the generalized distance from the jth new facility to this extra existing facility for $j \neq k_0$, then we can consider the current GEMFL problem as a new GEMFL problem with $m + 1$ existing facilities and $n - 1$ new facilities. Taking one step of the HAP algorithm on this new problem will result in exactly the same values for the jth new facility for all $j \neq k_0$. Applying Lemma 1 to this new GEMFL problem, we get the desired inequality. □

THEOREM 2. *If $\nabla f_\varepsilon(x) \neq 0$, then $f_\varepsilon(y) < f_\varepsilon(x)$.*

Proof. Since $\nabla f_\varepsilon(x) \neq 0$, there exists $j_0 \in \{1, 2, \ldots, n\}$ such that $\nabla_{j_0} f_\varepsilon(x) \neq 0$. Since variable x_{j_0} is chained, there exist $j_1, j_2, \ldots, j_l \in \{1, 2, \ldots, n\}$ and $i_0 \in \{1, 2, \ldots, m\}$ such that $v_{j_0j_1} \ldots v_{j_{l-1}j_l} w_{j_l i_0} > 0$.

Let $r = \max\{i \mid \nabla_{j_i} f_\varepsilon(x) \neq 0,\ 0 \leq i \leq l\}$. If $r = l$, it follows from Lemma 1 that $f_\varepsilon(y) < f_\varepsilon(x)$. If $r < l$, it follows from Lemma 2 that $f_\varepsilon(y) < f_\varepsilon(x)$. □

THEOREM 3. *From any initial point $x^0 \in \mathbf{R}^{n\times d}$, the HAP either stops at the ε-optimal solution of GEMFL, or generates an infinite sequence $\{x^s\}$. If $\{x^s\}$ is bounded, then $\{x^s\}$ converges to the ε-optimal solution of GEMFL.*

Proof. If the HAP stops at some iteration, then $x^{s+1} = x^s$ for some integer s. It follows from the definition of the algorithm that $\nabla_j f_\varepsilon(x^s) = 0$, $j = 1, 2, \ldots, n$. Therefore, $\nabla f_\varepsilon(x) = 0$ and x^s is the ε-optimal solution of GEMFL.

Now suppose that HAP generates an infinite sequence $\{x^s\}$, which is bounded. Suppose that $\{x^s\}$ does not converge to the ε-optimal solution of GEMFL, there would exist a subsequence $\{x^{r_s}\}$ that converges to a point $\overline{x}$, which is not the ε-optimal solution of GEMFL. Without loss of generality, we may assume that the subsequent $\{x^{r_s+1}\}$ converges to some point $\widehat{x}$.

From the continuity, we have

$$\lim_{r_s\to\infty} f_\varepsilon(x^{r_s}) = f_\varepsilon(\overline{x}), \quad \lim_{r_s\to\infty} f_\varepsilon(x^{r_s+1}) = f_\varepsilon(\widehat{x}). \tag{24}$$

Since $\{f_\varepsilon(x^s)\}$ is monotonically decreasing, (24) implies

$$f_\varepsilon(\overline{x}) = \lim_{r_s\to\infty} f_\varepsilon(x^{r_s}) = \lim_{r_s\to\infty} f_\varepsilon(x^{r_s+1}) = f_\varepsilon(\widehat{x}). \tag{25}$$

It follows from the continuity of $T(x)$ that

$$\widehat{x} = \lim_{r_s \to \infty} x^{r_s+1} = \lim_{r_s \to \infty} T(x^{r_s}) = T(\overline{x}). \tag{26}$$

It follows from Theorem 2 that

$$f_\varepsilon(\widehat{x}) < f_\varepsilon(\overline{x}). \tag{27}$$

This is in contradiction with (25) and the proof is complete. □

Corollary 1. *If $B_{ji} = \alpha_{ji} B_j$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, and $D_{jk} = \beta_{jk} B_j$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, where $\alpha_{ji} > 0$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $\beta_{jk} > 0$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, and B_j is symmetric and positive definite, for any $j \in \{1, 2, \ldots, n\}$, then from any initial point $x^0 \in \mathbf{R}^{n \times d}$, the HAP either stops at the ε-optimal solution of GEMFL, or generates an infinite sequence $\{x^s\}$ converging to the ε-optimal solution of GEMFL.*

Proof. From Proposition 2 it follows that if HAP generates an infinite sequence $\{x^s\}$, then $\{x^s\}$ is bounded. We apply now Theorem 3. □

Corollary 2. *If $B_{ji} = \alpha_{ji} B_j$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, and $D_{jk} = \beta_{jk} B_j$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, where $\alpha_{ji} > 0$, for any $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $\beta_{jk} > 0$, for any $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, and B_j is symmetric and positive definite, for any $j \in \{1, 2, \ldots, n\}$, then the unique ε-optimal solution of GEMFL problem is in the convex hull of the existing facilities.*

Proof. Start the HAP with any point in the convex hull of the existing facilities as the initial point. From Proposition 2, the sequence $\{x^s\}$ is in the convex hull of the existing facilities. From Corollary 1, the unique ε-optimal solution for the GEMFL problem either is one of this points or the limit of this sequence. Therefore, it is in the convex hull of the existing facilities. □

A future direction of research is to look for other conditions on the matrices B_{ji} and D_{jk}, in order to assure the boundedness of the sequence $\{x^s\}$.

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